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HOMOGENEOUS STRUCTURES ON CONTACT HYPERSURFACES IN HERMITIAN SYMMETRIC SPACES

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Abstract. We find a 1-parameter family of homogeneous structure tensors on contact hypersurfaces in Hermitian symmetric spaces. Among their associated Ambrose-Singer connections, we prove that the Tanaka-Webster connection is the unique pseudo-homothetically invariant connection.

1. Introduction

A *contact manifold* (M, η) is a smooth manifold M^{2n-1} together with a global one-form η such that $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on M. Given a contact manifold, we have two associated structures interacting with each other. One is a Riemannian structure (or, metric) g. The other is an *almost CR structure* (η, L), where L is the *Levi form* associated with an endomorphism J on D such that $J^2 = -I$. In particular, if J on D is integrable, then we call it the (*integrable*) *CR structure*. There is a one-to-one correspondence between the two associated structures by the relation $g = L + \eta \otimes \eta$, where we denote by the same letter L the natural extension ($i_\xi L = 0$) of the Levi form to a (0,2)-tensor field on M. For this reason, we use a contact Riemannian structure together with a contact strongly pseudo-convex almost CR structure. Corresponding to the Levi-Civita connection with respect to g, there is a canonical affine connection, namely, the *Tanaka-Webster connection* on a strongly pseudo-convex CR manifold. Tanno [16] defined the generalized Tanaka-Webster connection in a contact Riemannian manifold. On the other hand, the so-called contact (k, μ)-spaces ([4]) are defined by the condition:

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y) \tag{1}$$

for $k, \mu \in \mathbb{R}$, where I denotes the identity transformation. It includes Sasakian spaces for $k = 1$ ($h = 0$) and it provides a plenty of strongly pseudo-convex CR manifolds. Indeed, the present author gave a complete classification of non K-contact (k, μ)-hypersurfaces which are realized in Hermitian symmetric

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spaces of rank 1 and of rank 2 (see, Theorem 8 in [8] and Theorem 4.1). In this article, we find a 1-parameter family of homogeneous structure tensors on them (Theorem 4.2). Moreover, we prove that the Tanno's generalized Tanaka-Webster connection is the unique pseudo-homothetically invariant connection among the 1-parameter family of affine connections (Theorem 3.7).

2. Preliminaries

We start by collecting some fundamental materials about (almost) contact manifolds and strongly pseudo-convex pseudo-Hermitian manifolds. All manifolds in the present paper are assumed to be connected, oriented, and of class C^∞.

Definition 2.1 ([10],[13]). *A $(2n-1)$-dimensional manifold M is said to be an almost contact manifold if its structure group of the linear frame bundle is reducible to $U(n-1)\times\{1\}$, or equivalently, if there exist a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying*

$$\eta(\xi) = 1 \text{ and } \phi^2 = -I + \eta\otimes\xi,$$

where I denotes the identity transformation. We call (η,ϕ,ξ) an almost contact structure.

Then we can always find a compatible Riemannian metric g:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y on M. Such a metric is called an *associated metric* and $(M;\eta,\phi,\xi,g)$ is said to be an *almost contact metric manifold.* The *fundamental 2-form* Φ is defined by $\Phi(X,Y) = g(X,\phi Y)$. If, in addition, M satisfies $d\eta = \Phi$, then M is called a *contact Riemannian manifold* or a *contact metric manifold*, where d is the exterior differential operator. On a contact metric manifold, η is a *contact form*, i.e., $\eta\wedge(d\eta)^{n-1}\neq 0$, which yields that every contact metric manifold is orientable. We call the structure vector field ξ the *Reeb vector field* or the *characteristic vector field.* Given a contact metric manifold M, we define the *structural operator* h by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}_\xi$ denotes Lie differentiation for ξ. Then we may observe that h is self-adjoint and it satisfies $h\phi = -\phi h$ and $h\xi = 0$. Moreover, we have

