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IRREDUCIBILITY OF THE MODULI SPACE FOR THE QUOTIENT SINGULARITY $\frac{1}{2k+1}(k+1,1,2k)$

Seung-Jo Jung

Abstract. A 3-fold quotient terminal singularity is of the type $\frac{1}{r}(b,1,-1)$ with $\gcd(r,b)=1$. In [6], it is proved that the economic resolution of a 3-fold terminal quotient singularity is isomorphic to a distinguished component of a moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations for a suitable θ. This paper proves that each connected component of the moduli space $\mathcal{M}_\theta$ has a torus fixed point and classifies all torus fixed points on $\mathcal{M}_\theta$. By product, we show that for $\frac{1}{2k+1}(k+1,1,-1)$ case the moduli space $\mathcal{M}_\theta$ is irreducible.

1. Introduction

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group. The group G acts on $\mathbb{C}^n$ naturally. If the quotient variety $X := \mathbb{C}^n/G$ is singular, we may consider the resolution of singularities $Y \to X$. A natural question in this line is whether Y has a modular interpretation in terms of G-equivariant objects on $\mathbb{C}^n$. A G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ is called a *G-constellation* if $\mathrm{H}^0(\mathcal{F})$ is isomorphic to $\mathbb{C}[G]$ as G-representations. The moduli spaces of G-constellations can be constructed via King's stability [8] where the stability parameter space

$$\Theta = \left\{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G),\mathbb{Q}) \,\middle|\, \theta\left(\mathbb{C}[G]\right)=0\right\},$$

where $R(G)$ is the representation ring of G. For fixed θ, a G-constellation is said to be θ-*(semi) stable* if $\theta(\mathcal{G})>0$ ($\theta(\mathcal{G})\geq 0$) for every non-zero proper G-eqiuivariant subsheaf $\mathcal{G}\subset\mathcal{F}$. We say a parameter θ is *generic* if all θ-semistable objects are θ-stable. The moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations has a special irreducible component Y_θ.

It is known that the 3-fold terminal quotient singularity is the quotient singularity of type $\frac{1}{r}(b,1,r-1)$ with $\gcd(b,r)=1$, which means that the quotient by

$$G = \{\mathrm{diag}(\epsilon^b,\epsilon,\epsilon^{-1}) \mid \epsilon^r = 1\} \subset \mathrm{GL}_3(\mathbb{C})$$

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(see e.g. [10]). This quotient singularity $X := \mathbb{C}^3/G$ has an economic resolution $\varphi\colon Y \to X$ whose discrepancy is minimal. More precisely, it satisfies

$$K_Y = \varphi^*(K_X) + \sum_{1\le i<r} \frac{i}{r} E_i,$$

where E_i's are prime exceptional divisors. In [7], Kędzierski proved that there is a parameter θ such that the normalisation of Y_θ is isomorphic to Y. Further, it is proved that the component Y_θ is actually smooth and a connected component of $\mathcal{M}_\theta$ in [6]. It is quite natural to ask whether $\mathcal{M}_\theta$ is irreducible or not. On the other hand, since G is abelian, there is a natural torus $\mathbf{T} = (\mathbb{C}^\times)^3$-action on $\mathcal{M}_\theta$.

In this paper, we first prove that for the group of the type $\frac{1}{r}(b, 1, -1)$ with $\gcd(r, b) = 1$ and for generic parameter θ each connected component of $\mathcal{M}_\theta$ has a $\mathbf{T}$-fixed point. Thus if we show that every $\mathbf{T}$-fixed point lies over Y_θ, then we can conclude that $\mathcal{M}_\theta = Y_\theta$ (see Proposition 3.9). Using this result, we focus on a partial case where G is of type $\frac{1}{2k+1}(k + 1, 1, 2k)$. In this case, we can classify all $\mathbf{T}$-invariant G-constellations and show that they lie over Y_θ. This implies that $\mathcal{M}_\theta$ is irreducible so the economic resolution Y is isomorphic to $\mathcal{M}_\theta$ itself, which was stated in [5] without complete proofs.

This paper is organized as follows. We begin with Section 2 to recall general theory of G-constellations and their moduli spaces. Section 3 is devoted to apply the result in [6] to our case. Then in Section 4, we state the irreducibility theorem and prove it.

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2. *G*-constellations and *G*-prebricks

2.1. Moduli spaces of *G*-constellations

This section reviews G-constellations and their moduli spaces (see e.g. [2, 3, 8]).

Let G be a finite diagonal cyclic subgroup of $\mathrm{GL}_3(\mathbb{C})$.

Definition 2.1. A *G-constellation* is a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^3$ with $\mathrm{H}^0(\mathcal{F})$ isomorphic to the regular representation $\mathbb{C}[G]$ of G.

Define $R(G) := \bigoplus_{\rho\in\operatorname{Irr} G} \mathbb{Z}\,\rho$. For a stability parameter $\theta \in \Theta$ where

$$\Theta = \left\{\theta \in \operatorname{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \,\middle|\, \theta\,(\mathbb{C}[G]) = 0\right\},$$

we say that:

(i) a G-constellation $\mathcal{F}$ is *θ-semistable* if $\theta(\mathcal{G}) \ge 0$ for every non-zero proper G-eqiuivariant subsheaf of $\mathcal{F}$;

(ii) a G-constellation $\mathcal{F}$ is *θ-stable* if $\theta(\mathcal{G}) > 0$ for every non-zero proper G-eqiuivariant subsheaf of $\mathcal{F}$;

(iii) θ is *generic* if every θ-semistable object is θ-stable.

