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RADIAL SYMMETRY OF POSITIVE SOLUTIONS TO A CLASS OF FRACTIONAL LAPLACIAN WITH A SINGULAR NONLINEARITY

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Abstract. In this paper, we consider the following nonlocal fractional Laplacian equation with a singular nonlinearity

$$(-\Delta)^s u(x) = \lambda u^\beta(x) + a_0 u^{-\gamma}(x),\ x \in \mathbb{R}^n,$$

where $0 < s < 1$, $\gamma > 0$, $1 < \beta \leq \frac{n+2s}{n-2s}$, $\lambda > 0$ are constants and $a_0 \geq 0$.

We use a direct method of moving planes which introduced by Chen-Li-Li to prove that positive solutions $u(x)$ must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$.

1. Introduction

The fractional Laplacian appears in diverse areas including mathematical finances, physics, biological modeling and so on. It has been the subject of great interest in recent years (see in [2, 3, 11] and the reference therein). Analytical problems with negative powers arise naturally in the thin film equations and electrostatic micro-electromechanical system derives [1, 10] and the reference therein. In this paper, we are interested in the existence and symmetry of positive solutions satisfying the fractional Laplacian equation with a singular nonlinearity

$$(-\Delta)^s u(x) = \lambda u^\beta(x) + a_0 u^{-\gamma}(x),\ x \in \mathbb{R}^n, \tag{1.1}$$

where $0 < s < 1$, $\gamma > 0$, $1 < \beta \leq \frac{n+2s}{n-2s}$, $\lambda > 0$ is a constant and $a_0 \geq 0$.

We first recall that the fractional Laplacian $(-\Delta)^s u(x)$ $(0 < s < 1)$ is well-defined as

$$\begin{aligned}(-\Delta)^s u(x) &= C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(z)}{|x - z|^{n+2s}} dz \\ &= C_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}} dz,\end{aligned}$$

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where PV stands for the Cauchy principal value.

In order to make sense the integral, we require that

$$u \in L_{2s} \cap C^{1,1}_{loc}$$

with

$$L_{2s} = \left\{ u \in L^1_{loc} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty \right\}.$$

The difficulties in the study of nonlinear fractional Laplacian equation with a singular nonlinearity (1.1) are the non-locality of the fractional Laplacian and the negative power. To circumvent this difficulty, Caffarelli and Silvestre [4] (see also [3,5]) introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions, they considered the properties of the positive solutions for

$$(-\Delta)^s u(x) = u^p(x), \ x \in \mathbb{R}^n, \tag{1.2}$$

and obtained:

Corollary 1.1 ([2]). *Assume that* $\frac{1}{2} \leq s < 1$ *and* $1 < p < \frac{n+2s}{n-2s}$. *Then the equation* (1.2) *possesses no bounded positive solution.*

Remark 1.1. Due to the technical restriction, they have to assume $s \geq \frac{1}{2}$. It seems that this condition cannot be weakened if one wants to carry the *method of moving planes* on extended solution.

Another useful method to study the fractional Laplacian is the integral equations method, which turns a given fractional Laplacian equation into its equivalent integral equation, and then various properties of the original equation can be obtained by investigating the integral equation, see [7,8] and references therein.

However, there has neither been any extension method nor the integral equations method that work for problems involving the non-local operators such as fractional p-Laplacian when $p \neq 2$. Recently, Chen, Li and Li in [9] developed a direct method of moving planes that can deal with directly the fractional Laplacian with subcritical and critical Sobolev exponent. This direct method of moving planes has some advantages, it overcomes the necessary of imposing extra assumptions on the solutions when using the extension method.

In [9], the authors considered the same problem as (1.2) and then derived the following results:

Theorem 1.2 ([9]). *Assume that* $0 < s < 1$ *and* $u \in L_{2s} \cap C^{1,1}_{loc}$ *is a nonnegative solution of the equation* (1.2). *Then*

(i) *in the critical case* $p = \frac{n+2s}{n-2s}$, *the positive solution* u *must be radially symmetric and monotone decreasing about some point.*

(ii) *in the subcritical case* $1 < p < \frac{n+2s}{n-2s}$, $u \equiv 0$.