$$\nabla_X\xi = -\phi X - \phi h X, \tag{2}$$

where ∇ denotes the Levi-Civita connection on M. It follows that each trajectory of ξ is a geodesic. A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold.* It is easy to see that a contact Riemannian manifold is K-contact if and only if h vanishes. On the other hand, we may define naturally an almost complex structure $J^\times$ on the product manifold $M\times\mathbb{R}$. In the case of $J^\times$ is integrable (the Nijenhuis torsion of $J^\times$ vanishes), M is said to be *normal.* The integrability condition for the almost complex structure is the vanishing of the tensor $[\phi,\phi]+2d\eta\otimes\xi$, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ. Then we find that the associated pseudo-Hermitian structure of a normal almost contact structure is CR-integrable (cf. [3]). A normal contact metric manifold is said to be a *Sasakian manifold*. A Sasakian manifold is characterized by a condition

$$(\nabla_X\phi)Y = g(X,Y)\xi-\eta(Y)X \tag{3}$$

or

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y$$

for all vector fields X and Y on the manifold.

Definition 2.2 ([9]). *Let M be a $(2n-1)$-dimensional manifold and TM be its tangent bundle. A CR structure on M is a complex rank $(n-1)$ subbundle $\mathcal{H}\subset\mathbb{C}TM=TM\otimes\mathbb{C}$ satisfying (i) $\mathcal{H}\cap\bar{\mathcal{H}}=\{0\}$ and (ii) $[\mathcal{H},\mathcal{H}]\subset\mathcal{H}$ (integrability), where $\bar{\mathcal{H}}$ denotes the complex conjugation of $\mathcal{H}$.*

For a CR structure $\mathcal{H}$, there exists a unique subbundle $D=\mathrm{Re}\{\mathcal{H}\oplus\bar{\mathcal{H}}\}$, which is maximally holomorphic subbundle of $(M,\mathcal{H})$, and a unique bundle map J such that $J^2=-I$ and $\mathcal{H}=\{X-iJX|X\in D\}$. We call (D,J) the real representation of $\mathcal{H}$. Let $E\subset T^*M$ be the conormal bundle of D. If M is an oriented CR manifold, then E is a trivial bundle, and hence it admits globally defined a nowhere zero section η, i.e., a real one-form on M such that $\mathrm{Ker}(\eta)=D$. For (D,J) we define the Levi form by

$$L:\Gamma(D)\times\Gamma(D)\to\mathcal{F}(M),\quad L(X,Y)=-d\eta(X,JY),$$

where $\mathcal{F}(M)$ denotes the algebra of differentiable functions on M. If the Levi form is non-degenerate (positive or negative definite, resp.) and hermitian, then (η,J) is called a *non-degenerate* (*strongly pseudo-convex*, resp.) *pseudo-Hermitian CR structure*. Then we have a unique globally defined nowhere zero tangent vector field ξ such that $\eta(\xi)=1$ and $i_\xi d\eta=0$. Here, i_X denotes the interior product with a vector field X on M. We define the Webster metric on M by

$$g_\eta=L+\eta\otimes\eta,$$

where $i_\xi L=0$. The transversal complex structure ϕ is deduced from J: $\phi|_D=J$ and $\phi\xi=0$.

Returning to an almost contact manifold $M=(M;\eta,\phi,\xi)$, the tangent space T_pM of M at each point $p\in M$ is decomposed as $T_pM=D_p\oplus\{\xi\}_p$, where we denote $D_p=\{v\in T_pM|\eta(v)=0\}$. Then $D:p\to D_p$ defines a distribution orthogonal to ξ, which is called a *contact distribution*, and the restriction $J=\phi|_D$ of ϕ to D defines an almost complex structure in D. Such (η,J) is called an *almost CR structure* if M satisfies