For generic θ, a fine moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations exists as a quasi-projective scheme by King [8]. Furthermore the moduli space $\mathcal{M}_\theta$ has a unique irreducible component Y_θ which is a not-necessarily-normal toric variety birational to the quotient variety $\mathbb{C}^n/G$ by Craw–Maclagan–Thomas [3].

Definition 2.2. The irreducible component Y_θ in $\mathcal{M}_\theta$ is called the *birational component* of $\mathcal{M}_\theta$.

Note that via the $\mathbf{T} = (\mathbb{C}^\times)^n$-action on $\mathbb{C}^n$, the algebraic torus $\mathbf{T}$ naturally acts on the moduli space $\mathcal{M}_\theta$. The $\mathbf{T}$-fixed points on $\mathcal{M}_\theta$ play a crucial role in the proof of irreduciblity of $\mathcal{M}_\theta$.

Definition 2.3. For a G-equivariant sheaf $\mathcal{G}$, the *support*, denoted by $\mathrm{supp}(\mathcal{G})$, of $\mathcal{G}$ is the set of irreducible representations which appear in $\mathrm{H}^0(\mathcal{G})$.

2.2. G-prebricks and T-fixed points on $\mathcal{M}_\theta$

This section introduces G-prebricks and their correspondence with $\mathbf{T}$-fixed points on $\mathcal{M}_\theta$.

Let G be the group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$, i.e.,

$$G = \langle \mathrm{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3}) \,\big|\, \epsilon^r = 1 \rangle \subset \mathrm{GL}_3(\mathbb{C}).$$

Define the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$$

is an overlattice of $\overline{L} := \mathbb{Z}^3$. We identify the dual lattices $\overline{M} := \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ with Laurent monomials and G-invariant Laurent monomials, respectively. The embedding of G into the torus $\mathbf{T} = (\mathbb{C}^\times)^3 \subset \mathrm{GL}_3(\mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt}\colon \overline{M} \longrightarrow G^\vee,$$

where $G^\vee$ denotes the character group $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ of G.

Let $\overline{M}_{\geq 0}$ denote genuine monomials in $\overline{M}$, i.e.,

$$\overline{M}_{\geq 0} = \{x^{m_1} y^{m_2} z^{m_3} \in \overline{M} \,\big|\, m_i \geq 0 \text{ for all } i\}.$$

For a subset $A \subset \mathbb{C}[x^\pm, y^\pm, z^\pm]$, let $\langle A \rangle$ denote the $\mathbb{C}[x,y,z]$-submodule of $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ generated by A.

2.2.1. G-clusters and G-graphs. For a G-invariant subscheme Z, if $\mathcal{O}_Z$ is a G-constellation, we call it a G-cluster.

Suppose that a G-cluster $\mathcal{O}_Z$ is $\mathbf{T}$-invariant. This means that Z is given by a monomial ideal I_Z. In this case, we define:

$$\Gamma := \{x^{m_1} y^{m_2} z^{m_3} \in \overline{M}_{\geq 0} \mid x^{m_1} y^{m_2} z^{m_3} \notin I\}.$$

Then Γ gives a basis of $\mathrm{H}^0(\mathcal{O}_Z)$. From the properties of Γ, the following definition is natural (due to Nakamura [9]).

Definition 2.4. A (*Nakamura*) *G-graph* is a subset of genuine monomials in $\overline{M}_{\geq 0}$ such that:

(i) the monomial $\mathbf{1}$ is in Γ;
(ii) for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ of weight ρ, i.e., $\mathrm{wt}\colon \Gamma \to G^\vee$ is bijective;
(iii) if $\mathbf{p}' \cdot \mathbf{p} \in \Gamma$ for $\mathbf{p}, \mathbf{p}' \in \overline{M}_{\geq 0}$, then $\mathbf{p} \in \Gamma$.

From the argument above, we have a one-to-one correspondence between the set of G-graphs and the set of $\mathbf{T}$-invariant G-clusters. Thus for classifying all $\mathbf{T}$-invariant G-clusters, we can classify all G-graphs.

Similarly, for G-constellations, we define the following (see [5, 6]).

Definition 2.5. A *G-prebrick* Γ is a subset of Laurent monomials in $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ such that:

(i) the monomial $\mathbf{1}$ is in Γ;
(ii) for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ of weight ρ, i.e., $\mathrm{wt}\colon \Gamma \to G^\vee$ is bijective;
(iii) if $\mathbf{p}' \cdot \mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{p}, \mathbf{p}' \in \overline{M}_{\geq 0}$, then $\mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma$;
(iv) the set Γ is *connected* in the sense that for any element $\mathbf{m}_\rho$, there is a (fractional) path in Γ from $\mathbf{m}_\rho$ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of x, y, z.

For a G-prebrick Γ, we define $\mathrm{wt}_\Gamma\colon \overline{M} \to \Gamma$ by $\mathrm{wt}_\Gamma := (\mathrm{wt})^{-1} \circ \mathrm{wt}$. Thus $\mathrm{wt}_\Gamma(\mathbf{m})$ is the unique element $\mathbf{m}_\rho$ in Γ of the same weight as $\mathbf{m} \in \overline{M}$.

Let Γ be a G-prebrick. Define

$$C(\Gamma) := \langle \Gamma \rangle / \langle B(\Gamma) \rangle,$$

where

$$B(\Gamma) := \left\{ x \cdot \mathbf{m}, y \cdot \mathbf{m}, z \cdot \mathbf{m} \,\middle|\, \mathbf{m} \in \Gamma \right\} \setminus \Gamma.$$

Since $B(\Gamma)$ is generated by monomials, the module $C(\Gamma)$ is a torus invariant G-constellation corresponding to a $\mathbf{T}$-fixed point in the moduli space by [6]. Here the G-prebrick Γ forms a monomial $\mathbb{C}$-basis of the G-constellation $C(\Gamma)$.