Recently Cai and Ma in [6] have used the direct *method of moving planes* to study the following fractional Laplacian equation with negative power

$$(-\Delta)^s u(x) + u^{-\beta}(x) = 0,\ x \in \mathbb{R}^n, \tag{1.3}$$

where $\gamma > 0$. Their main results are as follows:

Theorem 1.3 ([6]). *Let $\beta > 0$. Assume that the positive functions $u(x) \in L_{2s} \cap C^{1,1}_{loc}$ satisfies* (1.3) *with the growth/decay property*

$$u(x) = a|x|^m + o(1) \text{ as } |x| \to \infty, \tag{1.4}$$

where $\frac{2s}{\beta+1} < m < 1$, $a > 0$ is a constant. Then the positive solution $u(x)$ must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$.

In [12], the authors are interested in the existence and multiplicity properties of weak positive solutions satisfying the following singular elliptic boundary value problem involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s u(x) = \lambda u^\beta(x) + a(x)u^{-\gamma}(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $N > 2s$ $(0 < s < 1)$, $0 < \gamma < 1 < \beta < 2^*_s - 1 = \frac{N+2s}{N-2s}$ and $\lambda > 0$ is a real parameter, $a : \Omega \to \mathbb{R}$ is a given non-negative non-trivial function belonging to some Lebesgue space. They derive that:

Theorem 1.4 ([12]). *Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 3)$. Let $\gamma \in (0,1)$, $1 < \beta < 2^*_s - 1$, $a(x) \in L^2(\Omega)$ with $a(x) \geq 0$ in Ω. Then there exists a real number λ^* such that for any $\lambda \in (0, \lambda^*)$ the problem* (1.5) *possesses at least two weak positive $H^s_0(\Omega)$-solutions $u_\lambda, v_\lambda \in H^s_0(\Omega)$.*

Inspired by the ideas of [6, 9, 12], our main concern in this paper is the symmetry and monotonicity of positive solutions to (1.1) by means of the direct *method of moving planes*. The result is as follows:

Theorem 1.5. *Assume that $u(x) \in L_{2s} \cap C^{1,1}_{loc}$ is a positive solution of* (1.1). *Then the positive solution u must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$.*

Remark. Compared with the result of Cai and Ma [6], our radial symmetry and monotonicity results of solution need not the extra growth/decay condition (1.4).

2. Proof of Theorem 1.5

Proof. Because no decay condition on u near infinity is assumed, we are not able to carry the *method of moving planes* on u directly. To circumvent this difficulty, we make a Kelvin transform.

For $\forall\, x^0 \in \mathbb{R}^n$, let

$$\bar{u} = \frac{1}{|x-x^0|^{n-2s}} u\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right),\ x \in \mathbb{R}^n \backslash \{x^0\}, \tag{2.1}$$

be the Kelvin transform of u centered at x^0. It is well-known that

$$\begin{aligned}(-\Delta)^s \bar{u}(x) &= \frac{1}{|x-x^0|^{n+2s}}((-\Delta)^s u)\left(\frac{x-x^0}{|x-x^0|^2}\right)\\ &= \frac{\lambda \bar{u}^\beta(x)}{|x-x^0|^{\tau_1}} + \frac{a_0 \bar{u}^{-\gamma}(x)}{|x-x^0|^{\tau_2}},\end{aligned} \tag{2.2}$$

where $\tau_1 = n + 2s - \beta(n-2s)$ and $\tau_2 = n + 2s + \gamma(n-2s)$.

Choose any direction to be the x_1 direction. For $\lambda < x_1^0$, let

$$T_\lambda = \{x \in \mathbb{R}^n \,|\, x_1 = \lambda \ \text{ for some } \lambda \in \mathbb{R}\}$$

be the moving plane,

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \,|\, x_1 < \lambda\}$$

be the region to the left of the plane, and

$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$$

be the reflection of x about the plane T_λ.

To compare the values of $\bar{u}(x)$ and $\bar{u}(x^\lambda) = \bar{u}_\lambda(x)$, we denote

$$w_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x).$$

Otherwise, $w_\lambda(x^\lambda) = -w_\lambda(x)$, hence it is said to be anti-symmetric.

