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Maximum product of spacings under a generalized Type-II progressive hybrid censoring scheme

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Abstract

This paper proposes a new estimation method based on the maximum product of spacings for estimating unknown parameters of the three-parameter Weibull distribution under a generalized Type-II progressive hybrid censoring scheme which guarantees a constant number of observations and an appropriate experiment duration. The proposed approach is appropriate for a situation where the maximum likelihood estimation is invalid, especially, when the shape parameter is less than unity. Furthermore, it presents the enhanced performance in terms of the bias through the Monte Carlo simulation. In particular, the superiority of this approach is revealed even under the condition where the maximum likelihood estimation satisfies the classical asymptotic properties. Finally, to illustrate the practical application of the proposed approach, the real data analysis is conducted, and the superiority of the proposed method is demonstrated through a simple goodness-of-fit test.

Keywords: generalized Type-II progressive hybrid censored sample, maximum product of spacings, Weibull distribution

1. Introduction

Smith (1985) discussed statistical properties of the maximum likelihood estimation according to an exponent $\lambda(> 0)$ by extending the following distribution with a single unknown location parameter μ to distributions in which there are other unknown parameters

$$f(x;\mu) = f_0(x-\mu), \quad x > \mu,$$

where $f_0(x) \sim \lambda c x^{\lambda-1}$, $c > 0$. The extended distribution includes the Weibull, gamma, beta, and log gamma distributions with three parameters. Among them, the Weibull distribution is one of the most widely used lifetime distributions in various fields, and the probability density function (pdf) and cumulative distribution function (cdf) of the three-parameter Weibull distribution are given by

$$f(x;\Theta) = \lambda\sigma^{-\lambda}(x-\mu)^{\lambda-1} e^{-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}} \tag{1.1}$$

and

$$F(x;\Theta) = 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}}, \quad x > \mu,\ \sigma > 0,\ \lambda > 0, \tag{1.2}$$

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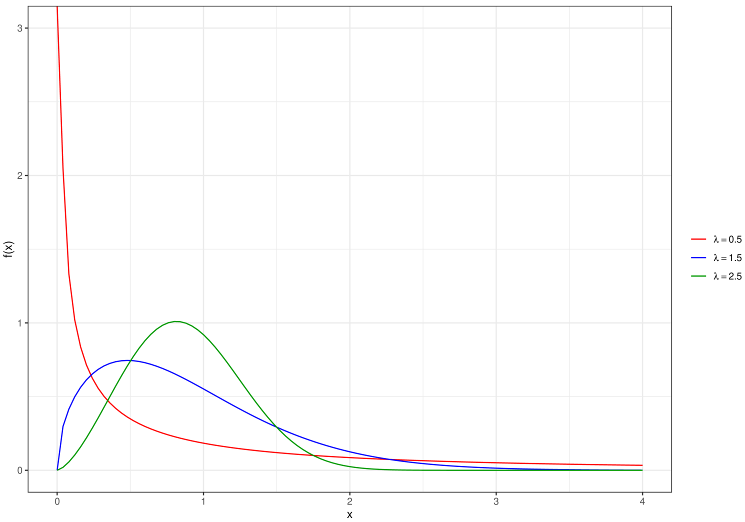


Figure 1: *The pdf of the three-parameter Weibull distribution.*

respectively, where $\Theta = (\mu, \sigma, \lambda)$. In Θ, σ is the scale parameter and λ serves as a shape parameter allowing different types of distributions.

For this distribution, it is worth noting that the domain of the random variable X depends on μ, so it does not satisfy one of the conditions called the regularity conditions (Cousineau, 2009). In addition, problems arising from λ have been addressed in some literature. Cheng and Amin (1983) pointed out that for $\lambda < 1$, the Weibull distribution is J-shaped, and the maximum likelihood estimation is bound to fail because no stationary point can yield a consistent estimator. Moreover, Smith (1985) mentioned that for $1 < \lambda < 2$, the maximum likelihood estimators (MLEs) exist but are not asymptotically normal since the estimators do not approximately follow a normal distribution. So, the MLEs have the classical asymptotic properties only for $\lambda > 2$. The behavior of the pdf (1.1) according to λ with $\mu = 0$ and $\sigma = 1$ is plotted in Figure 1.

Cheng and Amin (1983) and Ranneby (1984) overcame the problems of the Weibull distribution with the maximum product of spacings (MPS) method that gives consistent estimators even in a situation where the maximum likelihood method may fail. The former showed the superiority of the MPS approach under the assumption that a density function is strictly positive in the interval (a, b) and 0 outside this interval. The latter showed the consistency of the MPS estimator (MPSE) which is derived as an approximation of the Kullback-Leibler information. In addition, Ekström (1998) generalized a consistency theorem for the MPS method introduced by Ranneby (1984). Anatolyev and Kosenok (2005) demonstrated that the MPSE can be more efficient than the MLE for a small sample with a heavy tail and/or skewed distribution. Due to these research results, the MPS estimation method was applied to some probability distributions as an alternative to the maximum likelihood estimation method that is traditionally the most popular. Even, in the presence of censoring, the MPS estimation method was studied by some authors. Basu *et al*. (2017) developed the product of spacings function for the Type-I censored data. Basu *et al*. (2018) introduced the MPS estimation method under a Type-I progressive hybrid censoring scheme (Type-I PHCS) with binomial removals. These studies provided the MPSE of the scale parameter for the inverse Lindley distribution under the considered schemes, respectively, and proved that the MPSE is better than the MLE through the Monte Carlo simulation.

Based on this, it can be seen that the MPS estimation method provides better performance than the maximum likelihood estimation method in various situations.

However, despite these advantages of the MPS estimation method, it has hardly been applied to more advanced censoring schemes, and in particular, it has not been studied under a generalized Type-II PHCS which guarantees a constant number of observations and an appropriate experiment duration. This study provides an alternative to the maximum likelihood estimation method by proposing an appropriate MPS estimation method under the generalized Type-II PHCS. For detailed illustration, it is assumed that the generalized Type-II progressive hybrid censored sample is observed from the three-parameter Weibull distribution with the cdf (1.2).

The rest of the paper is organized as follows. Section 2 describes the generalized Type-II PHCS along with the previously heavily used censoring scheme and provides the corresponding maximum likelihood estimation method. Section 3 provides a new estimation method based on the MPS under the generalized Type-II PHCS, which improves the performance in terms of the bias, compared to the maximum likelihood estimation method. Section 4 shows the superiority of the proposed method through the Monte Carlo simulation and the real data analysis, and Section 5 concludes the paper.

2. Generalized Type-II progressive hybrid censoring scheme

Since the introduction of the Type-I and Type-II censoring schemes, various censoring schemes have been proposed for efficient experiments saving time and cost. Among these censoring schemes, the progressive Type-II censoring scheme (Type-II PCS) is the most widely used because it allows the removal of surviving units during the experiment, and recently its generalized hybrid versions have been proposed by some authors.

First, the Type-II PCS can be explained as follows: Suppose that n randomly selected units are put on a lifetime experiment, and $m(< n)$ is the predetermined observation number. Then, at the first failure $X_{1:m:n}$ occurrence, R_1 surviving units are randomly removed from the remaining $n - 1$ units. Subsequently, R_2 surviving units are randomly removed from the remaining $n - R_1 - 2$ units when the second failure $X_{2:m:n}$ occurs. The experiment is continued in this manner and terminated when the mth failure $X_{m:m:n}$ occurs, removing all remaining $R_m = n - m - R_1 - \cdots - R_{m-1}$ units. During this process, randomly removed $\mathscr{R} = (R_1, \ldots, R_m)$ is pre-fixed before the experiment.

To address the weakness of the Type-II PCS which can take a long time if the last observed failure time has long, the generalized hybrid versions of the Type-II PCS have been proposed by Cho *et al*. (2015) and Górny and Cramer (2016), namely the generalized Type-I and Type-II PHCSs, respectively. In the former, the experiment is terminated at $\max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}$, where $k < m$ and $T \in (0, \infty)$. The latter terminates the experiment at $\max\{\min\{X_{m:m+R_m:n}, T_2\}, T_1\}$ for the pre-fixed threshold times $T_1, T_2 \in (0, \infty)$ and $T_1 < T_2$.