Remark 2.6. In [6], it is shown that every $\mathbf{T}$-fixed point on the birational component Y_θ corresponds to a G-prebrick. However, we do not know if there is a G-prebrick corresponding to a $\mathbf{T}$-fixed point on $\mathcal{M}_\theta \setminus Y_\theta$.

Definition 2.7. A G-prebrick Γ is said to be *θ-stable* if the torus invariant G-constellation $C(\Gamma)$ is θ-stable.

For a G-prebrick $\Gamma = \{\mathbf{m}_\rho\}$, $S(\Gamma) \subset M$ is the subsemigroup generated by $\frac{\mathbf{p} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{p} \cdot \mathbf{m}_\rho)}$ for all $\mathbf{p} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. In [6], it is proved that $S(\Gamma)$ is finitely generated so it induces an affine toric variety.

Proposition 2.8 ([6, Lemma 2.11]). *For a G-prebrick Γ, the semigroup $S(\Gamma)$ is generated by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$ as a semigroup. In particular, $S(\Gamma)$ is finitely generated as a semigroup.*

Definition 2.9. For a G-prebrick Γ and a cone σ, we say that Γ *corresponds to* σ if $S(\Gamma) = \mathbb{C}[\sigma^\vee \cap M]$.

3. 3-fold terminal quotient singularities and G-constellations

This section reviews the results in [6] about 3-fold terminal quotient singularities. For simplicity, we restrict us for the type of $\frac{1}{2k+1}(k+1,1,2k)$ which is the main case of this paper. In this section, G is the group of type $\frac{1}{r}(k+1,1,2k)$ with $r = 2k+1$. Mainly we apply the method in [6] to this case directly.

The quotient singularity $X := \mathbb{C}^3/G$ does not have crepant resolutions but X has a certain toric resolution introduced by Danilov [4] (see also [10]). Consider the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(k+1,1,2k).$$

For each $1 \le i \le 2k$, let $v_i := \frac{1}{2k+1}(\overline{-ki}, i, r-i) \in L$ where $\overline{}$ denotes the residue modulo r. The *economic resolution* of $\mathbb{C}^3/G$ is the toric variety obtained by the consecutive weighted blowups at $v_2, v_4, \ldots, v_{2k}, v_1, v_3, \ldots, v_{2k-1}$ from $\mathbb{C}^3/G$. Each discrepancy of the economic resolution is in the interval $(0,1)$ (see [10]).

Let Σ be the toric fan of the economic resolution Y of $X = \mathbb{C}^3/G$. In the fan Σ, we have the following $(4k+1)$ 3-dimensional cones:

$$\begin{cases} \sigma_i = \mathrm{Cone}(e_1, v_{i-1}, v_i) & \text{for } 1 \le i \le 2k+1, \\ \sigma_i^{\triangle} = \mathrm{Cone}(v_{2i-1}, v_{2i-2}, v_{2i}) & \text{for } 1 \le i \le k, \\ \sigma_i^{\triangledown} = \mathrm{Cone}(e_2, v_{2i-2}, v_{2i}) & \text{for } 1 \le i \le k. \end{cases} \tag{3.1}$$

Example 3.2. Let G be the group of type $\frac{1}{7}(4,1,6)$. The fan of the economic resolution of the quotient variety is shown in Figure 3.1.

For the moduli description of the economic resolutions, we need to define

(i) an *admissible G-brickset*, and
(ii) an *admissible chamber* in Θ.

3.1. Stability parameter space

The index set $I := \{0, 1, \ldots, 2k\}$ is identified with $\mathbb{Z}/(2k+1)\mathbb{Z}$. For each $i \in I$, we define $\theta_i \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q})$ by $\theta_i(\rho_j) = \delta_{ij}$. Here ρ_j denotes the irreducible representation of weight j. Note that $\theta_i - \theta_j$ is an element of the stability parameter space Θ. Applying [6] to this case, we have the following.

Proposition 3.3 (cf. [5,6])**.** *Let us consider the group G of type $\frac{1}{2k+1}(k+1,1,2k)$. For the permutation*

$$\omega = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \ldots & 2k-2 & 2k-1 & 2k \\ 0 & 1 & k+1 & 2 & k+2 & \ldots & 2k-1 & k & 2k \end{pmatrix},$$

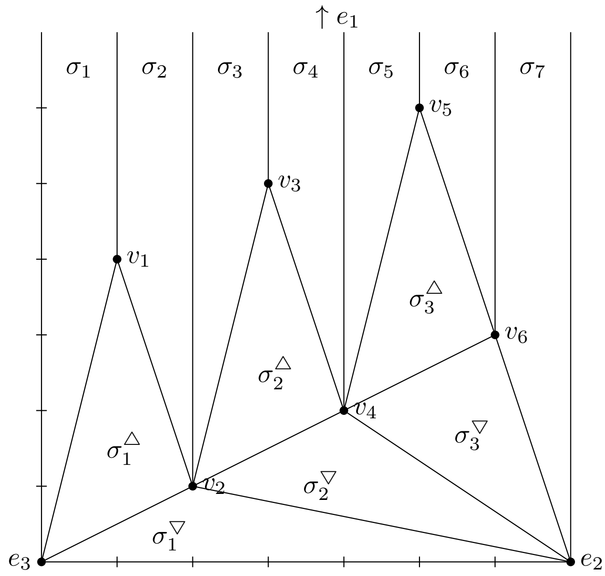

FIGURE 3.1. Fan of the economic resolution for $\frac{1}{7}(4,1,6)$

the admissible chamber $\mathfrak{C}$ *is the open cone generated by*

$$\varepsilon_i := \sum_{j=0}^{i-1} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right)$$

for $i = 1, \ldots, 2k$*, i.e.,* $\mathfrak{C} = \{a_1\varepsilon_1 + \cdots + a_{2k}\varepsilon_{2k} \mid a_i \in \mathbb{R}_{>0}\}$.