$$[JX,JY]-[X,Y]\in\Gamma(D)\ (\text{or } [JX,Y]+[X,JY]\in\Gamma(D))$$

for all $X,Y\in\Gamma(D)$. Furthermore, when it satisfies

$$[J,J](X,Y)=0,$$

where $[J, J]$ is the Nijenhuis torsion of J, the pair (η, J) is a pseudo-Hermitian (integrable) CR structure associated with the almost contact structure (η, ϕ, ξ). In fact, it should be notable that for a contact metric manifold $(M; \eta, g)$ it has a strongly pseudo-convex pseudo-Hermitian structure (η, J) but the CR-integrability does not hold in general. In terms of the structure tensors, CR-integrability condition is equivalent to the condition $\Omega = 0$, where Ω is the $(1, 2)$-tensor field on M defined by

$$\Omega(X, Y) = (\nabla_X \phi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{4}$$

for any vector fields X, Y on M (see [16, Proposition 2.1]). From (3) and (4), we see that the associated pseudo-Hermitian structure of a Sasakian manifold is strongly pseudo-convex and CR-integrable. The same is true for the associated CR structure of any three-dimensional contact Riemannian manifold.

Definition 2.3 ([15]). *A* pseudo-homothetic *or* D_a-homothetic transformation *of a contact metric manifold is a diffeomorphism f on M such that $f^*\eta = a\eta,\ f_*\xi = \frac{1}{a}\xi,\ \phi \circ f_* = f_* \circ \phi,\ f^*g = ag + a(a-1)\eta \otimes \eta$, where a is a positive constant.*

A *pseudo-homothetic* or D_a*-homothetic deformation* $(\bar\eta, \bar\phi, \bar\xi, \bar g)$ of a given contact metric structure (η, ϕ, ξ, g) is defined by

$$\bar\eta = a\eta,\ \bar\xi = \frac{1}{a}\ \xi,\ \bar\phi = \phi,\ \bar g = ag + a(a-1)\eta \otimes \eta. \tag{5}$$

From (5), we have $\bar h = (1/a)h$. By using the Koszul formula, we have

$$\bar\nabla_X Y = \nabla_X Y + C(X, Y), \tag{6}$$

where C is the (1,2)-type tensor defined by

$$C(X, Y) = -(a-1)\big(\eta(Y)\phi X + \eta(X)\phi Y\big) - \frac{a-1}{a} g(\phi hX, Y)\xi.$$

We remark that CR-integrability of the associated pseudo-Hermitian structure is preserved under pseudo-homothetic transformations. In fact, by direct computations, we have that $\Omega = 0$ implies $\bar\Omega = 0$ ([7]).

We review the *generalized Tanaka-Webster connection* $\hat\nabla^{(2)}$ ([16]) on a contact strongly pseudo-convex almost CR manifold $M = (M; \eta, J)$. It is defined by

$$\hat\nabla^{(2)}_X Y = \nabla_X Y + \eta(X)\phi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (2), $\hat\nabla^{(2)}$ is rewritten as

$$\hat\nabla^{(2)}_X Y = \nabla_X Y + A^{(2)}(X, Y), \tag{7}$$

where we put

$$A^{(2)}(X, Y) = \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi. \tag{8}$$

We see that the generalized Tanaka-Webster connection $\hat\nabla^{(2)}$ has the torsion

$$\hat T^{(2)}(X, Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi hX - \eta(X)\phi hY. \tag{9}$$

In particular, for a K-contact Riemannian manifold we get $\hat{T}^{(2)}(X,Y) = 2g(X,\phi Y)\xi$. The generalized Tanaka-Webster connection can also be characterized differently.