Note that the advantage of the generalized Type-II PHCS is that it guarantees a constant number of observations and an experimental duration that does not exceed time T_2 because of the following three possible scenarios:

$$\begin{aligned}
&\text{Case I} : X_{1:m+R_m:n}, \ldots, X_{D_1:m+R_m:n},\ D_1 \in \{m, \ldots, m + R_m\}, \quad \text{if } X_{m:m+R_m:n} \le X_{D_1:m+R_m:n} < T_1,\\
&\text{Case II} : X_{1:m+R_m:n}, \ldots, X_{m:m+R_m:n},\ D_1 \in \{0, \ldots, m - 1\},\ D_2 = m, \quad \text{if } T_1 < X_{m:m+R_m:n} < T_2, \qquad (2.1)\\
&\text{Case III} : X_{1:m+R_m:n}, \ldots, X_{D_2:m+R_m:n},\ D_2 \in \{0, \ldots, m - 1\}, \quad \text{if } T_2 \le X_{m:m+R_m:n}.
\end{aligned}$$

In (2.1), the discrete random variable $D_l(l = 1, 2)$ is defined by Górny and Cramer (2016) as

$$D_l = \sum_{j=1}^{m+R_m} \mathbb{1}_{(-\infty,T_l]}\left(X_{j:m+R_m:n}\right), \quad l = 1, 2,$$

which is the number of observed failures until the fixed time $T_l(l = 1, 2)$, where $\mathbb{1}_{(-\infty,T_l]}$ denotes the indicator function of $(-\infty, T_l]$. More specifically, for Case I, the experiment continues until time T_1 if the mth failure is observed before time T_1, which implies that more failures can be observed than m. The corresponding censored sample then is the extended progressive Type-II censored sample $X_{1:m+R_m:n}, \ldots, X_{m:m+R_m:n}, X_{m+1:m+R_m:n}, \ldots, X_{m+R_m:m+R_m:n}$ with the modified censoring scheme $\mathscr{R}^* = (R_1, \ldots, R_{m-1}, 0^{*R_m+1})$, where 0^{*R_m+1} denotes a vector of $R_m + 1$ zeros. For Case II, the experiment is terminated at $X_{m:m+R_m:n}$ if the mth failure is observed between times T_1 and T_2, and the progressive Type-II censored sample is considered as the censored sample. For Case III, the experiment is terminated at time T_2 if the mth failure is observed after T_2, and all the remaining units are removed at the termination point T_2.

The likelihood function under the generalized Type-II PHCS scenarios is given by

$$L(\Theta) \propto \begin{cases} [1-F(T_1;\Theta)]^{\gamma_{d_1+1}} \prod_{i=1}^{d_1} f(x_{i:m+R_m:n};\Theta)\,[1-F(x_{i:m+R_m:n};\Theta)]^{R_i}, & d_1 \in \{m, \ldots, m+R_m\}, \\ \prod_{i=1}^{m} f(x_{i:m+R_m:n};\Theta)\,[1-F(x_{i:m+R_m:n};\Theta)]^{R_i}, & d_1 \in \{0, \ldots, m-1\},\ d_2 = m, \\ [1-F(T_2;\Theta)]^{\gamma_{d_2+1}} \prod_{i=1}^{d_2} f(x_{i:m+R_m:n};\Theta)\,[1-F(x_{i:m+R_m:n};\Theta)]^{R_i}, & d_2 \in \{0, \ldots, m-1\}, \end{cases} \tag{2.2}$$

where $\gamma_{d_l+1}(l = 1, 2)$ is the number of units removed at the termination point $T_l(l = 1, 2)$, given by

$$\gamma_i = \begin{cases} \sum_{j=i}^{m}\left(R_j + 1\right), & i = 1, \ldots, m, \\ m + R_m - i + 1, & i = m+1, \ldots, m + R_m. \end{cases}$$

Then, assuming that the extended progressive Type-II censored sample $X_{1:m+R_m:n}, \ldots, X_{m:m+R_m:n}, X_{m+1:m+R_m:n}, \ldots, X_{m+R_m:m+R_m:n}$ is observed from the Weibull distribution with the cdf (1.2), the unified likelihood function for Cases I, II, and III can be expressed as

$$L(\Theta) \propto \lambda^{\omega}\sigma^{-\omega\lambda} \exp\left(-\sum_{i=1}^{\omega}\left(\frac{x_{i:m+R_m:n} - \mu}{\sigma}\right)^{\lambda}(1 + R_i) - h_1(\Theta)\right)\prod_{i=1}^{\omega}\left(x_{i:m+R_m:n} - \mu\right)^{\lambda-1}, \tag{2.3}$$

where

$$h_1(\Theta) = \begin{cases} \left(\frac{T_1 - \mu}{\sigma}\right)^{\lambda}\gamma_{\omega+1} & \text{for Case I}, \\ 0 & \text{for Case II}, \\ \left(\frac{T_2 - \mu}{\sigma}\right)^{\lambda}\gamma_{\omega+1} & \text{for Case III}, \end{cases} \qquad \omega = \begin{cases} d_1 & \text{for Case I}, \\ m & \text{for Case II}, \\ d_2 & \text{for Case III}. \end{cases}$$

The MLEs $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\lambda}$ can be obtained by maximizing the logarithm of the likelihood function (2.3).

However, the MLEs can be inconsistent or invalid when λ is less than unity, as mentioned earlier. To resolve this problem, a new estimation method based on the MPS under the generalized Type-II PHCS is introduced in the next section.

3. Maximum product of spacings

Let $x_1 < x_2 < \cdots < x_n$ be an ordered random sample drawn from the distribution with the pdf $f(x) > 0$ and cdf $F(x)$ for $\forall x \in (a, b)$, where $\forall x \in (a, b)$ denotes all x in the interval (a, b). In addition, let define the spacing as $S_i = F(x_i) - F(x_{i-1})$ for $i = 1, \ldots, n+1$, where $x_0 \equiv a$ and $x_{n+1} \equiv b$. Then, the MPS procedure achieves by maximizing the geometric mean of the spacings $G = (\prod_{i=1}^{n+1} S_i)^{1/(n+1)}$. However, in the presence of censoring, the MPS estimation method needs to be modified to match its censoring scheme.

The partitions induced by the sample information based on the generalized Type-II PHCS scenarios in the interval $(0, \infty)$ are summarized as:

$$\text{Case I}: \ (0, x_{1:m+R_m:n}],\ (x_{1:m+R_m:n}, x_{2:m+R_m:n}], \ldots, (x_{d_1:m+R_m:n}, T_1],\ (T_1, \infty),$$
$$\text{Case II}: \ (0, x_{1:m+R_m:n}],\ (x_{1:m+R_m:n}, x_{2:m+R_m:n}], \ldots, (x_{m-1:m+R_m:n}, x_{m:m+R_m:n}],\ (x_{m:m+R_m:n}, \infty),$$
$$\text{Case III}: \ (0, x_{1:m+R_m:n}],\ (x_{1:m+R_m:n}, x_{2:m+R_m:n}], \ldots, (x_{d_2:m+R_m:n}, T_2],\ (T_2, \infty),$$

where $(a, b]$ denotes the partition that does not include a and includes b. Note that there is no information in the last partition of each Case because all remaining units are removed at the termination point. The spacing corresponding to the partition $(x_{i-1:m+R_m:n}, x_{i:m+R_m:n}]$ is given by $F(x_{i:m+R_m:n}) - F(x_{i-1:m+R_m:n})$. However, it should include the information for R_i units that are removed at ith failure $X_{i:m+R_m:n}$. It can be treated as ties in terms of the survival function by borrowing from the idea of Basu *et al*. (2018), and the spacing is obtained by extending the method proposed by Shao and Hahn (1999) into a censoring scheme.

First, the total information for R_i units in the partition $(x_{i-1:m+R_m:n}, x_{i:m+R_m:n}]$ can be expressed as $[1 - F(x_{i:m+R_m:n})]$ in terms of the survival function. From it, the information for each of R_i units should be identified since R_i units can be treated as ties. It can easily resolve using the fact that the amount of information for each of R_i units is the same in the spacing because each of R_i units has the same probability of occurrence. Then, the probability information for each of R_i units is incorporated by $[1 - F(x_{i:m+R_m:n})]/R_i$ in the spacing, and the spacing for R_i units is derived as $\{[1 - F(x_{i:m+R_m:n})]/R_i\}^{R_i}$ by multiplying the information for each of R_i units. Thereby, the modified spacing with the censored information in the partition $(x_{i-1:m+R_m:n}, x_{i:m+R_m:n}]$ is given by

$$S_{i:m+R_m:n} = \left[F\left(x_{i:m+R_m:n}\right) - F\left(x_{i-1:m+R_m:n}\right)\right]\left[\frac{1 - F\left(x_{i:m+R_m:n}\right)}{R_i}\right]^{R_i}, \quad i = 1, \ldots, \omega, \tag{3.1}$$

where $F(x_{0:m+R_m:n}) = 0$. Additionally, the partition $(x_{d_l:m+R_m:n}, T_l](l = 1, 2)$ in Cases I and III leads to another spacing $S^*_{\xi_l} = [F(T_l) - F(x_{d_l:m+R_m:n})]\{[1 - F(T_l)]/\gamma_{d_l+1}\}^{\gamma_{d_l+1}}$ $(l = 1, 2)$ containing the information about $\gamma_{d_l+1}(l = 1, 2)$ units that are removed at the termination point $T_l(l = 1, 2)$. Note that the spacing $S^*_{\xi_l}$ may be approximated by $f(T_l)\{[1 - F(T_l)]/\gamma_{d_l+1}\}^{\gamma_{d_l+1}}$ if T_l and $x_{d_l:m+R_m:n}$ are very close (*i.e.* $|T_l - x_{d_l:m+R_m:n}| < \epsilon$), where ϵ is some small positive number.