First, notice that, by the definition of w_λ, we have

$$\lim_{|x|\to\infty} w_\lambda(x) = 0.$$

Hence if w_λ is negative somewhere in Σ_λ, then the negative minimum of w_λ was attained in the interior of Σ_λ.

Let

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \,|\, w_\lambda < 0\}.$$

Then from (2.2), we have, for $x \in \Sigma_\lambda^- \backslash \{(x^0)^\lambda\}$,

$$\begin{aligned}(-\Delta)^s w_\lambda(x) &= \frac{\lambda \bar{u}_\lambda^\beta(x)}{|x^\lambda - x^0|^{\tau_1}} + \frac{a_0 \bar{u}_\lambda^{-\gamma}(x)}{|x^\lambda - x^0|^{\tau_2}} - \left[\frac{\lambda \bar{u}^\beta(x)}{|x-x^0|^{\tau_1}} + \frac{a_0 \bar{u}^{-\gamma}(x)}{|x-x^0|^{\tau_2}}\right]\\ &\geq \frac{\lambda \bar{u}_\lambda^\beta(x) - \lambda \bar{u}^\beta(x)}{|x-x^0|^{\tau_1}} + \frac{a_0 \bar{u}_\lambda^{-\gamma}(x) - a_0 \bar{u}^{-\gamma}(x)}{|x-x^0|^{\tau_2}}\\ &= \left[\frac{\lambda\beta \xi_\lambda^{\beta-1}(x)}{|x-x^0|^{\tau_1}} + \frac{-\gamma a_0 \eta_\lambda^{-\gamma-1}(x)}{|x-x^0|^{\tau_2}}\right] w_\lambda(x)\\ &\geq \left[\frac{\lambda\beta \bar{u}^{\beta-1}(x)}{|x-x^0|^{\tau_1}} - \frac{\gamma a_0 \bar{u}^{-\gamma-1}(x)}{|x-x^0|^{\tau_2}}\right] w_\lambda(x),\end{aligned}$$

where $\xi_\lambda(x)$, $\eta_\lambda(x)$ are valued between $\bar{u}_\lambda(x)$ and $\bar{u}(x)$. That is

$$(-\Delta)^s w_\lambda(x) + c(x) w_\lambda(x) \geq 0,\ x \in \Sigma_\lambda^- \backslash \{(x^0)^\lambda\} \tag{2.3}$$

with

$$c(x) = \frac{\gamma a_0 \bar{u}^{-\gamma-1}(x)}{|x - x^0|^{\tau_2}} - \frac{\lambda \beta \bar{u}^{\beta-1}(x)}{|x - x^0|^{\tau_1}}. \tag{2.4}$$

From (2.1), it is easy to verify that, for $|x|$ sufficiently large

$$c(x) \sim \frac{1}{|x|^{4s}} \tag{2.5}$$

and it follows that $c(x)$ is clearly bounded from below in Σ_λ^-.

Step 1. In this step, we show that for λ sufficiently negative, it holds

$$w_\lambda(x) \geq 0,\ \forall x \in \Sigma_\lambda \backslash \{(x^0)^\lambda\}. \tag{2.6}$$

When λ sufficiently negative, there exist $\epsilon > 0$ and $c_\lambda > 0$ such that

$$w_\lambda(x) \geq c_\lambda,\ x \in B_\epsilon((x^0)^\lambda) \backslash \{(x^0)^\lambda\}.$$

For the proof, please see the proof of Appendix in Lemma 3.3. Then one can see that Σ_λ^- has no intersection with $B_\epsilon((x^0)^\lambda)$.

Since $w_\lambda(x)$ satisfies the following

$$\begin{cases} (-\Delta)^s w_\lambda(x) + c(x) w_\lambda(x) \geq 0, & x \in \Sigma_\lambda^- \backslash \{(x^0)^\lambda\}, \\ w_\lambda(x) \geq 0, & x \in \Sigma_\lambda \backslash \Sigma_\lambda^-, \\ w_\lambda(x^\lambda) = -w_\lambda(x), & x \in \Sigma_\lambda, \end{cases}$$

with $c(x) \sim \frac{1}{|x|^{4s}}$ for $|x|$ large, applying the maximum principle of decay at infinity to $w_\lambda(x)$, we conclude that, there exists a $R_0 > 0$ (independent of λ) such that if $\bar{x}$ is a negative minimum of $w_\lambda(x)$ in Σ_λ, then

$$|\bar{x}| \leq R_0 \tag{2.7}$$

that is, for λ sufficiently negative, (2.6) holds.