Proposition 2.4 ([16]). *The generalized Tanaka-Webster connection $\hat{\nabla}^{(2)}$ on a contact Riemannian manifold $M = (M;\eta,g)$ is the unique linear connection satisfying the following conditions:*

(*i*) $\hat{\nabla}^{(2)}\eta = 0$, $\hat{\nabla}^{(2)}\xi = 0$;

(*ii*) $\hat{\nabla}^{(2)}g = 0$;

(*iii*-1) $\hat{T}^{(2)}(X,Y) = 2L(X,JY)\xi$, $X,\ Y \in \Gamma(D)$;

(*iii*-2) $\hat{T}^{(2)}(\xi,\phi Y) = -\phi\hat{T}^{(2)}(\xi,Y)$, $Y \in \Gamma(D)$;

(*iv*) $(\hat{\nabla}^{(2)}_X\phi)Y = \Omega(X,Y)$, $X,\ Y \in \Gamma(TM)$, *where* Ω *is defined by (4).*

We note that the Tanaka-Webster connection ([14], [18]) was in origin defined for a nondegenerate integrable CR manifold, in which case condition (*iv*) reduces to $\hat{\nabla}^{(2)}J = 0$. So, the above definition is a natural generalization to the non-integrable case.

We may refer to [3] and [9] for more details about almost contact structures and their associated almost CR structures.

3. The Ambrose-Singer connections and contact (k,μ)-spaces

First, we review the notion of a locally homogeneous Riemannian manifold and its structure tensors.

Definition 3.1. *A Riemannian manifold $(M;g)$ is said to be a homogeneous Riemannian manifold if there exists a Lie group G of isometries which acts transitively on M. If there exists a local isometry which sends p to q for each $p,\ q \in M$, then M is said to be locally homogeneous Riemannian manifold*

Ambrose and Singer [1] gave an *infinitesimal characterisation* of the local homogeneity of Riemannian manifolds.

Definition 3.2. *A homogeneous Riemannian structure P on $(M;g)$ is a tensor field of type $(1,2)$ which satisfies*

$$\widehat{\nabla}g = 0, \quad \widehat{\nabla}R = 0, \quad \widehat{\nabla}P = 0.$$

Here $\widehat{\nabla}$ is a linear connection on M defined by $\widehat{\nabla} = \nabla + P$. The linear connection $\widehat{\nabla}$ is called the Ambrose-Singer connection.

Theorem 3.3 ([1]). *A Riemannian manifold $(M;g)$ with a homogeneous Riemannian structure P is locally homogeneous. Moreover, if $(M;g)$ is complete and simply connected locally homogeneous Riemannian space is (globally) homogeneous Riemannian space.*

For a homogeneous Riemannian structure tensor P of type $(1,1)$ on a Riemannian manifold $(M;g)$, we denote by the same P the $(0,3)$-tensor field in consideration of the isomorphism:

$$P(X,Y,Z) = g(P_X Y, Z)$$

for all $X,Y,Z \in \Gamma(TM)$. Tricerri and Vanhecke [17] obtained the following decompositions of all possible types of homogeneous Riemannian structures into eight classes:

Classes	Defining conditions
Symmetric	$P = 0$
$\mathfrak{T}_1$	$P(X,Y,Z) = g(X,Y)\omega(Z) - g(Z,X)\omega(Y)$ for some 1-form ω
$\mathfrak{T}_2$	$\mathfrak{S}_{X,Y,Z} P(X,Y,Z) = 0$ and $c_{12}(P) = 0$
$\mathfrak{T}_3$	$P(X,Y,Z) + P(Y,X,Z) = 0$
$\mathfrak{T}_1 \oplus \mathfrak{T}_2$	$\mathfrak{S}_{X,Y,Z} P(X,Y,Z) = 0$
$\mathfrak{T}_1 \oplus \mathfrak{T}_3$	$P(X,Y,Z) + P(Y,X,Z) = 2g(X,Y)\omega(Z) - g(Z,X)\omega(Y)$ $-g(y,Z)\omega(X)$ for some 1-form ω
$\mathfrak{T}_2 \oplus \mathfrak{T}_3$	$c_{12}(P) = 0$
$\mathfrak{T}_1 \oplus \mathfrak{T}_2 \oplus \mathfrak{T}_3$	no conditions

Here $\mathfrak{S}_{X,Y,Z} P$ denotes the cyclic sum of P, i.e.,

$$\mathfrak{S}_{X,Y,Z} P(X,Y,Z) = P(X,Y,Z) + P(Y,Z,X) + P(Z,X,Y),$$

and c_{12} denotes the contraction operator in $(1,2)$-entries:

$$c_{12}(P)(Z) = \sum_i P(e_i, e_i, Z),$$

where $\{e_i\}$, $i = 1, 2, \cdots, \dim$ M, is an orthonormal basis.