Then, using the spacings (3.1) and $S^*_{\xi_l}$, the conditional product of spacings for the unknown parameter Θ under the generalized Type-II PHCS is proposed as

$$G(\Theta|R) \propto \begin{cases} \text{Case I} & : f(T_1)\left[\frac{1-F(T_1)}{\gamma_{d_1+1}}\right]^{\gamma_{d_1+1}} \prod_{i=1}^{d_1} S_{i:m+R_m:n}, \quad \text{if } \left|T_1 - x_{d_1:m+R_m:n}\right| < \epsilon, \\ & : S^*_{\xi_1} \prod_{i=1}^{d_1} S_{i:m+R_m:n}, \quad \text{if } \left|T_1 - x_{d_1:m+R_m:n}\right| > \epsilon, \\ \text{Case II} & : \prod_{i=1}^{m} S_{i:m+R_m:n}, \\ \text{Case III} & : f(T_2)\left[\frac{1-F(T_2)}{\gamma_{d_2+1}}\right]^{\gamma_{d_2+1}} \prod_{i=1}^{d_2} S_{i:m+R_m:n}, \quad \text{if } \left|T_2 - x_{d_2:m+R_m:n}\right| < \epsilon, \\ & : S^*_{\xi_2} \prod_{i=1}^{d_2} S_{i:m+R_m:n}, \quad \text{if } \left|T_2 - x_{d_2:m+R_m:n}\right| > \epsilon. \end{cases}$$

The corresponding product of spacings of the Weibull distribution with the cdf (1.2) is given by

$$G(\Theta|R) \propto h_2(\Theta) \prod_{i=1}^{\omega} \left[e^{-\left(\frac{x_{i-1:m+R_m:n}-\mu}{\sigma}\right)^{\lambda}} - e^{-\left(\frac{x_{i:m+R_m:n}-\mu}{\sigma}\right)^{\lambda}}\right] e^{-\left(\frac{x_{i:m+R_m:n}-\mu}{\sigma}\right)^{\lambda} R_i},$$

where

$$h_2(\Theta) = \begin{cases} \lambda\sigma^{-\lambda}(T_l-\mu)^{\lambda-1} e^{-\left(\frac{T_l-\mu}{\sigma}\right)^{\lambda}(1+\gamma_{\omega+1})}, & \text{if } |T_l - x_{d_l:m+R_m:n}| < \epsilon; \\ & l = 1(\text{for Case I}),\ 2(\text{for Case III}), \\ \left[e^{-\left(\frac{x_{\omega:m+R_m:n}-\mu}{\sigma}\right)^{\lambda}} - e^{-\left(\frac{T_l-\mu}{\sigma}\right)^{\lambda}}\right] e^{-\left(\frac{T_l-\mu}{\sigma}\right)^{\lambda}\gamma_{\omega+1}}, & \text{if } |T_l - x_{d_l:m+R_m:n}| > \epsilon; \\ & l = 1(\text{for Case I}),\ 2(\text{for Case III}), \\ 1 & \text{for Case II}, \end{cases}$$

$$\omega = \begin{cases} d_1 & \text{for Case I}, \\ m & \text{for Case II}, \\ d_2 & \text{for Case III}, \end{cases}$$

and the MPSEs $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{\lambda}$ can be obtained by maximizing its logarithm. To demonstrate the superiority of the proposed MPS estimation method, the Monte Carlo simulations and the real data application are conducted in the next section.

4. Application

To examine how valid and excellent the proposed method is, the mean squared errors (MSEs) and biases of the MPSEs are computed and compared with those of the MLEs. In addition, a real dataset is analyzed to illustrate the actual application.

4.1. Simulation result

As mentioned earlier, the MLEs either cannot exist or become inconsistent estimators for $\lambda < 1$, and they have the classical asymptotic properties for $\lambda > 2$. To prove the superiority of the proposed MPS estimation method, the Monte Carlo simulations with 5,000 replications are conducted for $\lambda = 0.5$ and 2.5. In addition, μ and σ are assigned 0 and 1, respectively. Then, the generalized Type-II progressive hybrid censored samples are generated from the Weibull distribution with those parameter

Table 1: MSEs (biases) of estimators for $\lambda = 0.5$ when T_1 increases for the fixed T_2