Step 2. Inequality (2.6) provides a starting point, from which we move the plane T_λ toward the right as long as (2.6) holds to its limiting position to show that $\bar{u}$ are symmetric about the limiting plane. More precisely, let

$$\lambda_0 = \sup\{\lambda < x_1^0 \,|\, w_\mu(x) \geq 0,\ x \in \Sigma_\mu \backslash \{(x^0)^\mu\},\ \forall \mu \leq \lambda\},$$

and we will show that

$$\lambda_0 = x_1^0,$$

and

$$w_{\lambda_0}(x) \equiv 0,\ x \in \Sigma_{\lambda_0}. \tag{2.8}$$

The idea to prove this claim is below. Suppose $\lambda_0 < x_1^0$, we can show that the plane T_λ can be moved further right to cause a contradiction with the definition of λ_0 via the maximum principle of decay at infinity and the narrow region principle.

Again we need the fact that (see the proof of Appendix in Lemma 3.4) there exists $C > 0$ such that for sufficiently small η

$$w_{\lambda_0}(x) \geq C,\ x \in B_\eta((x^0)^{\lambda_0})\backslash\{(x^0)^{\lambda_0}\}. \tag{2.9}$$

The relation (2.7) tells us that the negative minimum of $w_{\lambda_0}(x)$ cannot attains in $B^c_{R_0}(0)$. We want to show that it can not be attained inside of $B_{R_0}(0)$. That is, we show that for λ sufficiently close to λ_0

$$w_\lambda(x) \geq 0,\ x \in (\Sigma_\lambda \cap B_{R_0}(0))\backslash\{(x^0)^\lambda\}. \tag{2.10}$$

When $\lambda_0 < x_1^0$, we already have $w_{\lambda_0}(x) \geq 0$, $x \in \Sigma_{\lambda_0}\backslash\{(x^0)^{\lambda_0}\}$. In fact, by the *strong maximum principle*, we have

$$w_{\lambda_0}(x) > 0,\ x \in \Sigma_{\lambda_0}\backslash\{(x^0)^{\lambda_0}\}. \tag{2.11}$$

To see this, we argue by contradiction.

In fact, if there exists some $\tilde{x}$ such that

$$w_{\lambda_0}(\tilde{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0}(x) = 0,$$

it follows that

$$\begin{aligned}
(-\Delta)^s w_{\lambda_0}(\tilde{x}) &= C_{n,2s} PV \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y)}{|\tilde{x}-y|^{n+2s}} dy \\
&= C_{n,2s} PV \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|\tilde{x}-y|^{n+2s}} dy + \int_{\mathbb{R}^n\backslash\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|\tilde{x}-y|^{n+2s}} dy \\
&= C_{n,2s} PV \left[\int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|\tilde{x}-y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(y)}{|\tilde{x}-y^{\lambda_0}|^{n+2s}} dy\right] \\
&= C_{n,2s} PV \int_{\Sigma_{\lambda_0}} \left[\frac{1}{|\tilde{x}-y^{\lambda_0}|^{n+2s}} - \frac{1}{|\tilde{x}-y|^{n+2s}}\right] w_{\lambda_0}(y) dy \\
&\leq 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(-\Delta)^s w_{\lambda_0}(\tilde{x}) &= \frac{\lambda \bar{u}^\beta_{\lambda_0}(\tilde{x})}{|\tilde{x}^{\lambda_0}-x^0|^{\tau_1}} + \frac{a_0 \bar{u}^{-\gamma}_{\lambda_0}(\tilde{x})}{|\tilde{x}^{\lambda_0}-x^0|^{\tau_2}} - \left[\frac{\lambda \bar{u}^\beta(\tilde{x})}{|\tilde{x}-x^0|^{\tau_1}} + \frac{a_0 \bar{u}^{-\gamma}(\tilde{x})}{|\tilde{x}-x^0|^{\tau_2}}\right] \\
&= \lambda \left[\frac{1}{|\tilde{x}^{\lambda_0}-x^0|^{\tau_1}} - \frac{1}{|\tilde{x}-x^0|^{\tau_1}}\right] \bar{u}^\beta(\tilde{x}) \\
&\quad + a_0 \left[\frac{1}{|\tilde{x}^{\lambda_0}-x^0|^{\tau_2}} - \frac{1}{|\tilde{x}-x^0|^{\tau_2}}\right] \bar{u}^{-\gamma}(\tilde{x}) \\
&> 0.
\end{aligned}$$

