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Private Equity Valuation under Model Uncertainty*

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Abstract

The study incorporates model uncertainty into the private equity (PE) valuation model (SWY model) (Sorensen et al., 2014) to evaluate how model uncertainty distorts the leverage and valuations of PE funds. This study applies a continuous-time model to PE project valuation, modeling the LPs' goal as multiplier preferences provided by Anderson et al. (2003), and assuming that LPs' aversion to model uncertainty causes endogenous belief distortions with entropy as a measure of model discrepancies. Concerns regarding model uncertainty, according to the theoretical model, have an unclear effect on LPs' risk attitude and GPs' decision, which is based on the value of the PE asset. It also demonstrates that model uncertainty lowers the certainty-equivalent valuation of the LPs. Finally, we compare the outcomes of the Full-spanning risk model with the Non-spanned risk model, and they match the intuitive economic reasoning. The most important implication is that model uncertainty will have negative effects on the LPs' certainty-equivalent valuation but has ambiguous effects on the portfolio allocation choice of liquid wealth. Our works contribute to two literature streams. The first is the literature that models the PE funds. The second is the literature introduces model uncertainty into standard finance models.

Keywords: Private Equity, Model Uncertainty, Valuations, Continuous-Time Model

JEL Classification Code: D81, G11, G2, G32

1. Introduction

The rise of Private Equity (PE) is inextricably linked to the financial system and the politics of debt. Debt, according to Jensen and Meekling (1976), lowers free cash flow agency costs, and the popularity of LBOs (Leveraged Buyouts) is largely attributable to this debt control function. Axelson et al. (2013) also showed that the use of leverage is strongly associated with higher valuation levels and lower PE return. Essentially, Private Equity is a closed-ended, finite partnership whereby the limited partner (LPs) and the general partner (GPs) share the residual profits after paying debt holders (Liu et al., 2018). Unlike hedge funds, which are mostly short-term traders, PE takes ownership and management control of corporations. Nowadays, Private Equity has emerged as an important area of the economy in the last decades and accounts for a substantial share of aggregate investment and production. The phenomenon that the average return of Private Equity exceeds the return on the market is widely accepted (Harris et al., 2014). So the academic literature focuses on whether this out-performance is sufficient to LPs due to the cost of risks and long-term illiquidity.

In response to the global financial crisis that began in 2007, governments are rethinking their approach to regulating financial institutions. For its features of illiquid and long term, PE funds and the secondary markets for PE positions are opaque, making the LPs is hard to rebalance their investments. In addition, regulars, politicians, and labor organizers have long expressed concern about the impact of PE pointing to their need to rapidly return capital to investors and the potentially deleterious effects of such practices as the extensive leverage of firms (Bernstein et al., 2017).

Security Exchange Commission (SEC) inquiries have examined the possibility of PE general partners (GP) overstating portfolio net asset values (NAV) in an attempt to attract investors to future funds. Chung et al. (2012)

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found that GPs trade off the short-term profits with long-term consequences and the information asymmetry about the valuation bias persists even after a fund is resolved. Due to the different status and information asymmetry between LPs and GPs. They treat risk and ambiguity in different ways. Just as documented by Hansen and Sargent (2001), economic agents believe that the observed economic data come from a set of unspecified models. Concerns regarding model misspecification induce economic agents to make a robust decision.

To study how model uncertainty impacts the leverage and values of PE funds, this paper introduces model uncertainty into Sorensen et al. (2014) (SWY model). In the classic approach to agency contract difficulties, both the principal and the agent are assumed to believe in the same level of uncertainty. Concern about model misspecification, according to Miao and Rivera (2016), leads a decision-maker to demand robust decision rules that work across a variety of near models. We assume that the GPs are risk-averse and that they are aware of the actual output distribution, so they have faith in the reference model. We model the LPs' objective as the multiplier preferences proposed by Andersen et al. (2003) and our model has an essential building block, LPs' aversion to model uncertainty generates endogenous belief distortions and we use entropy to measure model discrepancies, which is wildly used in statistics and econometrics for model detection.

After introducing model uncertainty into the previous model, we find the following main innovative conclusions. First, we find that model uncertainty reduces LPs' total valuation of partnership interests, particularly when PE value is high. The higher the value of PE assets, the higher the LPs' valuation of partnership interests rises. This is understandable because taking model uncertainty into account will result in a more accurate model. This makes sense because taking into account model uncertainty causes LPs to lose faith in the ex-ante plan and become more risk-averse. Furthermore, including model uncertainty has an equivocal impact on liquid wealth dynamics portfolio allocation. With the private asset value increases LPs tend to allocate more weight to public equity rather than the risk-free asset.

Furthermore, while PE investments are dangerous, a portion of the risk can be mitigated by public liquid wealth, while the remaining risk cannot be mitigated by the market. As a result, we include the full-spanning situation in our model. Our findings show that, regardless of model uncertainty, the full-spanning situation has a greater valuation of the LP's partnership interest than the non-spanned case since there is no cost of illiquidity in this case. These findings show that avoiding unspanned risk is the most effective method to protect LP equity.