n	m	T_1	T_2	Scheme	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\hat{\lambda}$	$\tilde{\lambda}$
18	16	1.5	4	I	0.00020(0.00607)	0.00030(0.00017)	0.46769(0.07707)	0.40547(0.02620)	0.01936(−0.05754)	0.01985(0.00138)
				II	0.00023(0.00598)	0.00348(−0.00119)	0.87550(0.09281)	0.47098(0.09197)	0.02038(−0.06111)	0.02665(0.00119)
				III	0.00021(0.00598)	0.00053(−0.00038)	1.06663(0.10825)	0.44215(0.06004)	0.01974(−0.05787)	0.02125(0.00231)
				IV	0.00034(0.00583)	0.00345(−0.00083)	0.96141(0.09217)	0.46017(0.09636)	0.02076(−0.05848)	0.02887(0.00250)
		2		I	0.00020(0.00607)	0.00029(0.00022)	0.47240(0.07623)	0.39890(0.03344)	0.01919(−0.05731)	0.01901(0.00062)
				II	0.00023(0.00598)	0.00346(−0.00113)	0.87550(0.09281)	0.47052(0.09231)	0.02038(−0.06111)	0.02557(0.00063)
				III	0.00021(0.00599)	0.00051(−0.00031)	1.06688(0.10780)	0.43968(0.06222)	0.01952(−0.05777)	0.02041(0.00147)
				IV	0.00034(0.00583)	0.00344(−0.00076)	0.96141(0.09217)	0.45945(0.09701)	0.02076(−0.05848)	0.02755(0.00172)
		2.5		I	0.00020(0.00607)	0.00026(0.00038)	0.47233(0.07645)	0.39224(0.04315)	0.01876(−0.05606)	0.01777(-0.00113)
				II	0.00023(0.00598)	0.00347(−0.00112)	0.87550(0.09281)	0.46946(0.09340)	0.02038(−0.06111)	0.02524(0.00020)
				III	0.00021(0.00599)	0.00102(−0.00052)	1.09404(0.11004)	0.43649(0.06699)	0.01954(−0.05768)	0.02137(0.00096)
				IV	0.00034(0.00583)	0.00342(−0.00067)	0.96141(0.09217)	0.45811(0.09817)	0.02076(−0.05848)	0.02681(0.00086)
	12	1.5		I	0.00020(0.00607)	0.01265(−0.00244)	6.48637(0.13276)	0.36751(0.06068)	0.02335(−0.06556)	0.07339(0.00544)
				II	0.00024(0.00612)	0.00680(−0.00669)	4.72031(0.18706)	0.94724(0.14274)	0.02820(−0.07873)	0.05569(0.02122)
				III	0.00028(0.00581)	0.00376(−0.00344)	4.02903(0.12025)	0.50963(0.01608)	0.02518(−0.06719)	0.04989(0.01142)
				IV	0.00024(0.00611)	0.00925(−0.00428)	1102.41899(0.64893)	1.15736(0.15954)	0.02517(−0.07162)	0.08627(0.02814)
		2		I	0.00020(0.00607)	0.00080(−0.00077)	6.48144(0.12421)	0.36472(0.07517)	0.02179(−0.06357)	0.02551(0.00156)
				II	0.00024(0.00612)	0.00666(−0.00647)	4.72031(0.18706)	0.94515(0.14431)	0.02820(−0.07873)	0.05070(0.01920)
				III	0.00028(0.00581)	0.00189(−0.00233)	3.99084(0.11640)	0.49734(0.03684)	0.02509(−0.06684)	0.03313(0.00661)
				IV	0.00024(0.00611)	0.00722(−0.00347)	1102.41899(0.64893)	1.15185(0.16213)	0.02517(−0.07162)	0.06887(0.02323)
		2.5		I	0.00020(0.00607)	0.00044(−0.00023)	1.04461(0.10279)	0.36963(0.08153)	0.02102(−0.06223)	0.02098(−0.00072)
				II	0.00024(0.00612)	0.00660(−0.00629)	4.72031(0.18706)	0.94241(0.14675)	0.02820(−0.07873)	0.04801(0.01715)
				III	0.00028(0.00582)	0.00200(−0.00216)	3.87469(0.11489)	0.49616(0.05932)	0.02498(−0.06750)	0.03086(0.00467)
				IV	0.00024(0.00611)	0.00668(−0.00285)	1102.41899(0.64893)	1.14733(0.16629)	0.02517(−0.07162)	0.05767(0.01820)
36	32	2.5	4	I	0.00001(0.00162)	0.00001(0.00053)	0.02003(0.04613)	0.16404(0.01931)	0.00822(−0.02288)	0.00717(−0.00541)
				II	0.00001(0.00151)	0.00001(0.00041)	0.02078(0.04724)	0.17017(0.04462)	0.00929(−0.02921)	0.00740(−0.00723)
				III	0.00002(0.00170)	0.00002(0.00060)	0.02419(0.04661)	0.16557(0.03712)	0.00876(−0.02603)	0.00733(−0.00644)
				IV	0.00001(0.00151)	0.00001(0.00046)	0.02241(0.04555)	0.16760(0.04849)	0.00897(−0.02647)	0.00720(−0.00677)
		3		I	0.00001(0.00162)	0.00001(0.00054)	0.01960(0.04437)	0.16075(0.02517)	0.00815(−0.02384)	0.00704(−0.00578)
				II	0.00001(0.00151)	0.00001(0.00041)	0.02078(0.04724)	0.17005(0.04472)	0.00929(−0.02921)	0.00739(−0.00727)
				III	0.00002(0.00170)	0.00002(0.00060)	0.02405(0.04745)	0.16449(0.03843)	0.00872(−0.02616)	0.00727(−0.00665)
				IV	0.00001(0.00151)	0.00001(0.00046)	0.02241(0.04555)	0.16747(0.04862)	0.00897(−0.02647)	0.00718(−0.00681)
		3.5		I	0.00001(0.00162)	0.00001(0.00054)	0.01939(0.04615)	0.15778(0.03364)	0.00808(−0.02367)	0.00691(−0.00614)
				II	0.00001(0.00151)	0.00001(0.00041)	0.02078(0.04724)	0.16993(0.04487)	0.00929(−0.02921)	0.00738(−0.00730)
				III	0.00002(0.00170)	0.00002(0.00060)	0.02490(0.04652)	0.16280(0.04081)	0.00870(−0.02645)	0.00719(−0.00693)
				IV	0.00001(0.00151)	0.00001(0.00046)	0.02241(0.04555)	0.16723(0.04889)	0.00897(−0.02647)	0.00717(−0.00688)
	26	2.5		I	0.00001(0.00162)	0.00001(0.00050)	0.02135(0.04437)	0.15061(0.03637)	0.00905(−0.02655)	0.00787(−0.00630)
				II	0.00002(0.00159)	0.00002(0.00030)	0.03784(0.05711)	0.21988(0.05274)	0.01188(−0.03731)	0.00953(−0.00702)
				III	0.00001(0.00155)	0.00001(0.00035)	0.02844(0.04981)	0.17702(0.01031)	0.01032(−0.03075)	0.00862(−0.00473)
				IV	0.00002(0.00159)	0.00001(0.00048)	0.03367(0.05099)	0.24728(0.07329)	0.01056(−0.03118)	0.00870(−0.00640)
		3		I	0.00001(0.00162)	0.00001(0.00052)	0.02344(0.04753)	0.15493(0.04242)	0.00888(−0.02830)	0.00739(−0.00680)
				II	0.00002(0.00159)	0.00002(0.00030)	0.03784(0.05711)	0.21964(0.05297)	0.01188(−0.03731)	0.00951(−0.00708)
				III	0.00001(0.00155)	0.00001(0.00037)	0.02782(0.04955)	0.17390(0.02495)	0.01008(−0.03054)	0.00824(−0.00553)
				IV	0.00002(0.00159)	0.00001(0.00048)	0.03367(0.05099)	0.24684(0.07373)	0.01056(−0.03118)	0.00865(−0.00656)
		3.5		I	0.00001(0.00162)	0.00001(0.00054)	0.02007(0.04561)	0.15537(0.04238)	0.00820(−0.02505)	0.00707(−0.00657)
				II	0.00002(0.00159)	0.00002(0.00030)	0.03784(0.05711)	0.21926(0.05340)	0.01188(−0.03731)	0.00948(−0.00720)
				III	0.00001(0.00155)	0.00001(0.00039)	0.02715(0.04938)	0.17520(0.03640)	0.01000(−0.03078)	0.00793(−0.00588)
				IV	0.00002(0.00159)	0.00001(0.00048)	0.03367(0.05099)	0.24618(0.07446)	0.01056(−0.03118)	0.00854(−0.00690)

values under the following Type-II PCS:

$$
\begin{aligned}
\text{Scheme}\quad & \text{I}: R_m = n - m \ \text{ and } \ R_i = 0 \ \text{ for } i \neq m,\\
& \text{II}: R_1 = n - m \ \text{ and } \ R_i = 0 \ \text{ for } i \neq 1,\\
& \text{III}: R_1 = R_m = \frac{n-m}{2} \ \text{ and } \ R_i = 0 \ \text{ for } i \neq 1 \text{ and } m,\\
& \text{IV}: R_{m/2} = n - m \ \text{ and } \ R_i = 0 \ \text{ for } i \neq \frac{m}{2}.
\end{aligned}
$$

The fixed time $T_l(l = 1, 2)$ is chosen to include all possible scenarios (Cases I, II, and III) of the generalized Type-II PHCS for each scheme. Here, various values of T_2 are considered for the fixed T_1 and vice versa, to examine the effect of time $T_l(l = 1, 2)$. Some small positive number ϵ required in Cases I and III of the MPS method is assigned 0.0001. Then, the MLEs and MPSEs are computed under all considered conditions and the corresponding MSEs and biases are empirically obtained over

5,000 replications as

$$\mathrm{MSE}\left(\breve{\Theta}\right) = \frac{1}{5000}\sum_{i=1}^{5000}\left(\breve{\Theta}_i - \Theta\right)^2$$

and

$$\mathrm{Bias}\left(\breve{\Theta}\right) = \frac{1}{5000}\sum_{i=1}^{5000}\breve{\Theta}_i - \Theta,$$

respectively, where $\breve{\Theta}$ is an estimator of Θ. These results are reported in Tables 1–4.

For $\lambda = 0.5$ (Tables 1–2), the MPSEs are generally more efficient than the MLEs counterpart in terms of the bias. In particular, the bias of the MPSE $\tilde{\mu}$ is almost zero in most cases. In addition, for the sample size $n = 36$, the MPSE $\tilde{\lambda}$ is generally more efficient than the MLE $\hat{\lambda}$ in terms of the MSE. In terms of the MSE, the MLEs generally show superior performance to the corresponding MPSEs for μ and λ when the sample size is small ($n = 18$). However, for σ, the MLE $\hat{\sigma}$ often yields absurd MSE values compared to the MPSE $\tilde{\sigma}$ when the size of the observed sample is small ($m = 12$). It may be a problem that arises because the maximum likelihood estimation method fails as mentioned earlier, which reveals that the proposed MPS estimation method yields a more stable estimation result than the maximum likelihood estimation method when the size of the observed sample is small.

One noteworthy point is that the MPSEs show better results than the MLEs counterpart in terms of the MSE and bias for $\lambda = 2.5$ (Tables 3–4). These results reveal that the proposed MPS estimation method is superior to the maximum likelihood estimation method under the generalized Type-II PHCS even when the MLEs have the classical asymptotic properties.

As expected, the MSEs for all estimators generally decrease as the sample size n increases, and the MSEs generally decrease as the predetermined observation number m increases for the fixed sample size n, time $T_l (l = 1, 2)$, and Scheme. In addition, the effect of time $T_l (l = 1, 2)$ reveals in terms of the MSE. The MSEs of the MPSE $\tilde{\lambda}$ generally decrease as time $T_l (l = 1, 2)$ increases and the MSEs of the MLE $\hat{\lambda}$ generally decrease as time T_2 increases for the fixed sample size n, predetermined observation number m, and Scheme.