This is a contradiction.

It follows from (2.11) that there exist a small δ and a constant $c_0 > 0$ such that

$$w_{\lambda_0}(x) \geq c_0,\ x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}\backslash\{(x^0)^\lambda\}.$$

Since w_λ depends on λ continuously, there exist $\epsilon > 0$ and $\epsilon < \delta$ such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_\lambda(x) \geq 0,\ x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)} \backslash \{(x^0)^\lambda\}. \tag{2.12}$$

Since $w_\lambda(x)$ satisfies the following:

$$\begin{cases} (-\Delta)^s w_\lambda(x) + c(x) w_\lambda(x) \geq 0, & x \in \Sigma_\lambda^- \backslash \Sigma_{\lambda_0-\delta}, \\ w_\lambda(x) \geq 0, & x \in \Sigma_{\lambda_0-\delta}, \\ w_\lambda(x^\lambda) = -w_\lambda(x), & x \in \Sigma_\lambda, \end{cases}$$

where $\Sigma_\lambda^- \backslash \Sigma_{\lambda_0-\delta}$ is a narrow region and the lower bounded $c(x)$ can be see from (2.5), applying narrow region principle, we obtain

$$w_\lambda(x) \geq 0,\ x \in \Sigma_\lambda \backslash \Sigma_{\lambda_0-\delta}. \tag{2.13}$$

Combining (2.7), (2.12) and (2.13), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$w_\lambda(x) \geq 0,\ x \in \Sigma_\lambda \backslash \{(x^0)^\lambda\}$$

this contradicts the definition of λ_0, therefore, we must have

$$\lambda_0 = x_1^0, \quad \text{and} \quad w_{\lambda_0}(x) \equiv 0,\ x \in \Sigma_{\lambda_0}.$$

So far, we have proved that $\bar{u}$ is symmetric about the plane $T_{x_1^0}$. Since the x_1 direction can be chosen arbitrarily, we have actually show that $\bar{u}$ is radically symmetric about x^0, so is u. So we obtain that u is radically symmetric about some point in $\mathbb{R}^n$. The monotonicity is a consequence of the fact that (2.6) holds for all $-\infty < \lambda \leq \lambda_0$.

This completes the proof of Theorem 1.5. □

3. Appendix

In this section, we present two theorems which play important roles in applying the standard moving planes method, and that have been explained in [9] with detailed proof. We also give two lemmas about the proofs of w_λ is strictly positive and bounded away from 0 in a small neighborhood of x_0.

Theorem 3.1 (Narrow Region Principle)**.** *Let Ω be a bounded narrow region in Σ_λ such that it is contained in $\{x \mid \lambda - l < x_1 < \lambda\}$ with small $l > 0$. Suppose that $w_\lambda(x) \in C^{1,1}_{loc}(\Omega) \cap L_{2s}(\mathbb{R}^n)$ and is lower semi-continuous on $\bar{\Omega}$. If $c(x)$ is bounded from below in Ω and*

$$\begin{cases} (-\Delta)^s w_\lambda(x) + c(x) w_\lambda(x) \geq 0, & x \in \Omega, \\ w_\lambda(x) \geq 0, & x \in \Sigma_\lambda \backslash \Omega, \\ w_\lambda(x^\lambda) = -w_\lambda(x), & x \in \Sigma_\lambda, \end{cases}$$