2. Literature Review

Our research is related to the growing literature that introduces model uncertainty into standard fund valuation models. Model uncertainty has been extensively discussed in asset pricing and corporate finance. Uppal and Wang (2003) study the problem of investors' intertemporal portfolio selection under model uncertainty and obtain steady investment strategies under continuous time. Maenhout (2004) presented an approach to the dynamic portfolio and consumption problem of an investor with model uncertainty based on the Anderson et al. (2003) model. Nishimura and Ozaki (2007) introduced Knightian uncertainty into the standard real options framework. Ju and Miao (2012) proposed a novel smooth ambiguity model that permits a three-way separation among risk aversion, ambiguity aversion, and intertemporal substitution, and showed that ambiguity aversion and model uncertainty plays a key role in explaining asset pricing puzzles. Szydlowski and Yoon (2021) investigated model uncertainty based on dynamic contract theory. Chen et al. (2014) studied the investors' optimal consumption and portfolio choice problem when he was confronted with two possibly misspecified models of stock returns. Miao and Rivera (2016) studied how to design robust contracts under hidden actions in a dynamic environment. However, the impact of model uncertainty in the valuation of funds has not received enough attention. Therefore, our paper introduces model uncertainty into SWY to study the leverage and valuation of PE funds.

Our work is also related to the literature about agency contracts and private equity investment. DeMarzo and Sannikov (2006) derived the optimal contract in a continuous-time setting. Axelson et al. (2009) developed a model and demonstrated that profit-sharing arrangements in PE funds should be nonlinear. Vijayakumaran and Vijayakumaran (2019) investigated the governance and capital structure decisions of Chinese listed companies; Saleem and Usman (2021) studied the Role of Stock Price Crash Risk with the consideration of information risk and equity cost; Kakinuma (2020) examined financial distress and investigated the return premium and negative book value in the emerging financial market; Lee (2020) examined R&D investment lagged effect on firm value using evidence from manufacturing firms listed in Chinese markets, and Sukesti et al. (2021) examined the factors affecting stock price and firm performance.

3. Model Setup

3.1. Public Equity

An institutional investor with an infinite horizon invests in three assets: risk-free asset, public equity, and private equity. The risk-free asset and public equity represent the standard investment opportunities as in the classic Merton (1975) model. The risk-free asset pays a constant interest rate r. Public equity can be interpreted as the public market portfolio, and its value, S_t, follows the geometric Brownian motion (GBM):

$$\frac{\mathrm{dS}_t}{S_t} = \mu_S dt + \sigma_S \mathrm{dB}_t^S, \tag{1}$$

where B_t^S is a standard Brownian motion, and μ_S and σ_S are the constant drift and volatility parameters. The Sharpe ratio for the public equity is: $\eta = \dfrac{\mu_S - r}{\sigma_S}$.

3.2. Private Equity

3.2.1. PE Asset Risk

PE asset is illiquid, and must hold it to maturity time, T. We assume that maturity equals the life of the PE fund. The value of the PE asset is the total value of the portfolio companies. Between times 0 and T, the value of PE asset, A_t, follows the GBM:

$$\frac{\mathrm{dA}_t}{A_t} = \mu_A dt + \sigma_A \mathrm{dB}_t^A, \tag{2}$$

where B_t^A is a standard Brownian motion, μ_A is the drift, and σ_A is the volatility. At times T, the PE asset is liquidated for total proceeds of A_t, and these proceeds are divided among the creditors, LPs, and GPs according to the waterfall structure specified below.

The correlation between the B_t^S, and B_t^A processes are denoted ρ to capture its systematic risk. When $|\sigma| < 1$, the two processes are not perfectly correlated; the risk of the PE asset is not fully spanned by the market, and the LPs cannot fully hedge the risk of the PE investment by dynamically trading the public equity and risk-free asset.

The unlevered beta (or asset beta) of the PE asset is given as $\beta = \dfrac{\rho\sigma_A}{\sigma_S}$, The total volatility of the PE asset is σ_A. The fraction of this volatility that is spanned by the public market is $\rho\sigma_A$. The remaining unspanned volatility is denoted, given as $\epsilon = \sqrt{\sigma_A^2 - \rho^2\sigma_A^2} = \sqrt{\sigma_A^2 - \beta^2\sigma_S^2}$.

3.2.2. Return of PE Asset

An important feature of our model is that it allows the value of the underlying PE asset to appreciate faster than the overall market and earn an excess risk-adjusted return, called alpha Formally, alpha is defined as:

$$\alpha = \mu_A - r - \beta\,(\mu_S - r). \tag{3}$$

3.2.3. Structure and Waterfall

Following the definition of waterfall in Sorensen et al. (2014), GPs' compensation can be demonstrate as management fees and an incentive fee. The management fee is an ongoing payment by the LPs to the GPs, specified as a fraction m of the committed capital, X_0. The committed capital is the sum of the initial investment, I_0, and the total management fee paid over the life of the fund: $X_0 = I_0 + mTX_0$.

When the fund matures, the final proceeds, A_t, are divided among the creditors, the LPs, and the GPs according to the waterfall schedule. For the creditors, let y denote the continuous yield on the debt. Assuming balloon debt, the payment due to the creditors, at maturity T, is: $Z_0 = D_0 e_{yT}$. Which includes both principals D_0 and interest payments. Any remaining proceeds after repaying the creditors, Z_0, and returning the LP's committed capital, X_0, constitute the funds' profits, given as: $A_t - X_0 - Z_0$.

3.2.4. Region 0: Debt Repayment ($A_T \le Z_0$)

Our model applies to general forms of debt, but for simplicity, we consider balloon debt with no intermediate payments. The principal and accrued interest are due at maturity T. Let y denote the debt yield, which we derive below to ensure creditors break even. At maturity T, the payment to the creditors is:

$$D(A_T, T) = \min\,(A_T, Z_0). \tag{4}$$

The debt is senior, and when the final proceeds, A_T, fall below this boundary, the LP and GP receive nothing.