4.2. Real data

To restrict the ruthless use of brittle fibers such as carbon, ceramic, and jute fibers in various structural applications, the experiment on strength reliability is essential. Fiber breaks randomly during applying a tensile load, which arises mainly from the variability in its fracture strength. For this reason, the breaking strength of fibers can be commonly characterized by statistical distributions.

To illustrate the actual application of the proposed method, data (Table 5) on the breaking strengths of jute fiber at gauge length 10mm reported by Xia *et al.* (2009) are considered. For simplicity, the data are divided by 100, and the Type-II PCS is set to $\mathscr{R} = (0^{*5}, 2^{*5}, 0^{*4}, 5)$ with $m = 15$ to generate an appropriate censored sample for the analysis. The extended progressive Type-II censored sample for Case I under the generalized Type-II PHCS is generated by extending the censoring scheme $\mathscr{R} = (0^{*5}, 2^{*5}, 0^{*4}, 5)$ to $\mathscr{R}^* = (0^{*5}, 2^{*5}, 0^{*10})$ and it is given in Table 6. For a fixed time pair (T_1, T_2), three pairs are assigned to consider all scenarios (Cases I, II, and III) occurring under the generalized Type-II PHCS. The time pairs leading to each Case and the estimated results for each Case are reported in Table 7.

Here, the estimates are compared through a simple goodness-of-fit test based on the replicated data $X^{rep}_{i:m+R_m:n}$ that are generated from the three-parameter Weibull distribution with the estimates.

Table 2: MSEs (biases) of estimators for $\lambda = 0.5$ when T_2 increases for the fixed T_1

n	m	T_1	T_2	Scheme	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\hat{\lambda}$	$\tilde{\lambda}$
18	16	1.5	2.5	I	0.00020(0.00607)	0.00058(−0.00058)	1.04499(0.10430)	0.50179(0.08820)	0.02186(−0.06436)	0.02347(0.00211)
				II	0.00023(0.00599)	0.03286(−0.00374)	3.25942(0.13719)	1.71151(0.14757)	0.02280(−0.06597)	0.06691(0.00491)
				III	0.00021(0.00598)	0.00066(−0.00101)	1.02466(0.10614)	0.87578(0.11921)	0.02233(−0.06355)	0.02502(0.00265)
				IV	0.00023(0.00599)	0.00144(−0.00096)	0.74091(0.10827)	0.68237(0.14219)	0.02228(−0.06591)	0.02792(0.00217)
			3.5	I	0.00020(0.00607)	0.00034(0.00003)	0.09112(0.07164)	0.42759(0.04718)	0.02018(−0.05989)	0.02053(0.00120)
				II	0.00023(0.00598)	0.00142(−0.00103)	2.83344(0.10812)	0.49546(0.10100)	0.02123(−0.06368)	0.02523(0.00134)
				III	0.00021(0.00598)	0.00213(−0.00090)	1.37997(0.10887)	0.47596(0.07778)	0.02052(−0.05909)	0.02368(0.00200)
				IV	0.00023(0.00597)	0.00122(−0.00056)	4.88991(0.14218)	0.51995(0.10766)	0.02099(−0.06130)	0.02555(0.00204)
			4.5	I	0.00020(0.00607)	0.00031(0.00019)	0.59097(0.08192)	0.39008(0.00951)	0.01882(−0.05554)	0.01968(0.00185)
				II	0.00023(0.00598)	0.00124(−0.00075)	1.12099(0.09874)	0.42841(0.08162)	0.01986(−0.05889)	0.02376(0.00126)
				III	0.00021(0.00598)	0.00151(−0.00068)	0.54935(0.09158)	0.41949(0.04677)	0.01910(−0.05593)	0.02229(0.00328)
				IV	0.00034(0.00583)	0.00343(−0.00083)	1.91043(0.10902)	0.43470(0.08745)	0.02018(−0.05692)	0.02907(0.00372)
	12		2.5	I	0.00020(0.00607)	0.01275(−0.00257)	6.49139(0.13366)	0.48644(0.08431)	0.02359(−0.06621)	0.07382(0.00575)
				II	0.00024(0.00612)	0.02873(−0.00821)	1023.68079(0.80365)	2.59952(0.21056)	0.02993(−0.08194)	0.08301(0.01993)
				III	0.00022(0.00594)	0.01385(−0.00474)	16.09833(0.18698)	1.99165(0.12963)	0.02597(−0.07231)	0.06815(0.01238)
				IV	0.00024(0.00611)	0.00434(−0.00367)	1452.009(0.80072)	2.60202(0.24987)	0.02694(−0.07407)	0.06450(0.02122)
			3.5	I	0.00020(0.00607)	0.01265(−0.00244)	6.48638(0.13267)	0.38909(0.06423)	0.02342(−0.06570)	0.07338(0.00545)
				II	0.00024(0.00612)	0.02667(−0.00813)	4.80471(0.19113)	1.02276(0.15609)	0.02807(−0.07846)	0.06922(0.02142)
				III	0.00028(0.00581)	0.00376(−0.00347)	7.68558(0.14880)	0.58315(0.03865)	0.02537(−0.06789)	0.05007(0.01133)
				IV	0.00024(0.00611)	0.00919(−0.00416)	15.2035(0.22215)	1.28823(0.18280)	0.02598(−0.07268)	0.08421(0.02563)
			4.5	I	0.00020(0.00607)	0.01265(−0.00244)	6.48619(0.13245)	0.35557(0.05836)	0.02333(−0.06552)	0.07338(0.00541)
				II	0.00024(0.00612)	0.00442(−0.00588)	180.70125(0.35552)	1.00664(0.13665)	0.02752(−0.07743)	0.05309(0.02178)
				III	0.00028(0.00581)	0.00376(−0.00343)	4.02200(0.11852)	0.46178(−0.00093)	0.02459(−0.06553)	0.04980(0.01150)
				IV	0.00024(0.00611)	0.00946(−0.00462)	13.30192(0.18511)	0.99173(0.14811)	0.02498(−0.07083)	0.09332(0.03151)
36	32	2.5	3	I	0.00001(0.00162)	0.00001(0.00052)	0.02396(0.04962)	0.17384(0.04271)	0.00901(−0.02767)	0.00754(−0.00645)
				II	0.00001(0.00151)	0.00001(0.00038)	0.02543(0.05263)	0.19270(0.05280)	0.01038(−0.03217)	0.00809(−0.00735)
				III	0.00002(0.00170)	0.00002(0.00058)	0.02402(0.04764)	0.18481(0.05061)	0.00979(−0.02817)	0.00789(−0.00707)
				IV	0.00001(0.00151)	0.00001(0.00042)	0.02462(0.04687)	0.18299(0.05345)	0.01000(−0.02962)	0.00781(−0.00654)
			3.5	I	0.00001(0.00162)	0.00001(0.00053)	0.02071(0.04563)	0.16947(0.03080)	0.00838(−0.02443)	0.00733(−0.00590)
				II	0.00001(0.00151)	0.00001(0.00039)	0.02361(0.04605)	0.17628(0.04677)	0.00984(−0.03004)	0.00769(−0.00695)
				III	0.00002(0.00170)	0.00002(0.00059)	0.02354(0.04653)	0.17144(0.04203)	0.00948(−0.02904)	0.00755(−0.00659)
				IV	0.00001(0.00151)	0.00001(0.00045)	0.02309(0.04576)	0.17348(0.04992)	0.00934(−0.02773)	0.00742(−0.00691)
			4	I	0.00001(0.00162)	0.00001(0.00053)	0.02003(0.04613)	0.16404(0.01931)	0.00822(−0.02288)	0.00717(−0.00541)
				II	0.00001(0.00151)	0.00001(0.00041)	0.02078(0.04724)	0.17017(0.04462)	0.00929(−0.02921)	0.00740(−0.00723)
				III	0.00002(0.00170)	0.00002(0.00060)	0.02419(0.04661)	0.16557(0.03712)	0.00876(−0.02603)	0.00733(−0.00644)
				IV	0.00001(0.00151)	0.00001(0.00046)	0.02241(0.04555)	0.16760(0.04849)	0.00897(−0.02647)	0.00720(−0.00677)
	26		3	I	0.00001(0.00162)	0.00001(0.00050)	0.02144(0.04471)	0.16623(0.04253)	0.00911(−0.02688)	0.00789(−0.00633)
				II	0.00002(0.00159)	0.00002(0.00026)	0.04019(0.05812)	0.24160(0.05804)	0.01286(−0.03851)	0.01020(−0.00665)
				III	0.00001(0.00155)	0.00001(0.00035)	0.02616(0.04969)	0.19975(0.04193)	0.01066(−0.03220)	0.00878(−0.00561)
				IV	0.00002(0.00159)	0.00001(0.00045)	0.04347(0.05302)	0.25869(0.08136)	0.01132(−0.03321)	0.00928(−0.00631)
			3.5	I	0.00001(0.00162)	0.00001(0.00050)	0.02135(0.04441)	0.15845(0.03912)	0.00909(−0.02672)	0.00787(−0.00636)
				II	0.00002(0.00159)	0.00002(0.00028)	0.03384(0.05751)	0.22635(0.05330)	0.01238(−0.03831)	0.00979(−0.00675)
				III	0.00001(0.00155)	0.00001(0.00035)	0.02433(0.04793)	0.18740(0.02414)	0.01024(−0.03074)	0.00863(−0.00508)
				IV	0.00002(0.00159)	0.00001(0.00047)	0.04093(0.05387)	0.24990(0.07747)	0.01126(−0.03406)	0.00892(−0.00651)
			4	I	0.00001(0.00162)	0.00001(0.00050)	0.02135(0.04437)	0.15061(0.03637)	0.00905(−0.02655)	0.00787(−0.00630)
				II	0.00002(0.00159)	0.00002(0.00030)	0.03784(0.05711)	0.21988(0.05274)	0.01188(−0.03731)	0.00953(−0.00702)
				III	0.00001(0.00155)	0.00001(0.00035)	0.02844(0.04981)	0.17702(0.01031)	0.01032(−0.03075)	0.00862(−0.00473)
				IV	0.00002(0.00159)	0.00001(0.00048)	0.03367(0.05099)	0.24728(0.07329)	0.01056(−0.03118)	0.00870(−0.00640)