then for sufficiently small, we have

$$w_\lambda(x) \geq 0 \text{ in } \Omega. \tag{3.1}$$

if Ω is unbounded, the conclusion still holds under the conditions

$$\varliminf_{|x| \to \infty} w_\lambda(x) \geq 0.$$

Theorem 3.2 (Decay at infinity)**.** *Let Ω be a bounded or unbounded domain in Σ_λ. Assume*

$$\begin{cases} (-\Delta)^s w_\lambda(x) + c(x) w_\lambda(x) \geq 0, & x \in \Omega, \\ w_\lambda(x) \geq 0, & x \in \Sigma_\lambda \backslash \Omega, \\ w_\lambda(x^\lambda) = -w_\lambda(x), & x \in \Sigma_\lambda, \end{cases}$$

with

$$\varliminf_{|x| \to \infty} |x|^{2s} c(x) \geq 0. \tag{3.2}$$

Then there exists a constant $R_0 > 0$ (depending only on $c(x)$, but independent of w_λ) such that if

$$w_\lambda(\tilde{x}) = \min_{\Omega} w_\lambda(x) < 0, \tag{3.3}$$

then

$$|\tilde{x}| \leq R_0. \tag{3.4}$$

Lemma 3.3. *Assume that $u(x) \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$ is a positive solution for*

$$(-\Delta)^s u(x) = \lambda u^\beta(x) + a_0 u^{-\gamma}(x), \ x \in \mathbb{R}^n. \tag{3.5}$$

Let $v(x) = \frac{1}{|x|^{n-2s}} u(\frac{x}{|x|^2})$ be a Kelvin transform of u, and

$$w_\lambda(x) = v_\lambda(x) - v(x).$$

Then there exists a constant $C > 0$ such that for any small $\epsilon > 0$ and λ sufficiently negative,

$$w_\lambda(x) \geq C > 0, \ x \in B_\epsilon(0^\lambda) \backslash \{0^\lambda\}.$$

Proof. The proof is similar to Lemma A.1 in [9]. Let η be a smooth cutoff function such that $\eta(x) \in [0,1]$ in $\mathbb{R}^n$, supp$\eta \subset B_2$ and $\eta(x) \equiv 1$ in B_1. Let

$$(-\Delta)^s \varphi(x) = \eta(x)(\lambda u^\beta(x) + a_0 u^{-\gamma}(x)).$$

Then

$$\begin{aligned} \varphi(x) &= C_{n,-2s} \int_{\mathbb{R}^n} \frac{\eta(y)(\lambda u^\beta(y) + a_0 u^{-\gamma}(y))}{|x-y|^{n-2s}} dy \\ &= C_{n,-2s} \int_{B_1} \frac{\lambda u^\beta(y) + a_0 u^{-\gamma}(y)}{|x-y|^{n-2s}} dy \\ &\quad + C_{n,-2s} \int_{B_2 \backslash B_1} \frac{\eta(y)(\lambda u^\beta(y) + a_0 u^{-\gamma}(y))}{|x-y|^{n-2s}} dy \\ &\leq C_{n,-2s} \int_{B_2} \frac{\lambda u^\beta(y) + a_0 u^{-\gamma}(y)}{|x-y|^{n-2s}} dy. \end{aligned}$$

It is easy to see that for all $|x|$ sufficiently large, there exists a constant $C > 0$ such that

$$\frac{2C}{|x|^{n-2s}} \leq \varphi(x) \leq \frac{3C}{|x|^{n-2s}}. \tag{3.6}$$

For $R > 0$ large, let

$$g(x) = u(x) - \varphi(x) + \frac{3C}{R^{n-2s}}.$$

Immediately, we have

$$\begin{cases} (-\Delta)^s g(x) \geq 0 & \text{in } B_R, \\ g(x) \geq 0 & \text{in } B_R^c. \end{cases} \tag{3.7}$$