3.2.5. Region 1: Preferred Return ($Z_0 \le A_T \le Z_1$)

After repaying the creditors, the investors and managers share the residual benefits. First, the investors must receive a preferred (hurdle) return before any profits can be distributed to managers. Formally, let F denote the amount that investors require to meet the hurdle:

$$F = I_0 e^{hT} + \int_0^T mX_0 e^{hs} ds = I_0 e^{hT} - \frac{mX_0}{h}\left(1 - e^{hT}\right). \tag{5}$$

The first term is the required return based on the initial investment. The second term measures the required return due to the payment of management fees during the PE investment period. The upper boundary, Z_1, of the preferred-return region satisfies $Z_1 = F + Z_0$. The LPs payoff in this region, at maturity T, is:

$$LP_1(A_T, T) = \max\{A_T - Z_0, 0\} - \max\{A_T - Z_1, 0\}. \quad (6)$$

3.2.6. Region 2: Catch-up ($Z_1 \leq A_T \leq Z_2$)

With a positive hurdle rate, the LPs require some of the funds' initial profits to meet the hurdle. The catch-up region then awards a large fraction, denoted n (typically, 100%), of the subsequent profits to the GPs to catch up to the pre-scribed profit share, denoted k (typically, 20%). The upper boundary of this region, Z_2, is the amount of final proceeds that are required for the GP to fully catch up, and it solves: $k(Z_2 - (X_0 + Z_0)) = n(Z_2 - Z_1)$. When $n < 100\%$, LPs receive the following residual payoff in this region:

$$LP_2(A_T, T) = (1 - n)\max\{A_T - Z_1, 0\} - \max\{A_T - Z_2, 0\}. \quad (7)$$

3.2.7. Region 3: Profit-sharing ($A_T > Z_2$)

After the GPs catch up with the prescribed profit share, k, The LPs' payoff in this profit-sharing is given as:

$$LP_3(A_T, T) = (1 - k)\max\{A_T - Z_2, 0\}. \quad (8)$$

3.2.8. LPs' Partnership Interest

At maturity T, the value of the LPs' partnership interest is the sum of the values of the LPs' individual payoffs in the three regions:

$$LP(A_T, T) = LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T). \quad (9)$$

Before maturity T, the value of the LPs' partnership interest under full spanning is denoted $LP(A_t, t)$:

$$\begin{aligned} LP(A_t, t) &= LP_1(A_t, t) + LP_2(A_t, t) \\ &\quad + LP_3(A_t, t) - MF(A_t, t). \end{aligned} \quad (10)$$

3.3. LPs' Problem

3.3.1. Preferences

For simplicity, the LPs and GPs are both risk-neutral and we use risk-free rate r to discount the LPs' utility function. Thus the LPs have standard time-separable preferences, represented by:

$$\mathbb{E}\left[\int_0^\infty e^{-rt}U(C_t)dt\right]. \quad (11)$$

For tractability, we choose $U(C) = -\frac{e^{-\gamma C}}{C}$, where $\gamma > 0$ is the coefficient of absolute risk aversion (CARA).

3.3.2. Liquid Wealth Dynamics

Let W_t denote the LPs' liquid wealth process, which excludes the value of the LPs' partnership interest. The LPs allocate πt to public equity and the remaining $W_t - \pi_t$ to the risk-free asset. Over the life of the PE investment, the liquid wealth evolves as:

$$\begin{aligned} dW_t &= (rW_t - mX_0 - C_t)dt + \pi_t\left((\mu_S - r)dt + \sigma_S dB_t^S\right), \\ &\quad t < T. \end{aligned} \quad (12)$$

The first term in (11) is the wealth accumulation when the LPs are fully invested in the risk-free asset, net of management fees, mX_0, and the LPs' consumption, C_t. The second term is the excess return from the LPs' investment in public equity. At time T, when the fund is liquidated and the proceeds are distributed, the LPs' liquid wealth jumps: $W_T = W_{T-} + LP(A_T, T)$.

And the liquid wealth process simplifies to:

$$dW_t = (rW_t - C_t)dt + \pi_t\left((\mu_S - r)dt + \sigma_S dB_t^S\right), \ t > T. \quad (13)$$

3.3.3. Certainty-equivalent Valuation

Let $J(W, A, t)$ be the LPs' value function before the PE investment matures. Given $J^*(W)$ the value function is:

$$J\left(W_0, A_0, 0\right) = \max_{C,\pi} \mathbb{E}\left[\int_0^T e^{-rt}U\left(C_t\right)dt + e^{-rT}J^*\left(W_T\right)\right]. \quad (14)$$

The LPs' optimal consumption and public equity allocation solve the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rJ\left(W, A, t\right) = \max_{C,\pi} \; & U\left(C\right) + J_t + \begin{pmatrix} rW + \pi\left(\mu_S - r\right) \\ -mX_0 - C \end{pmatrix} J_W \\ & + \frac{1}{2}\pi^2\sigma_S^2 J_{WW} + \mu_A A J_A + \frac{1}{2}\sigma_A^2 A^2 J_{AA} \\ & + \rho\sigma_S\sigma_A \pi A J_{WA}. \end{aligned} \quad (15)$$

In Appendix, we verify that the value function takes the exponential form:

$$J\left(W, A, t\right) = -\frac{1}{\gamma r} exp\left[-\gamma r\left(W + b + V\left(A, t\right)\right)\right]. \quad (16)$$

$V(A, t)$ is the LPs certainty-equivalent valuation of the partnership interest.