Proof. We proceed by induction on k. □

Note that

$$\omega(i) = \begin{cases} 0 & \text{if } i = 0, \\ l+1 & \text{if } i = 2l+1 \text{ is odd,} \\ k+l & \text{if } i = 2l \text{ is even.} \end{cases}$$

Using this, we can describe ε_i as follows.

$$\varepsilon_i = \begin{cases} \theta_{k+l+1} + \theta_{l+1} - \theta_{k+1} - \theta_0 & \text{if } i = 2l+1 \text{ is odd,} \\ \theta_{k+l} + \theta_{l+1} - \theta_{k+1} - \theta_0 & \text{if } i = 2l \text{ is even.} \end{cases} \tag{3.4}$$

Note that

$$\varepsilon_0 = \theta_1 - \theta_0, \quad \varepsilon_{2k} = \theta_{2k} - \theta_0.$$

Corollary 3.5. *For* $\theta \in \mathfrak{C}$*, every* θ*-stable* G*-constellation* $\mathcal{F}$ *is generated by* ρ_0 *and* ρ_{k+1}*, i.e., every subsheaf of* $\mathcal{F}$ *whose support contains* ρ_0 *and* ρ_{k+1} *is equal to* $\mathcal{F}$.

Proof. From the discussion above, we find that $\theta(\rho_i) < 0$ if and only if $i = 0$ or $k+1$, because θ is a positive linear combination of ε_i. Assume that $\mathcal{F}$ has a subsheaf $\mathcal{G}$ with $\operatorname{supp}\mathcal{G}$ containing ρ_0 and ρ_{k+1}. Since we have $\theta(\mathcal{G}) \leq 0$, from the stability, we have $\mathcal{F} = \mathcal{G}$. □

3.2. Admissible *G*-brickset

For the toric cones in (3.1), we define corresponding G-prebricks.

Proposition 3.6. *For the cones in* (3.1)*, the following G-prebricks Γ_l, $\Gamma_i^{\triangle}$, and $\Gamma_i^{\triangledown}$ correspond to σ_l, $\sigma_i^{\triangle}$, and $\sigma_i^{\triangledown}$, respectively*:

(i) $\Gamma_1 = \{1, z, \dots, z^{2k}\}$.

(ii) $\Gamma_{2i+1} = \left\{ \begin{matrix} 1, z, \dots, z^{k-i}, y, \dots, y^{i-1}, y^i \\ \frac{y^{i+1}}{z^{k-i}}, \frac{y^{i+1}}{z^{k-i-1}}, \dots, y^{i+1}, \frac{y^{i+2}}{z^{k-i}}, \dots, \frac{y^{l-1}}{z^{k-i}} \end{matrix} \right\}$.

(iii) $\Gamma_{2i} = \left\{ \begin{matrix} 1, z, \dots, z^{k-i}, y, \dots, y^{i-1}, y^i \\ \frac{z^{k-i+1}}{y^{i-1}}, \frac{z^{k-i+1}}{y^{i-2}}, \dots, z^{k-i+1}, \frac{z^{k-i+2}}{y^{i-1}}, \dots, \frac{z^{2k-2i+1}}{y^{i-1}} \end{matrix} \right\}$.

(iv) $\Gamma_i^{\triangle} = \left\{ \begin{matrix} 1, & x, & y, & \dots, & y^{k-i}, & y^{k-i+1} \\ z, & xz, & xy, & \dots, & xy^{k-i} & \\ \dots, & \dots & & & & \\ z^{i-1}, & xz^{i-1} & & & & \end{matrix} \right\}$.

(v) $\Gamma_i^{\triangledown} = \left\{ \begin{matrix} 1, & x, & x^2, & x^3 & \dots, & x^{2k-2i+1}, & x^{2k-2i+2} \\ z, & xz & & & & & \\ \dots, & \dots & & & & & \\ z^{i-1}, & xz^{i-1} & & & & & \end{matrix} \right\}$.

Proof. The proof goes through a direct calculation (see Proposition 5.3.2 in [5]). Here we show it for Γ_{2i}. First note that

$$\langle B(\Gamma) \rangle = \langle y^{i+1}, yz, \frac{z^{k-i+2}}{y^{i-2}}, \frac{z^{2k-2i+2}}{y^{i-1}}, x, \frac{xz^{k-i+1}}{y^{i-1}} \rangle.$$

Thus the semigroup $S(\Gamma)$ is generated by

$$\frac{y^{2i}}{z^{2k-2i+1}}, yz, \frac{z^{2k-2i+2}}{y^{2i-1}}, \frac{xy^{i-1}}{z^{k-i+1}}, \frac{xz^{k-i+1}}{y^i},$$

and then $S(\Gamma) = \mathbb{C}[\frac{y^{2i}}{z^{2k-2i+1}}, \frac{z^{2k-2i+2}}{y^{2i-1}}, \frac{xy^{i-1}}{z^{k-i+1}}] = \mathbb{C}[\sigma_{2i}^{\vee} \cap M]$. Thus the assertion is proved. □

Remark 3.7. Note that $\Gamma_i^{\triangle}$ and $\Gamma_i^{\triangledown}$ are Nakamura G-graphs.

3.3. *G*-constellations and representations of the McKay quiver

It is well-known that the language of G-constellations is the same as the language of the McKay quiver representations with relations. In this section, we briefly review the McKay quiver representations for our group G of type $\frac{1}{2k+1}(k+1,1,2k)$ with $r=2k+1$.