Next, we review the *contact* (k,μ)*-spaces*, introduced by Blair, Koufogiorgos and Papantoniou [4], are defined by the curvature condition (1). This class includes Sasakian manifolds (for $k = 1$ and $h = 0$) and the trivial sphere bundle $\mathbb{R}^n \times S^{n-1}(4)$ (for $k = \mu = 0$). Typical examples of non-Sasakian (k,μ)-contact spaces are the unit tangent sphere bundles of Riemannian manifolds of constant curvature $\neq 1$. Boeckx [6] proved the equivalence theorems of contact (k,μ)-spaces.

Theorem 3.4 ([6])**.** *Let* $(M^{2n-1}; \eta, \phi, \xi, g)$ *and* $(M'^{2n-1}; \eta', \phi', \xi', g')$ *be two non-Sasakian* (k,μ)*-spaces. Then they are locally isometric as contact metric spaces. In particular, if they are simply connected and complete, then they are globally isometric.*

The *Boeckx invariant* $\mathbf{I}_M$ of a *non-Sasakian* contact (k,μ)-space M is defined by $\mathbf{I}_M = (1-\mu/2)/\sqrt{1-k}$, which determines completely a contact (k,μ)-space with the fixed dimension up to equivalence.

Theorem 3.5 ([6]). *Let $(M_i; \eta_i, \phi_i, \xi_i, g_i)$, $i = 1, 2$, be two contact (k_i, μ_i)-spaces of the same dimension. Then $\mathbf{I}_{M_1} = \mathbf{I}_{M_2}$ if and only if, up to a pseudo-homothetic deformation of the contact metric structure, the two spaces are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, then they are globally isometric up to a pseudo-homothetic transformation.*

Based on the above fundamental theorem of contact (k, μ)-spaces, Boeckx himself gave examples of contact (k, μ)-spaces by a two-parameter family of solvable Lie groups which admit a left-invariant contact metric structure (other than the unit tangent sphere bundles of Riemannian manifolds of constant curvature). However, their geometric description or explicit realization has been desirable. Very recently, the present author [8] proved a complete classification theorem of non-Sasakian (k, μ)-spaces which are realized as real hypersurfaces in the complex quadric Q^n, its non-compact dual space Q^{*n}, and the complex Euclidean space $\mathbb{C}^n$ (Theorem 4.1). It provides a new geometric description and moreover unifies all the contact (k, μ)-spaces by real hypersurfaces in Hermitian symmetric spaces.

On the other hand, in [5], using the following (1,2)-type tensor field $A^{(\mu)}$:

$$A^{(\mu)}(X, Y) = \frac{\mu}{2}\eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi \tag{10}$$