3.3.4. Belief Distortions and Model Uncertainty

We now introduce belief distortions and concerns about model uncertainty. The LPs may not trust this model and consider alternative models to protect themselves from model misspecification. Let $\mathbb{P}$ reflect the previously definitized measure of the model, and $\mathbb{G}$ denotes the probability measure of the alternative model. Where η_t is its Radon-Nikodym derivative with the respect to $\mathbb{P}$.

$$\frac{d\eta_t}{\eta_t} = g_t \mathrm{dB}_t^A, \tag{17}$$

where g_t is a real-valued process satisfying $\int_0^t g_s^2 ds < \infty$ for all $t > 0$, and $\eta_0 = 1$. Then, we can define the standard Brownian process B_t^{Ag} by $dB_t^{Ag} = dB_t^A - g_t dt$. Thus, under the new measure $\mathbb{G}$, the value of a public asset and private value can be given by:

$$\begin{aligned} \frac{dA_t}{A_t} &= \mu_A dt + \sigma_A \left(g_t dt + dB_t^{Ag} \right) \\ &= \left(\mu_A + \sigma_A g_t \right) dt + \sigma_A dB_t^{Ag}. \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_S dt + \sigma_S \left(\rho dB_t^A + \sqrt{1-\rho^2}\, d\widehat{B}_t^A \right) \\ &= \left(\mu_S + \rho \sigma_S g_t \right) dt + \sigma_S \left(\rho dB_t^{Ag} + \sqrt{1-\rho^2}\, d\widehat{B}_t^A \right). \end{aligned} \tag{19}$$

Following Anderson et al. (2003), Hansen et al. (2006), and Hansen and Sargent (2012), we can calculate the discounted relative entropy to measure the discrepancy between $\mathbb{G}$ and $\mathbb{P}$:

$$\begin{aligned} r\mathbb{E}^{\mathbb{P}} \left[\int_0^\infty e^{-rt} \eta_t ln\eta_t dt \right] &= \frac{1}{2} \mathbb{E}^{\mathbb{P}} \left[\int_0^\infty e^{-rt} \eta_t g_t^2 dt \right] \\ &= \frac{1}{2} \mathbb{E}^{\mathbb{G}} \left[\int_0^\infty e^{-rt} g_t^2 dt \right]. \end{aligned} \tag{20}$$

To incorporate a concern about the robustness of belief distortions, we present the LP's objective to maximize:

$$\inf_g \left\{ \begin{aligned} &\mathbb{E}^{\mathbb{G}} \left[\int_t^T e^{-rt} U\left(C_t\right) dt + e^{-rT} J^*\left(W_T\right) \right] \\ &+ \frac{1}{2\theta} \mathbb{E}_t^{\mathbb{G}} \left[\int_t^T e^{-rt} g_t^2 \left| J\left(W, A, t\right) \right| dt \right] \end{aligned} \right\}. \tag{21}$$

Where the parameter θ can be interpreted as an ambiguity aversion parameter. A small θ implies a small degree of concern about robustness. According to the definite formulation of $J(W, A, t)$ in (15), the absolute value of $J(W, A, t)$ is negative. But in our model, the entropy penalty term of (21) should be positive. So that's the reason why we add the absolute value sign in (21). Additionally, as θ converges to zero, the manager's objective will reduce to the case without model uncertainty. In addition, when considering belief distortions and model uncertainty, the liquid wealth dynamics conforms to:

$$\begin{aligned} dW_t &= \left(rW_t - mX_0 - C_t + \pi\left(\mu_S - r\right) + \pi\sigma_S \rho g \right) dt \\ &\quad + \pi_t \sigma_S \left(\rho dB_t^{Ag} + \sqrt{1-\rho^2}\, d\widehat{B}_t^A \right), \; t < T. \end{aligned} \tag{22}$$

4. Solution

4.1. Complete-markets Solution

First, consider the case with complete markets and the risk of PE can be fully spanned by the public equity and risk-free asset. Thus the PE investment can be perfectly replicated and investors demand neither idiosyncratic nor illiquidity risk premia by dynamically trading a few long-lived assets. In this circumstance, under complete markets, there cannot be excess returns, adjusting for systematic risks. $\alpha = 0$, and the equilibrium expected rate of return μ_A for the PE asset is $\mu_A = r + \beta\left(\mu_R - r\right)$. LPs will trust the GPs investment strategy and it's unnecessary to talk about incorporating model uncertainty in our model.

4.2. Incomplete Markets with Non-spanned Risk

Different from the full-spanning case, with non-spanned risks, the risk of PE asset is not fully spanned by the public market, so the PE investment illiquidity will impact the cost of LP. In our model, we consider model uncertainty with the incorporation into SWY.

We can solve the LPs' optimization problem under model uncertainty using dynamic programming. We can obtain the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rJ(W,A,t)=\max_{C,\pi}\inf_{g}\begin{array}{l}U(C)+J_t+(rW+\pi(\mu_S-r)\\ +\pi\sigma_S\rho g-mX_0-C)J_W\\ +\frac{1}{2}\pi^2\sigma_S^2J_{WW}+(\mu_A A+\sigma_A Ag_t)J_A\\ +\frac{1}{2}\sigma_A^2A^2J_{AA}+\rho\sigma_S\sigma_A\pi AJ_{WA}\\ -\frac{g^2J(W,A,t)}{2\theta}.\end{array} \quad (23)$$

And the $J(W, A, t)$ takes the form defined in (15).