It follows from the narrow region principle that

$$g(x) \geq 0 \text{ in } B_R.$$

Thus

$$u(x) - \varphi(x) + \frac{3C}{R^{n-2s}} \geq 0, \ x \in B_R.$$

Fix x, as $R \to \infty$,

$$u(x) \geq \varphi(x) \text{ in } \mathbb{R}^n. \tag{3.8}$$

Combining (3.6) and (3.8), we derive that

$$u(x) \geq \frac{2C}{|x|^{n-2s}}.$$

It follows that

$$u(x) = \frac{1}{|x|^{n-2s}} v\left(\frac{x}{|x|^2}\right) \geq \frac{2C}{|x|^{n-2s}},$$

that is

$$v_\lambda(x) \geq 2C > 0, \ x \in B_\epsilon(0^\lambda) \backslash \{0^\lambda\}.$$

For λ sufficiently negative, we have

$$v(x) \leq C, \ x \in \Sigma_\lambda.$$

Therefore,

$$w_\lambda(x) \geq C > 0, \ x \in B_\epsilon(0^\lambda) \backslash \{0^\lambda\}.$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let the w_λ be defined as before in the previous theorem and let*

$$\lambda_0 = \sup\{\lambda \,|\, w_\mu(x) \geq 0, \ x \in \Sigma_\mu \backslash \{(0)^\mu\}, \ \mu \leq \lambda\}.$$

If $w_{\lambda_0} \not\equiv 0$, then

$$w_{\lambda_0} \geq c > 0, \ x \in B_\eta((0)^{\lambda_0}) \backslash \{(0)^{\lambda_0}\}.$$

Proof. By the definition, if

$$w_{\lambda_0} \not\equiv 0, \ x \in \Sigma_{\lambda_0},$$

then there exists a point x^0 such that

$$w_{\lambda_0}(x^0) > 0.$$

And further, there exists a small positive η such that

$$w_{\lambda_0}(x) > 0, \ x \in B_\eta(x^0).$$

From the integral equation that $w_{\lambda_0}(x)$ satisfies, we can derive that. If $u(x)$ satisfy the following equation

$$(-\Delta)^s u(x) = \lambda u^\beta(x) + a_0 u^{-\gamma}(x),\ x \in \mathbb{R}^n,$$

then

$$u(x) = \int_{\mathbb{R}^n} \frac{\lambda u^\beta(y) + a_0 u^{-\gamma}(y)}{|x-y|^{n-2s}} dy,$$

that is

$$\begin{aligned} u\left(\frac{x}{|x|^2}\right) &= \int_{\mathbb{R}^n} \frac{\lambda u^\beta(y) + a_0 u^{-\gamma}(y)}{|\frac{x}{|x|^2} - y|^{n-2s}} dy \\ &= \int_{\mathbb{R}^n} \frac{\lambda u^\beta(\frac{z}{|z|^2}) + a_0 u^{-\gamma}(\frac{z}{|z|^2})}{|\frac{x}{|x|^2} - \frac{z}{|z|^2}|^{n-2s}} \frac{1}{|z|^{2n}} dz \\ &= |x|^{n-2s} \int_{\mathbb{R}^n} \frac{\lambda u^\beta(\frac{z}{|z|^2}) + a_0 u^{-\gamma}(\frac{z}{|z|^2})}{|x-z|^{n-2s}} \frac{1}{|z|^{n+2s}} dz, \end{aligned}$$

then

$$\begin{aligned} v(x) = &\int_{\mathbb{R}^n} \frac{\lambda |z|^{\beta(n-2s)} v^\beta(z) + a_0 |z|^{-\gamma(n-2s)} v^{-\gamma}(z)}{|x-z|^{n-2s}} \frac{dz}{|z|^{n+2s}} \\ = &\int_{\mathbb{R}^n} \frac{1}{|x-z|^{n-2s}} \left[\frac{\lambda v^\beta(z)}{|z|^{\tau_1}} + \frac{a_0 v^{-\gamma}(z)}{|z|^{\tau_2}}\right] dz \\ = &\int_{\Sigma_\lambda} \frac{1}{|x-z|^{n-2s}} \left[\frac{\lambda v^\beta(z)}{|z|^{\tau_1}} + \frac{a_0 v^{-\gamma}(z)}{|z|^{\tau_2}}\right] dz \\ &+ \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{1}{|x-z|^{n-2s}} \left[\frac{\lambda v^\beta(z)}{|z|^{\tau_1}} + \frac{a_0 v^{-\gamma}(z)}{|z|^{\tau_2}}\right] dz \\ = &\int_{\Sigma_\lambda} \left\{ \frac{1}{|x-z|^{n-2s}} \left[\frac{\lambda v^\beta(z)}{|z|^{\tau_1}} + \frac{a_0 v^{-\gamma}(z)}{|z|^{\tau_2}}\right] \right. \\ &\left. + \frac{1}{|x-z^\lambda|^{n-2s}} \left[\frac{\lambda v_\lambda^\beta(z)}{|z^\lambda|^{\tau_1}} + \frac{a_0 v_\lambda^{-\gamma}(z)}{|z^\lambda|^{\tau_2}}\right] \right\} dz, \end{aligned}$$