for all vector fields X, Y, we define $\hat{\nabla}^{(\mu)}$ by

$$\hat{\nabla}^{(\mu)}_X Y = \nabla_X Y + A^{(\mu)}(X, Y).$$

In case that $\mu = 2$, that is, $\hat{\nabla}^{(2)}$ is the generalized Tanaka-Webster connection. Then we can show that the tensor field $A^{(\mu)}$ gives a Riemannian homogeneous structure, that is, $\hat{\nabla}^{(\mu)}$ satisfies $\hat{\nabla}^{(\mu)}g = 0$, $\hat{\nabla}^{(\mu)}R = 0$, $\hat{\nabla}^{(\mu)}A^{(\mu)} = 0$. In addition to them, it holds $\hat{\nabla}^{(\mu)}\xi = 0$, $\hat{\nabla}^{(\mu)}\eta = 0$, $\hat{\nabla}^{(\mu)}\phi = 0$. Then due to Kiričenko's generalization ([11]) of the Ambrose-Singer theorem we have a transitive pseudo-group of local automorphisms of the associated contact Riemannian structure (η, ϕ, ξ, g). Moreover, from (10) we have that among the 1-parameter family of homogeneous structures $A^{(\mu)}$ the case $\mu = 4$ is the only homogeneous Riemannian structure of type $\mathfrak{T}_2$.

Proposition 3.6. *For a contact (k, μ)-space, $A^{(\mu)}$ is a homogeneous Riemannian structure. In particular, a contact $(k, 4)$-space has a homogeneous Riemannian structure of type $\mathfrak{T}_2$.*

Now, we may define the linear connections $\hat{\nabla}^{(\mu)}$ on contact strongly pseudo-convex almost CR manifolds, in general. Then we have

Theorem 3.7. *Among the linear connections $\hat{\nabla}^{(\mu)}$, the generalized Tanaka-Webster connection $\hat{\nabla}^{(2)}$ is the unique pseudo-Hermitian invariant connection.*

Proof. From (5) and (6) we have

$$\begin{aligned}\hat{\bar{\nabla}}^{(\mu)}_X Y &= \bar{\nabla}_X Y + \bar{A}^{(\mu)}(X,Y)\\ &= \nabla_X Y + C(X,Y) + \bar{A}^{(\mu)}(X,Y)\end{aligned}$$

and

$$\bar{A}^{(\mu)}(X,Y) = a\big(\eta(Y)\phi X + \frac{\mu}{2}\eta(X)\phi Y\big) + \eta(Y)\phi hX - g(\phi X,Y)\xi - \frac{1}{a}g(\phi hX,Y)\xi.$$

Together with the definition of the tensor C we see that $C(X,Y)+\bar{A}^{(\mu)}(X,Y) = A^{(\mu)}(X,Y)$ for any positive a if and only if $\mu = 2$. Hence, we find that among the linear connections $\hat{\nabla}^{(\mu)}$ the generalized Tanaka-Webster connection is the unique pseudo-homothetically invariant connection.

□

4. Homogeneous structures on contact hypersurfaces

In this section, we treat real hypersurfaces of the complex quadric $Q^n = SO_{n+2}/SO_nSO_2$ and its noncompact dual space $Q^{n*} = SO_{n,2}/SO_nSO_2$. Then Q^n (resp. Q^{n*}) has two fundamental geometric structures which completely describe its Riemannian curvature tensor $\widetilde{R}$. The first one is the Kähler structure $(\tilde{J},\tilde{g})$ and the second one is a rank two vector bundle $\mathcal{A}$ over Q^n(resp. Q^{n*}) which contains an S^1-bundle of real structures on the tangent spaces. The complex quadric Q^n is also realized as a complex hypersurface in the $(n+1)$-dimensional complex projective space $\mathbb{C}P^{n+1}$. Then the bundle $\mathcal{A}$ is just the family of shape operators with respect to the normal vectors in the normal bundle of rank two. We should remark that $Q^{n*} = SO_{n,2}/SO_nSO_2$ is not realized as a homogeneous complex hypersurface in the $(n+1)$-dimensional complex hyperbolic space $\mathbb{C}H^{n+1}$ (cf. [12]). Now we consider Q^n (resp. Q^{n*}) as a Hermitian symmetric space equipped with the Kähler structure $(\tilde{J},\tilde{g})$ for which the maximal (resp. minimal) sectional curvature $c > 0$ (resp. $c < 0$). Then we have