Proposition 1. Consumption and portfolio rules:

$$C\left(W,A,t\right)=r\left(W+V\left(A,t\right)+b\right), \quad (24)$$

$$\pi=\frac{\eta J_W J\left(W,A,t\right)+\theta\sigma_A A\rho J_W J_A+\rho\sigma_A AJ_W J\left(W,A,t\right)}{\theta\sigma_S\rho^2J_W^2+\sigma_S J_{WW}J\left(W,A,t\right)}. \quad (25)$$

The certainty-equivalent valuation of the LPs' partnership interest $V(A, t)$, given in (24), solves the partial differential equation (PDE):

$$\begin{aligned}rV(A,t)=&V_t-mX_0+\frac{1}{2}\sigma_A^2A^2V_{AA}+(\alpha+r)AV_A\\ &-\frac{\theta\rho^2}{2\gamma r(\theta\rho^2+1)}\eta^2+\frac{\rho\eta\theta(\rho^2-1)}{\theta\rho^2+1}\sigma_A AV_A \quad (26)\\ &-\frac{\gamma r}{2}\frac{\in^2(\theta+1)}{\theta\rho^2+1}A^2V_A^2.\end{aligned}$$

And the PDE equation is subject to two boundary conditions. First, at maturity T, the value of the LPs claim equals the LPs payoff: $V(A_T, T) = LP(A_T, T)$; Second, when the value of the underlying PE asset converges to zero, the value of the LP s partnership interest converges to the (negative) PV of the remaining management fees:

$$V\left(0,t\right)=-\int_t^T e^{-r(T-s)}mX_0ds=-\frac{mX_0}{r}\left(1-e^{-r(T-t)}\right). \quad (27)$$

In the second situation, the LP would receive nothing from the PE fund at maturity and there is no possibility that the value of the PE assets will increase.

5. Quantitative Results and Discussion

5.1. Parameter Choices and Calibration

First, under complete markets, the idiosyncratic risks of PE can be hedged perfectly, in this circumstance, the risk attitude of LPs has no influence on the model. However, in the case of incomplete markets, the LPs' valuation now depends on the LPs endogenous absolute risk-aversion, γ. As shown in SWY, let γ_R denote the LPs' relative risk aversion. In terms of the value function $J(W, A, t)$, the relative risk aversion is defined as: $\gamma_R = -[J_{WW}(W, A, t)/J_W(W, A, t)]W$. Using the FOC with respect to consumption, we can write γ_R as: $\gamma_R=\left[\gamma rU'(Ct)/U'(Ct)\right]W_t=\gamma rW_t$. We choose $\gamma = 0.5$ ($\gamma_R = 2.5$ and the LPs risk aversion attitude is at a high-efficiency level according to SWY).

Then following Metrick and Yasuda (2011), annual volatility for PE investments, $\sigma_A = 25\%$ for PE assets. We use the annual volatility of $\sigma_S = 25\%$ with an expected return of $\mu_S = 11\%$ for the public market. In addition, considering the initial investment leverage, Axelson et al. (2013) show that the mean range of debt divided by enterprise value in LBO over the period 1980–2008 is between 0.65 and 0.89. So they report that equity accounted for 25% of the purchase price. In this case, we choose $l = 3$ as our benchmark leverage ratio.

Besides these parameters, the next is the unlevered asset beta. In the literature of empirical study, Ewens et al. (2013) examined US data from 1980 to 2007 then find that the betas of VC are 1.24 while the BO's are 0.72. We calibrate the unlevered asset beta as 0.5. For this beta, the correlation between PE asset and the public market is $\rho = 0.4$ and the unspanned volatility is 23%.

In the field of empirical study, Robinson and Sensoy (2012) evaluated 837 funds and found that 37% of them had an initial management charge of 2% and a mean carried interest of 20.13 percent. Therefore we choose $m = 2\%$ and $k = 20\%$ as the basic compensation terms. In addition, we choose $T=10$ and $n=100\%$. Metrick and Yasuda (2011) analyzed PE funds in the 1993–2006 period and fiound that 92.4% of BO (44.7% of VC) funds employ a hurdle return mechanism for the LP, and most funds use 8% as their hurdle rate. Most funds The investment cost I equal to 100 and we use $r = 5\%$.

5.2. Valuation and Analysis

Figure 1 shows the portfolio distribution of liquid wealth and the certainty-equivalent valuation of the partnership stake. The LPs' Certainty-equivalent value can be greatly reduced when model uncertainty is incorporated into SWY, as seen in Panel A of Figure 1. When LPs don't trust the existing model and prefer other models, they need to defend themselves from information asymmetry and model