where $\tau_1 = n + 2s - \beta(n-2s)$, $\tau_2 = n + 2s + \gamma(n-2s)$, one can derive that

$$\begin{aligned} w_{\lambda_0}(x) = C_{n,2s} &\int_{\Sigma_\lambda} \left\{ \left[\frac{1}{|x^\lambda - z|^{n-2s}} - \frac{1}{|x-z|^{n-2s}}\right] \left[\frac{\lambda v^\beta(z)}{|z|^{\tau_1}} + \frac{a_0 v^{-\gamma}(z)}{|z|^{\tau_2}}\right] \right. \\ &\left. + \left[\frac{1}{|x^\lambda - z^\lambda|^{n-2s}} - \frac{1}{|x-z^\lambda|^{n-2s}}\right] \left[\frac{\lambda v_\lambda^\beta(z)}{|z^\lambda|^{\tau_1}} + \frac{a_0 v_\lambda^{-\gamma}(z)}{|z^\lambda|^{\tau_2}}\right] \right\} dz \\ = C_{n,2s} &\int_{\Sigma_\lambda} \left[\frac{1}{|x-z|^{n-2s}} - \frac{1}{|x-z^\lambda|^{n-2s}}\right] \\ &\left[\frac{\lambda v_\lambda^\beta(z)}{|z^\lambda|^{\tau_1}} - \frac{\lambda v^\beta(z)}{|z|^{\tau_1}} + \frac{a_0 v_\lambda^{-\gamma}(z)}{|z^\lambda|^{\tau_2}} - \frac{a_0 v^{-\gamma}(z)}{|z|^{\tau_2}}\right] dz \end{aligned}$$

$$\geq C_{n,2s}\int_{\Sigma_\lambda}\left[\frac{1}{|x-z|^{n-2s}}-\frac{1}{|x-z^\lambda|^{n-2s}}\right]$$
$$\left[\frac{\lambda\beta v^{\beta-1}(z)}{|z|^{\tau_1}}+\frac{-\gamma a_0 v^{-\gamma-1}(z)}{|z|^{\tau_2}}\right]w_\lambda(z)dz$$

the last inequality is due to $\lambda < x_1^0 = 0$.

It follows the positivity of w_{λ_0} that

$$w_{\lambda_0}(x) \geq \int_{B_\eta(x^0)} c_0 dy > 0,$$

if and only if

$$\frac{\lambda\beta v^{\beta-1}(z)}{|z|^{\tau_1}}-\frac{\gamma a_0 v^{-\gamma-1}(z)}{|z|^{\tau_2}} \geq 0,$$

that is,

$$v^{\beta+\gamma}(z) \geq \frac{a_0\gamma}{\lambda\beta}\frac{1}{|z|^{(n-2s)(\beta+\gamma)}},$$

we have

$$v(x) = \frac{1}{|x|^{n-2s}}u\left(\frac{x}{|x|^2}\right) \geq \left(\frac{a_0\gamma}{\lambda\beta}\right)^{\frac{1}{\beta+\gamma}}\frac{1}{|x|^{n-2s}},$$

i.e., $u(\frac{x}{|x|^2}) \geq (\frac{a_0\gamma}{\lambda\beta})^{\frac{1}{\beta+\gamma}} = C$, where $C > 0$ is a constant.

This completes the proof of Lemma 3.4. □

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