The vertex set of the McKay quiver for G is in one-to-one correspondence with $G^\vee$. We denote ρ_i the weight of y^i for each $0 \le i \le 2k$. The quiver has $3(2k+1)$ arrows as follows. For each $0 \le i \le 2k$, there are three arrows x_i, y_i, z_i which are arrows from ρ_i to ρ_{i+k+1}, ρ_{i+1}, ρ_{i-1}, respectively.

We impose the following commutation relations:

$$\begin{cases} x_i y_{i+k+1} - y_i x_{i+1}, \\ x_i z_{i+k+1} - z_i x_{i-1}, \\ y_i z_{i+1} - z_i y_{i-1}. \end{cases} \tag{3.8}$$

Note that by definition we are only interested in the representations of dimension vector (1^{2k+1}). After fixing a basis on the vector spaces attached to vertices, the McKay quiver representations are in one-to-one correspondence with points of the affine scheme

$$\mathrm{Rep}_G := \mathrm{Spec}\,\mathbb{C}[x_0,\dots,x_{2k},y_0,\dots,y_{2k},z_0,\dots,z_{2k}]/I_G,$$

where I_G is the ideal generated by the commutation relations (3.8).

The torus $\mathbf{T} = (\mathbb{C}^\times)^3$ acts on Rep_G by

$$(t_1,t_2,t_3)\cdot(x_i,y_i,z_i) = (t_1x_i, t_2y_i, t_3z_i).$$

This action corresponds to the action on G-constellations. There is a torus $T = (\mathbb{C}^*)^{3r}/\mathbb{C}^*$ acting on Rep_G as change of basis on quiver representations. Using this data, the GIT yields the moduli space $\overline{\mathcal{M}_\theta}$ of θ-semistable G-constellations as $\mathcal{M}_\theta \simeq \mathrm{Rep}_G /\!\!/_\theta T$.

Proposition 3.9. *Let G be the group of type $\frac{1}{2k+1}(k+1,1,2k)$ and θ generic parameter. Then each component of $\mathcal{M}_\theta$ has a $\mathbf{T}$-fixed point.*

Proof. Let G be the group of type $\frac{1}{2k+1}(k+1,1,2k)$ and θ generic parameter. By GIT construction in [8], the moduli space $\mathcal{M}_\theta$ is projective over $\overline{\mathcal{M}_0}$, where $\overline{\mathcal{M}_0}$ is the moduli space of 0-semistable objects with $0=(0,0,\dots,0)$ the trivial parameter in Θ. In general, $\mathbb{C}^3/G$ is an irreducible component of $\overline{\mathcal{M}_0}$. In Appendix of [5], it is proved that $\overline{\mathcal{M}_0}$ is irreducible. Thus we have a $\mathbf{T}$-equivariant projective morphism

$$\pi\colon \mathcal{M}_\theta \to \overline{\mathcal{M}_0} \simeq \mathbb{C}^3/G, \quad \mathcal{F} \mapsto [\operatorname{supp}\mathcal{F}],$$

where $[\operatorname{supp}\mathcal{F}]$ is the G-orbit which $\mathcal{F}$ supports on.

Let $\mathcal{M}$ be an irreducible component of $\mathcal{M}_\theta$. Since Y_θ has a $\mathbf{T}$-fixed point, we may assume $\mathcal{M}$ is not the birational component. This means that $\mathcal{M}$ consists of θ-stable G-constellations supported on the origin. Indeed, if $\mathcal{F}$

supports on a free G-orbit, then it is on the birational component Y_θ (see [6, Proposition 2.23]). By restricting the morphism π to $\mathcal{M}$, we know that $\mathcal{M}$ is projective over a point. Thus $\mathcal{M}$ itself is a projective scheme with $\mathbf{T}$-action. By Borel's fixed point theorem, we get the existence of a $\mathbf{T}$-fixed point on $\mathcal{M}$. □

Remark 3.10. In conclusion, each θ-stable G-prebrick yields a $\mathbf{T}$-fixed point in the moduli space $\mathcal{M}_\theta$. Even though each connected component of $\mathcal{M}_\theta$ has a $\mathbf{T}$-fixed point, in general, it is not clear that each $\mathbf{T}$-fixed point corresponds to a G-prebrick.

4. The irreducibility

Theorem 4.1. *Let G be the group of type $\frac{1}{2k+1}(k+1,1,2k)$. Let θ be in the admissible chamber $\mathfrak{C}$. The moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations is irreducible. Therefore the economic resolution Y of $\mathbb{C}^3/G$ is isomorphic to $\mathcal{M}_\theta$.*

First, a simple calculation shows the following lemma.

Lemma 4.2. *For the group of type $\frac{1}{2k+1}(k+1,1,2k)$, the G-invariant monomials are generated by*

- $1, yz$,
- $y^{2k+1}, xy^k, x^3y^{k-1}, x^5y^{k-2}, \dots, x^{2k-1}y, x^{2k+1}$, *and*
- z^{2k+1}, xz^{k+1}, x^2z.

4.1. Cases $x_0 \neq 0$

First note that G-constellations generated by ρ_0 are all G-clusters. From Corollary 3.5, for $\theta \in \mathfrak{C}$, θ-stable G-constellations with $x_0 \neq 0$ must be G-clusters. Therefore we have a one-to-one correspondence between the set

$$\{\theta\text{-stable } \mathbf{T}\text{-invariant } G\text{-constellations with } x_0 \neq 0\}$$

and the set

$$\{\theta\text{-stable Nakamura } G\text{-graphs containing } x\}.$$

By classifying all Nakamura G-graphs containing x, we show all such G-graphs are in Proposition 3.6.