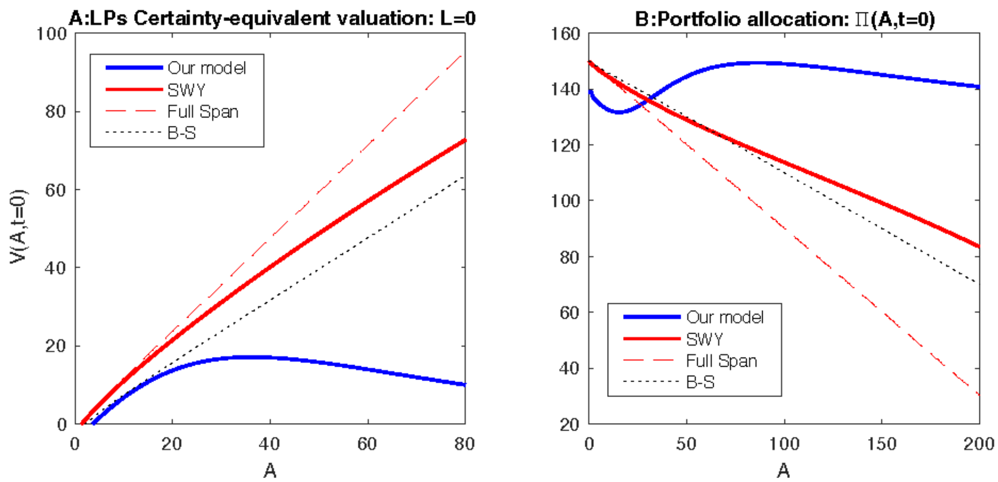

Figure 1: LPs Certainty-Equivalent Valuation and Portfolio Allocation Under L = 0

misspecification, which is easy to understand. Thus they will become more conservative, and choose the more risk-aversion investment strategy, and the results show that the LPs' Certainty-equivalent valuation is remarkably smaller than the model of SWY.

Panel B of Figure 1 indicates that model uncertainty will also affect the choice of portfolio allocation during the investment period. First, when the PE asset value is at a low level, the LP will choose to hold less public equity in liquid wealth rather than risk-free assets when considering model uncertainty. However, with the PE asset value increases, the impact of model uncertainty in LPs' liquid wealth will change to another situation. Without incorporating model uncertainty the LPs will choose the lower allocation of public equity with the growth of PE asset value, the LPs will allocate a lower rate to public equity compared to the model of SWY. However, when considering model uncertainty, the LPs will choose to allocate more public equity first and then decrease the amount of public equity with the PE asset value increased to a higher level. It is intuitive that when the PE asset value is low, the LPs' certainty-equivalent valuation is also at an extremely low rate, so the LPs will choose less public equity rather than the risk-free asset because they are relative extremely risk-aversion at this period. However, with the PE asset value and Certainty-equivalent valuation grow, the LPs will become less conservative and choose to allocate more public equity. In another circumstance, the PE asset value is at a high level, the LPs will certainty behave more confident about the return of the PE asset. They will choose a lower allocation of public equity to prevent the loss of liquid wealth due to the systematic risk of public equity.

In addition, we also compared the case of the full-spanning and Black-Scholes risk-neutral model. Our results show that under the full-spanning case, the LPs Certainty-equivalent valuation is highest compared to the other cases, and when considering model uncertainty, the LPs Certainty-equivalent valuation is not only smaller than the SWY model but also lower than the Black-Scholes risk-neutral model. That indicates that model uncertainty will extremely lower the LPs Certainty-equivalent valuation due to the more conservative strategy and more risk-aversion.

5.2.1. Break-even Alpha and Performance Measures

Due to management fees, carried interest, and idiosyncratic risks, LPs will only work with experienced managers in real-world investing situations. SWY defines break-even alpha as the LPs' incremental cost of capital of the PE investments; a higher break-even alpha indicates a higher cost of capital. Thus the break-even alpha equals the minimal level that GPs must generate to make the LPs participate $I_0 = V(A_0, 0)$.

However, in reality, the break-even alpha is difficult to estimate directly. Following Kaplan and Schoar(2005), Driessen et al. (2011), and Robinson and Sensoy (2011), we use the internal rate of return (IRR), the total-value-to-paid-in-capital multiple (TVPI), and the public market equivalent (PME) these three common empirical performance measures to measure the GPs performance.

Following SWY, Let ϕ denote the internal rate of return (IRR), the solution of ϕ is:

$$I_0 + \int_0^T m X_0 e^{-\phi t} dt = e^{-\phi T} \mathbb{E}\left[LP\left(A_T, T\right)\right]. \tag{28}$$

And the ex-ante expected TVPI and PME are defined as:

$$\mathbb{E}\left[TVPI\right]=\frac{\mathbb{E}\left[LP_1\left(A_T,T\right)+LP_2\left(A_T,T\right)+LP_3\left(A_T,T\right)\right]}{X_0}. \quad (29)$$

$$PME=\frac{\mathbb{E}\left[e^{-\mu_R T}(LP_1\left(A_T,T\right)+LP_2\left(A_T,T\right)+LP_3\left(A_T,T\right))\right]}{I_0+\frac{mX_0}{\mu_R}\left(1-e^{-\mu_R T}\right)}. \quad (30)$$

$$Adj.PME=\frac{\mathbb{E}\left[e^{-rT}(LP_1\left(A_T,T\right)+LP_2\left(A_T,T\right)+LP_3\left(A_T,T\right))\right]}{\mathbb{E}\left[I_0+\int_0^T mX_0 e^{-rt}dt\right]}. \quad (31)$$

As noted by Kaplan and Schoar (2004), empirical studies typically interpret $PME > 1$ as that GPs performance outperforming the market.

Table 1 reports break-even alphas under various levels of effective risk aversion of the different performance measures. Break-even alphas for LPs = 0 without model uncertainty are shown in the first row of Panel A. The LPs are effectively risking neutral in this situation, thus there is no additional cost of illiquidity or non-spanned risks, and the break-even alphas are 2.61 percent without leverage. For LPs to break even, GPs must provide an excess return of 2.61 percent. Risk-averse LPs expect a higher premium for taking idiosyncratic risks, which means the break-even alpha threshold will rise as the risk aversion level rises. The IRR, TVPI, and PME performance levels will all increase as the break-even alpha level rises. Thus that means the LPs break-even alpha values depend on the LPs risk-aversion magnitudes.