The goodness-of-fit test is conducted through the correlation between the generated censored sample $x_{i:m+R_m:n}$ and the expectation of the replicated data $E\left(X^{rep}_{i:m+R_m:n}\right)$. From the three-parameter Weibull distribution with the estimates reported in Table 7, the replicated data $X^{rep}_{i:m+R_m:n}$ are randomly generated over $N = 20{,}000$. Then, the expectation $E\left(X^{rep}_{i:m+R_m:n}\right)$ is obtained as

$$E\left(X^{rep}_{i:m+R_m:n}\right) = \frac{1}{20000}\sum_{j=1}^{20000} X^{rep,j}_{i:m+R_m:n}.$$

In addition, the 95% predictive region for the replicated data $X^{rep}_{i:m+R_m:n}$ is also computed to examine the uncertainty. The lower and upper bounds of the 95% predictive region are given by

$$\left(X^{rep,([0.025N])}_{i:m+R_m:n},\ X^{rep,([0.975N])}_{i:m+R_m:n}\right),$$

where $X^{rep,([0.025N])}_{i:m+R_m:n}$ denotes the $[0.025N]$ smallest of $\left\{X^{rep,j}_{i:m+R_m:n}\right\}$. These results represent in Figure 2 which shows the 95% predictive regions for the replicated data $X^{rep}_{i:m+R_m:n}$ and the scatter plot between

Table 3: MSEs (biases) of estimators for $\lambda = 2.5$ when T_1 increases for the fixed T_2

n	m	T_1	T_2	Scheme	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\hat{\lambda}$	$\tilde{\lambda}$
18	16	1.1	1.4	I	0.05747(0.17559)	0.02132(0.07131)	0.08326(−0.19844)	0.04119(−0.10687)	1.05033(−0.53640)	0.54612(−0.26881)
				II	0.05961(0.18232)	0.02156(0.07079)	0.09121(−0.20772)	0.03974(−0.09219)	1.14718(−0.61454)	0.55262(−0.27072)
				III	0.05763(0.17614)	0.02128(0.07059)	0.08541(−0.20044)	0.04079(−0.10140)	1.07555(−0.55458)	0.52828(−0.26166)
				IV	0.05921(0.18123)	0.02105(0.06959)	0.08423(−0.19697)	0.03784(−0.08782)	1.11937(−0.59035)	0.55700(−0.25339)
		1.2		I	0.05800(0.17748)	0.02125(0.07109)	0.08371(−0.19966)	0.03929(−0.10164)	1.03585(−0.55580)	0.52844(−0.27117)
				II	0.05961(0.18232)	0.02155(0.07080)	0.09121(−0.20772)	0.03949(−0.09160)	1.14718(−0.61454)	0.54974(−0.27228)
				III	0.05789(0.17695)	0.02115(0.07027)	0.08572(−0.20106)	0.03973(−0.09870)	1.06760(−0.56245)	0.52315(−0.26334)
				IV	0.05921(0.18123)	0.02107(0.06967)	0.08423(−0.19697)	0.03755(−0.08706)	1.11937(−0.59035)	0.55185(−0.25635)
		1.3		I	0.05841(0.17900)	0.02119(0.07068)	0.08387(−0.20053)	0.03705(−0.09296)	1.02303(−0.56971)	0.50428(−0.27392)
				II	0.05961(0.18232)	0.02164(0.07114)	0.09121(−0.20772)	0.03906(−0.08994)	1.14718(−0.61454)	0.53953(−0.27977)
				III	0.05831(0.17845)	0.02133(0.07060)	0.08584(−0.20197)	0.03803(−0.09290)	1.05987(−0.57474)	0.50661(−0.27255)
				IV	0.05921(0.18123)	0.02122(0.07012)	0.08423(−0.19697)	0.03717(−0.08520)	1.11937(−0.59035)	0.53950(−0.26596)
	12	1.1		I	0.06372(0.19212)	0.02363(0.07721)	0.09239(−0.18815)	0.03564(−0.08860)	1.43236(−0.72270)	0.70308(−0.32615)
				II	0.06065(0.18365)	0.02247(0.07228)	0.11500(−0.21739)	0.04978(−0.10475)	1.51627(−0.69966)	0.67956(−0.28694)
				III	0.05901(0.17981)	0.02345(0.07587)	0.09536(−0.20350)	0.04511(−0.11250)	1.39241(−0.63865)	0.69265(−0.32733)
				IV	0.05813(0.17709)	0.02101(0.06749)	0.09777(−0.16899)	0.04335(−0.08850)	1.36983(−0.58216)	0.71866(−0.20955)
		1.2		I	0.06286(0.19072)	0.02256(0.07438)	0.08538(−0.19408)	0.03576(−0.08529)	1.28684(−0.69481)	0.61078(−0.30369)
				II	0.06065(0.18365)	0.02255(0.07250)	0.11500(−0.21739)	0.04907(−0.10301)	1.51627(−0.69966)	0.66733(−0.29343)
				III	0.06068(0.18423)	0.02285(0.07431)	0.09477(−0.20494)	0.04220(−0.10213)	1.35353(−0.67502)	0.64734(−0.31968)
				IV	0.05813(0.17709)	0.02117(0.06797)	0.09777(−0.16898)	0.04265(−0.08620)	1.36985(−0.58216)	0.68834(−0.22382)
		1.3		I	0.06086(0.18568)	0.02194(0.07189)	0.08481(−0.19969)	0.03565(−0.08363)	1.12970(−0.63555)	0.53123(−0.28969)
				II	0.06065(0.18365)	0.02247(0.07238)	0.11497(−0.21737)	0.04740(−0.09850)	1.51634(−0.69968)	0.64533(−0.30505)
				III	0.05981(0.18281)	0.02178(0.07136)	0.09428(−0.20536)	0.04036(−0.09111)	1.28424(−0.65885)	0.58346(−0.30557)
				IV	0.05813(0.17709)	0.02136(0.06881)	0.09779(−0.16900)	0.04133(−0.08129)	1.36994(−0.58220)	0.65085(−0.24978)
36	32	1.2	1.4	I	0.02707(0.11465)	0.01049(0.04722)	0.03661(−0.12736)	0.01894(−0.06518)	0.43324(−0.33046)	0.26204(−0.18816)
				II	0.02653(0.11352)	0.01003(0.04631)	0.03741(−0.12694)	0.01812(−0.05536)	0.45681(−0.35342)	0.25948(−0.20325)
				III	0.02790(0.11695)	0.01071(0.04842)	0.03763(−0.12969)	0.01883(−0.06213)	0.44927(−0.35010)	0.26686(−0.19220)
				IV	0.02643(0.11345)	0.00991(0.04587)	0.03554(−0.12332)	0.01732(−0.05261)	0.44646(−0.34773)	0.25608(−0.19682)
		1.25		I	0.02718(0.11510)	0.01057(0.04765)	0.03671(−0.12760)	0.01872(−0.06385)	0.43147(−0.33549)	0.25955(−0.19093)
				II	0.02653(0.11352)	0.01003(0.04630)	0.03741(−0.12694)	0.01811(−0.05532)	0.45681(−0.35342)	0.25930(−0.20338)
				III	0.02789(0.11693)	0.01070(0.04834)	0.03759(−0.12961)	0.01863(−0.06152)	0.44939(−0.35066)	0.26610(−0.19255)
				IV	0.02643(0.11345)	0.00991(0.04587)	0.03554(−0.12332)	0.01730(−0.05255)	0.44646(−0.34773)	0.25586(−0.19698)
		1.3		I	0.02730(0.11573)	0.01065(0.04803)	0.03671(−0.12783)	0.01830(−0.06146)	0.42883(−0.34209)	0.25403(−0.19420)
				II	0.02653(0.11352)	0.01004(0.04636)	0.03741(−0.12694)	0.01811(−0.05529)	0.45681(−0.35342)	0.25874(−0.20386)
				III	0.02800(0.11731)	0.01069(0.04840)	0.03778(−0.12999)	0.01839(−0.06050)	0.44864(−0.35360)	0.26409(−0.19424)
				IV	0.02643(0.11345)	0.00991(0.04585)	0.03554(−0.12332)	0.01726(−0.05242)	0.44646(−0.34773)	0.25549(−0.19731)
	26	1.2		I	0.02962(0.12348)	0.01104(0.04909)	0.03687(−0.12821)	0.01754(−0.05539)	0.53682(−0.41374)	0.29902(−0.21446)
				II	0.02900(0.12154)	0.01134(0.05094)	0.04424(−0.13977)	0.02176(−0.06350)	0.57106(−0.40812)	0.31282(−0.23090)
				III	0.02804(0.11743)	0.01111(0.05055)	0.03984(−0.13180)	0.02039(−0.06994)	0.51422(−0.36445)	0.30474(−0.21863)
				IV	0.02846(0.11989)	0.01098(0.04977)	0.03801(−0.12452)	0.01969(−0.05568)	0.53065(−0.38135)	0.30406(−0.20958)
		1.25		I	0.02928(0.12245)	0.01096(0.04892)	0.03679(−0.12859)	0.01770(−0.05480)	0.50894(−0.40342)	0.28737(−0.21113)
				II	0.02900(0.12154)	0.01134(0.05094)	0.04424(−0.13977)	0.02172(−0.06340)	0.57106(−0.40812)	0.31253(−0.23116)
				III	0.02827(0.11870)	0.01086(0.04999)	0.03981(−0.13229)	0.01961(−0.06604)	0.50726(−0.37548)	0.29574(−0.21714)
				IV	0.02846(0.11989)	0.01098(0.04975)	0.03801(−0.12452)	0.01962(−0.05547)	0.53065(−0.38135)	0.30338(−0.21025)
		1.3		I	0.02867(0.12037)	0.01085(0.04853)	0.03677(−0.12838)	0.01754(−0.05486)	0.47677(−0.38459)	0.26980(−0.20498)
				II	0.02900(0.12154)	0.01134(0.05095)	0.04424(−0.13977)	0.02167(−0.06320)	0.57106(−0.40812)	0.31213(−0.23176)
				III	0.02835(0.11914)	0.01083(0.04982)	0.03981(−0.13244)	0.01915(−0.06264)	0.50256(−0.37971)	0.28768(−0.21640)
				IV	0.02846(0.11989)	0.01098(0.04978)	0.03801(−0.12452)	0.01951(−0.05503)	0.53065(−0.38135)	0.30123(−0.21215)