$$\begin{aligned}\widetilde{R}(X,Y)Z =&\frac{c}{4}\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \tilde{g}(\tilde{J}Y,Z)X - \tilde{g}(\tilde{J}X,Z)\tilde{J}Y\\ &- 2\tilde{g}(\tilde{J}X,Y)\tilde{J}Z\} + \epsilon\{\tilde{g}(AY,Z)AX - \tilde{g}(AX,Z)AY\\ &+ \tilde{g}(\tilde{J}AY,Z)\tilde{J}AX - \tilde{g}(\tilde{J}AX,Z)\tilde{J}AY\},\end{aligned}\tag{11}$$

where $X,Y,Z \in \Gamma(TQ^n)$ (resp. $\Gamma(TQ^{n*})$) and $\epsilon = \pm 1$ for Q^n (resp. Q^{n*}). For more details about the geometric structure of Q^n and Q^{n*}, and the fundamental properties of their real hypersurfaces, we refer to [2], [12].

Theorem 4.1. *The simply connected, complete, non K-contact, contact metric space M is a $(0,\mu)$-space if and only if it is globally isometric (as a contact metric space) to one of (i), (ii), (iii), (iv) and (v) in the following:*

(i) *a tube of radius* $r = \sqrt{\frac{2}{c}} \arctan \frac{2\sqrt{2}}{\sqrt{c}} \in (0, \pi/\sqrt{2c})$ *around a real form* S^n *of the complex quadric* Q^n *of maximal curvature* $c > 0$*;*

(ii) $\mathbb{R}^n \times S^{n-1}(4)$ *in* $\mathbb{C}^n$, $c = 0$*;*

(iii-1) *a tube of radius* $r = \sqrt{\frac{2}{|c|}} \coth^{-1} \frac{2\sqrt{2}}{\sqrt{|c|}}$ *around a real form* $\mathbb{R}H^n$ *in the dual space* Q^{n*} *of the complex quadric* Q^n *of minimal curvature* c, $-4 < c < 0$*;*

(iii-2) *a tube of radius* $r = \frac{\sqrt{2}}{2} \coth^{-1} \sqrt{2}$ *around a real form* $\mathbb{R}H^n$ *in* Q^{n*}, $c = -4$*;*

(iii-3) *a tube of radius* $r = \sqrt{\frac{2}{|c|}} \coth^{-1} \frac{2\sqrt{2}}{\sqrt{|c|}}$ *around a real form* $\mathbb{R}H^n$ *in* Q^{n*}, $-8 < c < -4$*;*

(iv) *a horosphere in* Q^{n*} *whose center at infinity is determined by an* $\mathcal{A}$*-principal geodesic in* Q^{n*}, $c = -8$*;*

(v) *a tube of radius* $r = \sqrt{\frac{2}{|c|}} \tanh^{-1} \frac{2\sqrt{2}}{\sqrt{|c|}}$ *around* $Q^{(n-1)*}$ *in* Q^{n*} $c < -8$.

Proof. From (11), we have the Gauss equation, and then we can compute $R(X, Y)\xi$ for real hypersurfaces (i), (iii), (iv), (v), which shows that they are $(0, \mu)$-spaces, where $\mu = -\frac{c}{2}$. Moreover, we already know that (ii) $\mathbb{R}^n \times S^{n-1}(4)$ (in $\mathbb{C}^n$) is a contact $(0, 0)$-space. Then, due to Theorem 3.4, we have completed the proof. For the detailed computation, we refer to [8].

□

Due to Proposition 3.6 and Theorem 4.1, we have

Theorem 4.2. *Real hypersurfaces (i), (ii), (iii), (iv) and (v) in Theorem (4.1) are homogeneous and the corresponding (1-parameter family of) homogeneous Riemannian structures* $P = A^{(-\frac{c}{2})}$*. In particular, (iv) a horosphere in* Q^{n*} *whose center at infinity is determined by an* $\mathcal{A}$*-principal geodesic in* Q^{n*}, $c = -8$ *has the homogeneous Riemannian structure* $P = A^{(4)}$ *of type* $\mathfrak{T}_2$.

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