Break-even alphas for different levels of effective risk aversion under model uncertainty are shown in Panel B. For LPs = 0, the first row demonstrates that the LPs are effective risk-neutral, and we won't include model uncertainty, in this case, thus the performance measures results won't change. Under model uncertainty, however, risk-averse LPs will seek larger break-even alpha to absorb the higher cost of capital. As the risk-aversion parameter $|\gamma = 2$, the break-even alpha increases from 3.08% to 3.68, IRR increases from 8.41% to 9.36%, and TVPI increases from 2.16 to 2.37. Moreover, the PME will also increase from 0.78 to 0.84. The break-even alpha and performance measures findings will likewise increase to greater magnitudes under the case with $y = 5$. It is self-evident in economics that when model uncertainty is taken into account, the LPs would act more cautiously to protect themselves from the loss of model misspecification and uncertainty.

To recapitulate, effective risk-averse LPs will invest in PE when the investment excess return alpha surpasses the break-even alpha, and the performance measures IRR, TVPI, and PME are all dependent on the break-even alpha with the given investment beta. However, the most essential argument in our paper is that, when model uncertainty is taken into account, the LPs' estimate of future return probability is unclear; as a result, LPs will become more cautious and choose for the riskier approach. Our results indicate that model uncertainty will extremely lower the LPs Certainty-equivalent valuation due to the more conservative strategy. In contrast, the LPs will choose higher break-even alpha to participate in the PE investment, higher break-even means the PE investment performance measures will increase to a higher level.

5.3. Leverage

According to the basic Modigliani-Miller hypothesis, increasing leverage allows investors to earn a larger expected return while taking on more risk. The benefit of leverage in

Table 1: Break-Even Values of Empirical Performance Measures for Various Levels of Effective Risk Aversion Without Leverage

Panel A					
Risk Aversion	**Alpha (α)**	**IRR (ϕ)**	**E[TVPI]**	**PME**	**Adj. PME**
$\gamma l_0 = 0$	2.61%	7.90%	2.07	0.75	1.00
$\gamma l_1 = 2$	3.08%	8.41%	2.16	0.78	1.04
$\gamma l_2 = 5$	3.74%	9.02%	2.30	0.83	1.11
Panel B					
$\gamma l_0 = 0$	2.61%	7.90%	2.07	0.75	1.00
$\gamma l_0 = 2$	3.68%	9.36%	2.37	0.84	1.13
$\gamma l_0 = 5$	4.03%	9.71%	2.44	0.88	1.17

the PE market is that GPs can have a larger asset base, but LPs pay a greater price for idiosyncratic and systematic risks. When evaluating model uncertainty, it's crucial to examine how leverage may affect valuation, break-even alpha, and other PE investment performance metrics.

Figure 2 depicts the LPs' Certainty-equivalent valuation of the partnership interest and the portfolio of liquid wealth for the PE asset investment leverage $L = 3$ in addition to the unique case with leverage $L = 0$. We can draw similar conclusions from this graph as we did with the previous scenario $L = 0$. However, in this circumstance, the certainty-equivalent valuation is significantly lower than in the absence of leverage. It makes sense because the debt carrier will receive a big portion of the PE asset value. Furthermore, while comparing Figures 1 and 2, we can see that the curve of panel A in Figure 1 is more concave than the curve of panel A in Figure 2.

This is an intriguing and logical result, indicating that when leverage is taken into account, the marginal utility of LP value increases, whereas when leverage is not taken into account, it decreases. This is also in line with behavior literature studies. Furthermore, regardless of whether we incorporate model uncertainty, the valuation of LPs in the Full spanning situation is substantially higher than in the general case with non-spanned risk. Furthermore, in the Full-spanning case, the LPs will allocate less public equity in liquid wealth, corresponding to a greater valuation of the LPs' Certainty-equivalent valuation. This is simple to comprehend since, in the full-spanning situation, the PE asset's risk can be completely hedged by LPs dynamically trading public equity and the risk-free asset. That means there are no illiquidity or risk costs. As a result, the LPs' certainty-equivalent valuation will rise significantly, and they will opt for less public stock to avoid systematic risk.

5.3.1. Effect of Leverage on Break-even Alpha and Performance Measures

Table 2 reports break-even alphas under various levels of effective risk aversion of the different performance measures with leverage. Table 2 shows in the previous paragraph, the first row of Panel A shows break-even alphas for LPs $\gamma = 0$ without model uncertainty but with leverage $L = 3$. In this circumstance, the LPs are effective risk-neutral, thus there is no additional cost of illiquidity and non-spanned risks, and the break-even alphas of 1.01% with leverage $L = 3$. Compared with the break-even alpha 2.61 without leverage, the effect of leverage on the break-even alpha is substantial. With the LPs' risk aversion level increasing to $\gamma = 2$ or $\gamma = 5$, the break-even alpha also significantly decreases with leverage. It is intuitive that with greater leverage, the total investment assets are larger thus the alpha required to generate to compensate LPs is lower. In addition, when considering leverage, the IRR, TVPI, and PME increase with leverage. With leverage $L = 3$ and $\gamma = 2$, IRR increases from 8.41% to 13.81%, and TVPI increases from 2.16 to 3.61. Moreover, the PME increases from 0.78 to 1.30.