the generated censored sample $x_{i:m+R_m:n}$ and the expectation of the replicated data $E\left(X^{rep}_{i:m+R_m:n}\right)$ for all Cases. From Figure 2, it can be seen that the generated censored sample $x_{i:m+R_m:n}$ and the expectation of the replicated data $E\left(X^{rep}_{i:m+R_m:n}\right)$ are close to a straight line with high correlations, which indicates that the censored sample generated from the real data fits the three-parameter Weibull distribution for all estimates. In addition, the MPSEs show the smaller predictive regions in terms of the uncertainty along with a higher correlation, which indicates that the MPSEs have better performance than the MLEs.

5. Conclusion

This study proposes a new approach based on the MPS to overcome the shortcomings that the maximum likelihood method may fail or yield an inconsistent estimator when the shape parameter λ of the three-parameter Weibull distribution is less than unity under the generalized Type-II PHCS.

The proposed approach was proved to be superior through the Monte Carlo simulation and the application for the real data. According to our simulation, in the case of $\lambda = 0.5$, the performance of proposed MPSEs was enhanced in terms of the bias. In addition, for the large sample size, the MPSE

Table 4: MSEs (biases) of estimators for $\lambda = 2.5$ when T_2 increases for the fixed T_1

n	m	T_1	T_2	Scheme	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\hat{\lambda}$	$\tilde{\lambda}$
18	16	1.1	1.2	I	0.06275(0.18995)	0.02244(0.07415)	0.09057(−0.18688)	0.03815(−0.08653)	1.32663(−0.68862)	0.63062(−0.29953)
				II	0.06463(0.19592)	0.02347(0.07612)	0.10495(−0.18794)	0.04072(−0.08319)	1.47525(−0.76579)	0.66851(−0.32353)
				III	0.06309(0.19233)	0.02266(0.07417)	0.09203(−0.18619)	0.03886(−0.08343)	1.37664(−0.72068)	0.63726(−0.30622)
				IV	0.06437(0.19482)	0.02250(0.07357)	0.10918(−0.17747)	0.03795(−0.07752)	1.43117(−0.74010)	0.64821(−0.30338)
			1.3	I	0.06004(0.18250)	0.02197(0.07239)	0.08477(−0.19699)	0.03991(−0.09676)	1.16203(−0.60459)	0.57228(−0.28422)
				II	0.06240(0.18976)	0.02292(0.07438)	0.09495(−0.20531)	0.04012(−0.08944)	1.28633(−0.68941)	0.59174(−0.30251)
				III	0.05988(0.18318)	0.02196(0.07255)	0.08719(−0.19803)	0.03937(−0.09303)	1.19789(−0.62682)	0.57348(−0.28420)
				IV	0.06132(0.18699)	0.02226(0.07234)	0.09460(−0.18799)	0.03807(−0.08369)	1.23868(−0.65408)	0.59126(−0.28407)
			1.4	I	0.05747(0.17559)	0.02132(0.07131)	0.08326(−0.19844)	0.04119(−0.10687)	1.05033(−0.53640)	0.54612(−0.26881)
				II	0.05961(0.18232)	0.02156(0.07079)	0.09121(−0.20772)	0.03974(−0.09219)	1.14718(−0.61454)	0.55262(−0.27072)
				III	0.05763(0.17614)	0.02128(0.07059)	0.08541(−0.20044)	0.04079(−0.10140)	1.07555(−0.55458)	0.52828(−0.26166)
				IV	0.05921(0.18123)	0.02105(0.06959)	0.08423(−0.19697)	0.03784(−0.08782)	1.11937(−0.59035)	0.55700(−0.25339)
	12		1.2	I	0.06415(0.19324)	0.02344(0.07676)	0.09804(−0.18217)	0.03613(−0.08461)	1.45767(−0.73591)	0.70525(−0.32436)
				II	0.06611(0.19940)	0.02337(0.07497)	4.11610(−0.14584)	0.05194(−0.08498)	1.86260(−0.88804)	0.76729(−0.34770)
				III	0.06234(0.18941)	0.02331(0.07569)	0.12453(−0.17969)	0.04495(−0.09095)	1.58851(−0.74612)	0.72767(−0.33565)
				IV	0.06295(0.18964)	0.02205(0.07157)	0.51634(−0.11845)	0.05195(−0.06633)	1.61932(−0.72865)	0.75293(−0.28429)
			1.3	I	0.06384(0.19236)	0.02353(0.07707)	0.09296(−0.18754)	0.03576(−0.08783)	1.43738(−0.72505)	0.70230(−0.32579)
				II	0.06308(0.19065)	0.02282(0.07382)	0.15676(−0.20395)	0.05015(−0.09530)	1.67115(−0.78385)	0.70855(−0.31799)
				III	0.06022(0.18318)	0.02328(0.07559)	0.11004(−0.19787)	0.04510(−0.10555)	1.45958(−0.67340)	0.69935(−0.32937)
				IV	0.06103(0.18423)	0.02188(0.07096)	0.16623(−0.15524)	0.04420(−0.08022)	1.47737(−0.65691)	0.71649(−0.25766)
			1.4	I	0.06372(0.19212)	0.02363(0.07721)	0.09239(−0.18815)	0.03564(−0.08860)	1.43236(−0.72270)	0.70308(−0.32615)
				II	0.06065(0.18365)	0.02247(0.07228)	0.11500(−0.21739)	0.04978(−0.10475)	1.51627(−0.69966)	0.67956(−0.28694)
				III	0.05901(0.17981)	0.02345(0.07587)	0.09536(−0.20350)	0.04511(−0.11250)	1.39241(−0.63865)	0.69265(−0.32733)
				IV	0.05813(0.17709)	0.02101(0.06749)	0.09777(−0.16899)	0.04335(−0.08850)	1.36983(−0.58216)	0.71866(−0.20955)
36	32	1.2	1.3	I	0.02845(0.11931)	0.01068(0.04764)	0.03669(−0.12765)	0.01826(−0.05824)	0.48228(−0.37382)	0.27765(−0.19881)
				II	0.02778(0.11760)	0.01014(0.04641)	0.03788(−0.12698)	0.01808(−0.05365)	0.51445(−0.39229)	0.27812(−0.21178)
				III	0.02939(0.12212)	0.01110(0.04950)	0.03789(−0.13046)	0.01861(−0.05769)	0.49994(−0.39626)	0.28303(−0.20872)
				IV	0.02760(0.11693)	0.01015(0.04646)	0.03563(−0.12233)	0.01733(−0.05129)	0.50316(−0.38283)	0.27730(−0.20656)