Lemma 4.3. *Let G be a group of type $\frac{1}{2k+1}(k+1,1,2k)$. If a Nakamura G-graph Γ contains x, then the following hold.*

(i) $yz \notin \Gamma$, $x^2z \notin \Gamma$, $y^{k+1} \notin \Gamma$, $z^k \notin \Gamma$.
(ii) *if* $y \in \Gamma$, *then* $x^2 \notin \Gamma$.

Moreover, suppose that Γ is θ-stable for $\theta \in \mathfrak{C}$. If $z^l \in \Gamma$, then $xz^l \in \Gamma$.

Proof. Since the weight of yz and x^2z is the same as $\mathbf{1}$ and the weight of y^{k+1} and z^k is the same as x, by the definition of G-graphs, (i) follows. Similarly the y and x^2 have the same weight so Γ cannot contain both.

Suppose that a θ-stable Γ contains z^l for $1 \leq l \leq k-1$. In (3.4), the $\varepsilon_{2k+1-2l}$ is

$$\theta_{2k-l+1} + \theta_{k-l+1} - \theta_{k+1} - \theta_0.$$

By stability, there should be a non-zero path from ρ_{k+1} to ρ_{2k+1-l} or ρ_{k+1-l}. Note that Γ contains z^l which is the weight ρ_{2k+1-l}. Since x does not divides z^l, there is no non-zero path from ρ_{k+1} to ρ_{2k+1-l}. Therefore there is a non-zero path from ρ_{k+1} to ρ_{k+1-l}.

Assume that the genuine monomial $x^\alpha y^\beta z^\gamma$ gives the non-zero path. Then $x^{\alpha+1} y^\beta z^\gamma \in \Gamma$. Since $yz \notin \Gamma$, either β or γ is zero. The weight of $x^{\alpha+1} y^\beta z^\gamma \in \Gamma$ is ρ_{k+1-l} so if $\gamma = 0$, then $\beta = 2k+1-l \geq k+1$. This contradicts to $y^{k+1} \notin \Gamma$. Thus we can conclude that

$$\alpha = 0, \ \beta = 0, \ \gamma = l. \qquad \square$$

Proposition 4.4. *With the notation above, if a Nakamura G-graph Γ containing x is θ-stable, then Γ is one of the $\Gamma_i^{\triangle}$ and $\Gamma_i^{\triangledown}$ in Proposition 3.6.*

Proof. Let Γ be a θ-stable G-graph containing x. There exists $1 \leq i \leq k$ such that $1, z, z^2, \ldots, z^{i-1} \in \Gamma$ but $z^i \notin \Gamma$. By Lemma 4.3, this induces that

$$x, xz, xz^2, \ldots, xz^{i-1} \in \Gamma, \text{ and } xz^i \notin \Gamma.$$

We have two cases: (i) $y \in \Gamma$, (ii) $y \notin \Gamma$.

Case (i). In this case, by Lemma 4.3, $x^2 \notin \Gamma$. Since $xz^{i-1} \in \Gamma$ is of weight $\rho_{k+1-i+1}$, the monomial $y^{k+1-i+1}$ of the same weight cannot be in Γ. Since we need $(2k+1)$ monomials, only possible case is:

$$\Gamma = \left\{ \begin{matrix} y^{k-i+1} & \\ y^{k-i}, & xy^{k-i} \\ \ldots, & \ldots \\ y, & xy \\ 1, & x \\ z, & xz \\ \ldots, & \ldots \\ z^{i-1}, & xz^{i-1} \end{matrix} \right\}.$$

This is $\Gamma_i^{\triangle}$ in Proposition 3.6.

Case (ii). In this case, Γ consists of monomials in x, z. We need $(2k+1)$ monomials, but $z^i \notin \Gamma$ and $x^2 z \notin \Gamma$. By the definition of G-graphs, there is only one choice:

$$\Gamma = \left\{ \begin{matrix} 1, & x, & x^2, & x^3 & \ldots, & x^{2k-2i+1}, & x^{2k-2i+2} \\ z, & xz & & & & & \\ \ldots, & \ldots & & & & & \\ z^{i-1}, & xz^{i-1} & & & & & \end{matrix} \right\}.$$

This is equal to $\Gamma_i^{\bigtriangledown}$ in Proposition 3.6. □

4.2. Cases $x_0 = 0$

It is known that the moduli spaces of θ-stable G-constellations for $\frac{1}{2k+1}(1,2k)$ is irreducible if θ is generic (see e.g. [1,2]). Thus it is enough to show that the condition $x_0 = 0$ implies $x_i = 0$ for all i.

From Proposition 3.3, recall that for the permutation

$$\omega = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & 2k-2 & 2k-1 & 2k \\ 0 & 1 & k+1 & 2 & k+2 & \cdots & 2k-1 & k & 2k \end{pmatrix},$$

the admissible chamber $\mathfrak{C}$ is the open cone generated by

$$\varepsilon_i := \sum_{j=0}^{i-1} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right)$$

for $i = 1, \ldots, 2k$ (see (3.4)).

In this section, we mainly use the language of the McKay quiver representations in Section 3.3.

Rules of the Game.

• Since G-invariant monomials act trivially on $\mathbf{T}$-invariant G-constellations, any path induced by a G-invariant monomial except $\mathbf{1}$ is zero. In particular, the action of yz is zero. This means in terms of the McKay quiver representations, if y_i is non-zero, then z_{i+1} is zero.

• If a path induced by a monomial $x^\alpha y^\beta z^\gamma$ from ρ_i is non-zero, then so are any path induced by $x^\alpha y^\beta z^\gamma$ from ρ_i by the commutative relation (3.8). For example, suppose that xy induces a non-zero path from ρ_0. This means $x_0 y_{k+1}$ is non-zero. From the commutative relation, $y_0 x_1$ is non-zero as well.