Panel B shows break-even alphas for different levels of effective risk aversion under model uncertainty. $\gamma = 0$, With effective risk-neutral LPs $\gamma = 0$, we will not consider model uncertainty in this circumstance. Similar to the analysis in the last paragraph, under model uncertainty, the risk-averse LPs will demand higher break-even alpha tolerate the higher cost of capital. As the risk-aversion parameter $\gamma = 2$, the break-even alpha increases from 2.06% without model uncertainty to 3.10%, IRR increases from 13.80% to 16.22%, and TVPI increases from 3.61 to 4.45. Furthermore, the PME will rise from 1.30 to 1.64. The break-even alpha and performance measures findings will likewise increase

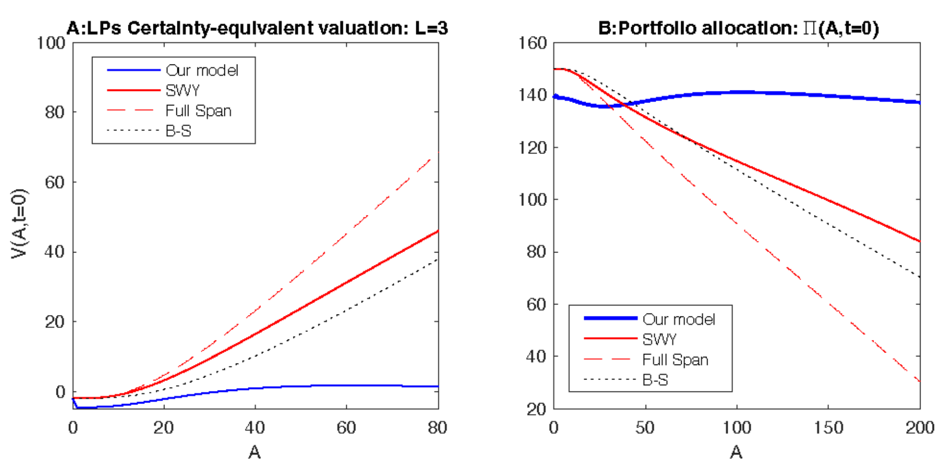

Figure 2: LPs Certainty-Equivalent Valuation and Portfolio Allocation Under $L = 3$

Table 2: Break-Even Values of Empirical Performance Measures for Various Levels of Effective Risk Aversion With Leverage $l = 3$

Panel A						
Risk Aversion	**Alpha (α)**	**IRR (ϕ)**	**Credit Spread**	**E[TVPI]**	**PME**	**Adj. PME**
$\gamma l_0 = 0$	1.01%	11.20%	3.48%	2.81	1.02	1.00
$\gamma l_0 = 2$	2.06%	13.81%	2.64%	3.61	1.30	1.34
$\gamma l_0 = 5$	3.33%	16.5%	1.91%	4.66	1.68	1.80
Panel B						
$\gamma l_0 = 0$	1.68%	13.02%	2.83%	3.35	1.21	1.23
$\gamma l_0 = 2$	3.10%	16.21%	1.97%	4.45	1.64	1.74
$\gamma l_0 = 5$	3.51%	16.81%	1.82%	4.82	1.74	1.87

to greater magnitudes with a value of = 5. Furthermore, when model uncertainty is taken into account, the effect of leverage produces the same results as in the absence of model uncertainty. Break-even alpha and these performance measurements will benefit from leverage. As a result, leverage will lower the break-even alpha while raising these various performance metrics. We now focus on the change in credit spread for debt, as opposed to the case without leverage. Our model's results show that model uncertainty causes the break-even alpha to rise, and a higher alpha raises the debt value, predicting a reduced spread.

In conclusion, when model uncertainty is combined with leverage, the effect of leverage reduces the LP's break-even alpha and increases the GP's performance since leverage allows the GPs to invest more assets. In addition, the impact of model uncertainty will cause LPs to become more risk-averse and opt for a more cautious strategy. And, as a result of this behavior and mindset, the break-even alpha and performance measurements will skyrocket.

6. Conclusion

PE valuation has become a very interesting and challenging topic as the academic literature has grown. We add model uncertainty into the asset allocation and valuation dilemma with investors that participate in both illiquid private equity and traditional liquid assets in our study. Model uncertainty will have negative consequences on the LPs' Certainty-equivalent valuation independent of leverage, but has equivocal effects on the portfolio allocation decision of liquid wealth, according to our model. We also compare our model's outcomes to those of the Full-spanning example. And the results show that the Full-spanning case's valuation is substantially higher than our model's. That is, the greatest strategy to preserve LPs' investments is to lower the cost of illiquidity and risk in PE assets, and the logical objective for GPs is to discover better ways to hedge PE assets' risk with public stock and risk-free assets.

Furthermore, the GP must create sufficient excess return for the LPs to break even and participate. We compare the break-even alpha values in our model to model uncertainty, and we find that the break-even alpha implied in our model is significantly larger than the model in SWY. We also take model uncertainty into account when investing with leverage, and we get identical outcomes without it. This suggests that, regardless of leverage, model uncertainty will cause LPs to be more cautious and opt for alternate ways to shield themselves from potential losses. Theoretically, model uncertainty raises the break-even alpha, and performance metrics rise in the mean-time, according to our findings. As a result, the private equity market must pay attention to the issue of model uncertainty during the investment phase.

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