			1.35	I	0.02769(0.11676)	0.01057(0.04739)	0.03651(−0.12768)	0.01863(−0.06164)	0.45143(−0.34949)	0.26846(−0.19339)
				II	0.02713(0.11553)	0.01025(0.04712)	0.03790(−0.12775)	0.01807(−0.05525)	0.48463(−0.37258)	0.26961(−0.21054)
				III	0.02874(0.11951)	0.01100(0.04921)	0.03803(−0.13021)	0.01868(−0.05964)	0.47284(−0.37372)	0.27491(−0.20267)
				IV	0.02699(0.11523)	0.01015(0.04668)	0.03549(−0.12331)	0.01728(−0.05244)	0.47438(−0.36484)	0.26749(−0.20350)
			1.4	I	0.02707(0.11465)	0.01049(0.04722)	0.03661(−0.12736)	0.01894(−0.06518)	0.43324(−0.33046)	0.26204(−0.18816)
				II	0.02653(0.11352)	0.01003(0.04631)	0.03741(−0.12694)	0.01812(−0.05536)	0.45681(−0.35342)	0.25948(−0.20325)
				III	0.02790(0.11695)	0.01071(0.04842)	0.03763(−0.12969)	0.01883(−0.06213)	0.44927(−0.35010)	0.26686(−0.19220)
				IV	0.02643(0.11345)	0.00991(0.04587)	0.03554(−0.12332)	0.01732(−0.05261)	0.44646(−0.34773)	0.25608(−0.19682)
	26		1.3	I	0.02963(0.12350)	0.01102(0.04901)	0.03690(−0.12795)	0.01761(−0.05505)	0.53791(−0.41460)	0.29886(−0.21434)
				II	0.03015(0.12535)	0.01133(0.05126)	0.04639(−0.13740)	0.02173(−0.06085)	0.65297(−0.45344)	0.33607(−0.24197)
				III	0.02880(0.12013)	0.01116(0.05050)	0.03954(−0.13027)	0.02016(−0.06425)	0.54916(−0.39294)	0.31110(−0.22324)
				IV	0.02973(0.12382)	0.01131(0.05032)	0.03764(−0.12236)	0.01947(−0.05225)	0.60486(−0.42208)	0.32384(−0.22344)
			1.35	I	0.02962(0.12348)	0.01104(0.04909)	0.03688(−0.12821)	0.01754(−0.05534)	0.53702(−0.41385)	0.29910(−0.21449)
				II	0.02960(0.12348)	0.01146(0.05157)	0.04445(−0.13914)	0.02165(−0.06247)	0.61447(−0.43193)	0.32480(−0.23934)
				III	0.02831(0.11839)	0.01112(0.05058)	0.03945(−0.13096)	0.02031(−0.06768)	0.52882(−0.37454)	0.30769(−0.22028)
				IV	0.02904(0.12173)	0.01108(0.05007)	0.03778(−0.12397)	0.01935(−0.05401)	0.56395(−0.39998)	0.31329(−0.21659)
			1.4	I	0.02962(0.12348)	0.01104(0.04909)	0.03687(−0.12821)	0.01754(−0.05539)	0.53682(−0.41374)	0.29902(−0.21446)
				II	0.02900(0.12154)	0.01134(0.05094)	0.04424(−0.13977)	0.02176(−0.06350)	0.57106(−0.40812)	0.31282(−0.23090)
				III	0.02804(0.11743)	0.01111(0.05055)	0.03984(−0.13180)	0.02039(−0.06994)	0.51422(−0.36445)	0.30474(−0.21863)
				IV	0.02846(0.11989)	0.01098(0.04977)	0.03801(−0.12452)	0.01969(−0.05568)	0.53065(−0.38135)	0.30406(−0.20958)

Table 5: Breaking strength of jute fiber of gauge length 10 mm

693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48	108.94	50.16	671.49	183.16	257.44	727.23	291.27
101.15	376.42	163.40	141.38	700.74	262.90	353.24	422.11	43.93	590.48	212.13	303.90	506.60	530.55	177.25

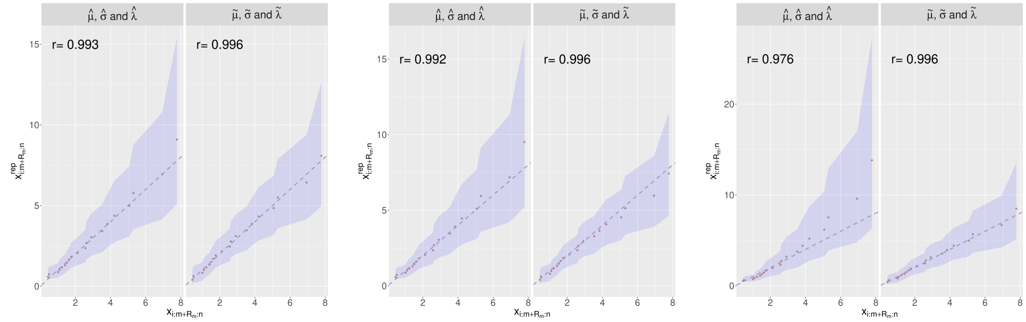
Table 6: Extended progressive Type-II censored sample generated for Case I

i	1	2	3	4	5	6	7	8	9	10
$x_{i:20:30}$	0.4393	0.5016	1.0115	1.0894	1.2306	1.4138	1.5148	1.6340	1.7725	2.1213
R_i	0	0	0	0	0	2	2	2	2	2
i	11	12	13	14	15	16	17	18	19	20
$x_{i:20:30}$	2.5744	2.6290	2.9127	3.5324	3.8343	4.2211	5.0660	5.3055	6.9373	7.7817
R_i	0	0	0	0	0	0	0	0	0	0

of λ was generally superior to the MLE of λ in terms of the MSE. One notable thing is that for a small m, the MLE of σ yielded absurd MSE values compared to the MPSE of σ. Even for $\lambda = 2.5$ where the MLEs have the classical asymptotic properties, the superiority of the proposed method was demonstrated through our simulation results that the MPSEs were generally more efficient than the

Table 7: Estimates of μ, σ, and λ for the real data

T_1	T_2	Case	$\hat{\mu}$	$\tilde{\mu}$	$\hat{\sigma}$	$\tilde{\sigma}$	$\hat{\lambda}$	$\tilde{\lambda}$
6	7	I	0.3999	0.0694	3.2071	3.5864	1.2103	1.4890
3	4	II	0.4103	0.0484	3.2358	3.4090	1.1686	1.5466
2.5	3.5	III	0.4393	0.1079	3.5495	3.6072	0.9225	1.4214



(a) (b) (c)

Figure 2: *95% predictive regions for the replicated data $X^{rep}_{i:m+R_m:n}$ and the scatter plot between the generated censored sample $x_{i:m+R_m:n}$ and the expectation of the replicated data $E\left(X^{rep}_{i:m+R_m:n}\right)$ for (a) Case I (b) Case II (c) Case III.*

MLEs counterpart in terms of the MSE and bias. Moreover, for the real data analysis, the MPSEs had a higher correlation between the generated censored sample and the expectation of the replicated data and the smaller predictive regions for the replicated data in terms of the uncertainty, compared to the MLEs. Based on these results, it can be seen that the proposed approach is superior to the maximum likelihood method under the generalized Type-II PHCS. In addition, the proposed approach can be applied to other real data and can be easily extended to other censoring schemes.

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