• The commutative relation can be used to show a linear map is zero. For example, if $x_0 = 0$ and $y_0 \neq 0$, then from $x_0 y_{k+1} = y_0 x_1$ we have $x_1 = 0$.

• Suppose that there is a nonzero path from ρ_i to ρ_j which is induced by $x^\alpha y^\beta z^\gamma$. If $x_i = 0$, which is a linear map from ρ_i, then $\alpha = 0$. If y_{j-1}, which is to ρ_j, then $\beta = 0$.

Remark 4.5. Note that the admissible chamber $\mathfrak{C}$ is a chamber in the GIT parameter space Θ. This means that θ-stable objects are the same for any $\theta \in \mathfrak{C}$. Since the admissible chamber $\mathfrak{C}$ is the open cone generated by ε_i's, i.e., $\mathfrak{C} = \{a_1\varepsilon_1 + \cdots + a_{2k}\varepsilon_{2k} \mid a_i \in \mathbb{R}_{>0}\}$, we conclude that it is enough to consider the stability with respect to ε_j for all j. Indeed, for each j, we may consider $\theta = \sum a_i\varepsilon_i$ with $a_i = 1$ for $i \neq j$ and $a_j \gg 0$, then θ is equivalent to ε_j.

Let $0 \leq j \leq 2k+1$ be the smallest number such that the linear map $y_{\omega(j)}$ is zero. We show that there is a unique $\mathbf{T}$-invariant θ-stable G-constellation for each j.

4.2.1. $\boldsymbol{j = 0}$. This means $y_0 = 0$. The first vector ε_1 is equal to

$$\theta_1 - \theta_0.$$

By stability, there should be a non-zero path from ρ_0 to ρ_1. Since $x_0 = y_0 = 0$, the path should be induced by z^{2k}. Here the path induced by z^{2k} is the linear map from ρ_0 to ρ_1 given by

$$z_0 z_{2k} z_{2k-1} \cdots z_3 z_2$$

which is non-zero. From Rule, $x_i = y_i = 0$ for each i. This corresponds to the G-prebrick Γ_1 in Proposition 3.6.

4.2.2. $\boldsymbol{j = 1}$. This means $y_1 = 0$ and y_0 is non-zero. The second vector ε_2 is

$$\theta_2 - \theta_0.$$

By stability, there should be a non-zero path from ρ_0 to ρ_2. Suppose that the path is given by $x^\alpha y^\beta z^\gamma$. Since $x_0 = 0$ and $y_1 = 0$, we have $\alpha = 0$ and $\beta \leq 1$. Thus $\gamma \geq 1$ so $\beta = 0$. Therefore the path is given by z^{2k-1} which induces a non-zero linear map

$$z_0 z_{2k} z_{2k-1} \cdots z_3.$$

This corresponds to Γ_2 in Proposition 3.6.

4.2.3. $\boldsymbol{j = 2}$. This means y_0, y_1 are non-zero and y_{k+1} is zero. The 3rd ray ε_3 is

$$\theta_{k+2} - \theta_{k+1} + \theta_2 - \theta_0.$$

Here we have $x_1 = x_2 = z_1 = z_2 = 0$.

Suppose that there is a non-zero path from ρ_{k+1} to ρ_{k+2}. This path should be given by a monomial of weight ρ_1. Since y_{k+1} is zero, the candidates are

$$x^2, xz^k, z^{2k}.$$

However, as we have $x_1 = z_1 = 0$, x^2 and xz^k cannot induce non-zero paths from ρ_{k+1}. Thus the non-zero path is induced by z^{2k} which is equal to

$$z_{k+1} z_k z_{k-1} \cdots z_1 z_0 \cdots z_{k+3}.$$

But it contradicts to $z_1 = 0$.

By stability with the discussion above, there should be two non-zero paths $\mathbf{p}$ from ρ_0 to ρ_{k+2} and $\mathbf{q}$ from ρ_{k+1} to ρ_2, respectively. Since we have $x_0 = y_{k+1} = 0$, the path $\mathbf{p}$ should be given by z^{k-1}. Since $y_{k+1} = 0 = x_1 = 0$, the path $\mathbf{q}$ is given by z^{k-1}. This corresponds to Γ_3 in Proposition 3.6.

4.2.4. $j = 2l + 1$ for $0 < l < k$. Note that $\omega(j) = l + 1$. The condition $j = 2l + 1$ means that

$$y_0, y_1, y_2, \ldots, y_l \neq 0, \quad y_{k+1}, y_{k+2}, \ldots, y_{k+l} \neq 0, \quad y_{l+1} = 0.$$

This implies that many linear maps are zero, for example

$$x_0 = x_1 = x_2 = \cdots = x_l = x_{l+1} = z_{k+2} = z_{k+3} = \cdots = z_{k+l+1} = 0.$$

The $(j+1)$-th vector ε_{j+1} is

$$\theta_{k+l+1} + \theta_{l+2} - \theta_{k+1} - \theta_0.$$

Suppose that there is a non-zero path from ρ_0 to ρ_{l+2}. Since $y_{l+1} = x_0 = 0$, we have the path is given by z^{2k-l-1}, which is equal to

$$z_0 z_{2k} z_{2k-1} \cdots z_{k+2} z_{k+1} \cdots z_{l+3}.$$

It contradicts to $z_{k+2} = 0$.

By stability, we have two non-zero paths $\mathbf{p}$ from ρ_0 to ρ_{k+1+l} and $\mathbf{q}$ from ρ_{k+1} to ρ_{l+2}, respectively. Since x_0 is zero, the path $\mathbf{p}$ is given by y^{k+1+l} or z^{k-l}. From the fact $y_{l+1} = 0$, we have $\mathbf{p}$ is given by z^{k-l}. Assume that the path $\mathbf{q}$ is given by $x^\alpha y^\beta z^\gamma$. Since $y_{l+1} = 0$ and $x_{k+1}x_1 = 0$, we have $\beta = 0$ and $\alpha \leq 1$. If $\alpha = 1$, then from $x_{k+1}z_1 = 0$, we have $\gamma = 0$. This means $\mathbf{q}$ is given by x, which cannot reach ρ_{l+2}. From this, we have the path $\mathbf{q}$ is given by z^{k-l-1}. This corresponds to Γ_{j+1} in Proposition 3.6.

4.2.5. $j = 2l$ for $0 < l < k$. Note that $\omega(j) = k + l$. The condition $j = 2l$ means that

$$y_0, y_1, y_2, \ldots, y_l \neq 0, \quad y_{k+1}, y_{k+2}, \ldots, y_{k+l-1} \neq 0, \quad y_{k+l} = 0.$$

This implies that many linear maps are zero, for example

$$x_0 = x_1 = x_2 = \cdots = x_l = x_{l+1} = z_{k+2} = z_{k+3} = \cdots = z_{k+l} = 0.$$

The $(j+1)$-th vector ε_{j+1} is

$$\theta_{k+l+1} + \theta_{l+1} - \theta_{k+1} - \theta_0.$$

Suppose that there is a non-zero path induced by $x^{m_1} y^{m_2} z^{m_3}$ from ρ_{k+1} to ρ_{k+l+1}. Since both y_{k+l} and x_1 are zero, we have $m_2 = 0$ and $m_1 \leq 1$. From $x_{k+1}z_1 = 0$, if $m_1 = 1$, then $m_3 = 0$. This means the path is given by x, which is to ρ_1. Thus we have $m_1 = 0$ and the path is given by z^{2k-l}, which is equal to

$$z_{k+1} z_k z_{k-1} \cdots z_2 z_1 z_0 \cdots z_{k+l+2}.$$

This should be zero because $z_1 = 0$.

By stability, there exist two non-zero paths $\mathbf{p}$ from ρ_0 to ρ_{k+l+1} and $\mathbf{q}$ from ρ_{k+1} to ρ_{l+1}, respectively. First note that the path $\mathbf{p}$ should be induced by z^{k-l} because $x_0 = y_{k+l} = 0$. From this, we can conclude that $x_{k+l+1} = 0$; otherwise the monomial xz^{k-l} induces a non-zero path from ρ_0 to ρ_{l+1}, which contradicts to $x_0 = 0$. Since $x_{k+l+1} = 0$, we have that $\mathbf{q}$ is induced by y^{k+l+1}

or z^{k-l}. From the fact $y_{k+l} = 0$, we get $\mathbf{q}$ is given by z^{k-l}. This corresponds to Γ_{j+1} in Proposition 3.6.

4.2.6. $j = 2k$. This means that all y_i's are non-zero except y_{2k}. This shows that all $x_i = z_i = 0$ for all i. This corresponds to Γ_{2k+1} in Proposition 3.6.

4.3. Conclusion

Through this section, we have seen that $\mathbf{T}$-invariant θ-stable G-constellations are all listed in Proposition 3.6. In other words, $\mathbf{T}$-invariant θ-stable G-constellations lie over the birational component Y_θ. In Proposition 4.10 in [6], it is shown that the birational component is a connected component. Therefore we can conclude that $Y_\theta = \mathcal{M}_\theta$, which means the moduli space $\mathcal{M}_\theta$ is irreducible. This proves Theorem 4.1.

Remark 4.6. Theorem 4.1 was first stated in [5] without rigorous proof. Note that without Proposition 3.9, the irreducibility of $\mathcal{M}_\theta$ does not follow from the classification of $\mathbf{T}$-invariant θ-stable G-constellations.

References

[1] T. Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554. https://doi.org/10.1090/S0894-0347-01-00368-X

[2] A. Craw and A. Ishii, *Flops of G-Hilb and equivalences of derived categories by variation of GIT quotient*, Duke Math. J. **124** (2004), no. 2, 259–307. https://doi.org/10.1215/S0012-7094-04-12422-4

[3] A. Craw, D. Maclagan, and R. R. Thomas, *Moduli of McKay quiver representations. I. The coherent component*, Proc. Lond. Math. Soc. (3) **95** (2007), no. 1, 179–198. https://doi.org/10.1112/plms/pdm009

[4] V. I. Danilov, *Birational geometry of three-dimensional toric varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 971–982, 1135.

[5] S.-J. Jung, *McKay quivers and terminal quotient singularities in dimension* 3, PhD thesis, University of Warwick, 2014.

[6] S.-J. Jung, *Terminal quotient singularities in dimension three via variation of GIT*, J. Algebra **468** (2016), 354–394. https://doi.org/10.1016/j.jalgebra.2016.08.032

[7] O. Kędzierski, *Danilov's resolution and representations of the McKay quiver*, Tohoku Math. J. (2) **66** (2014), no. 3, 355–375. https://doi.org/10.2748/tmj/1412783203

[8] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530. https://doi.org/10.1093/qmath/45.4.515

[9] I. Nakamura, *Hilbert schemes of abelian group orbits*, J. Algebraic Geom. **10** (2001), no. 4, 757–779.

[10] M. Reid, *Young person's guide to canonical singularities*, in Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

Seung-Jo Jung
Department of Mathematics Education, and
Institute of Pure and Applied Mathematics
Jeonbuk National University
Jeonju 54896, Korea
Email address: seungjo@jbnu.ac.kr