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DUALLY FLAT AND PROJECTIVELY FLAT FINSLER WARPED PRODUCT STRUCTURES

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Abstract. In this paper, we study the Finsler warped product metric which is dually flat or projectively flat. The local structures of these metrics are completely determined. Some examples are presented.

1. Introduction

The warped product metric was first introduced by Bishop-O'Neil as a generalization of the direct product of Riemannian metrics [1]. This concept was later generalised to the Finsler realm by the work of Kozma-Peter-Varga and Chen-Shen-Zhao [2, 8]. These metrics are called Finsler warped product metrics.

Chen-Shen-Zhao studied the warped product structures, gave the formulae of flag curvature and Ricci curvature of these metrics, obtained the characterization of such metrics to be Einstein, and constructed some new Einstein metrics by modifying spherically symmetric Finsler metrics with constant flag curvature [2]. Recently, Liu-Mo obtained the differential equations that characterize Finsler warped product metrics with vanishing Douglas curvature. By solving the equations, they obtained the complete list of warped product Douglas metrics [11]. In [13], Liu-Mo-Zhang simplified the equations of Finsler warped product metrics with constant flag curvature and improved Chen-Shen-Zhao's results on Einstein Finsler warped product metrics. As an application, they constructed new warped product Douglas metrics of constant Ricci curvature. For non-Riemannian quantities, Yang-Zhang [17] studied such metrics with relatively isotropic Landsberg curvature. It was also worth to point out that the warped product complex Finsler manifold was studied in [5, 6].

Besides the curvatures, there are two important concepts of flatness in Finsler geometry, one is the dual flatness and the other is the projective flatness. A Finsler metric $F = F(x, y)$ on an open domain $\mathbb{U} \subset \mathbb{R}^n$ is *dually flat* if

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and only if

$$[F^2]_{x^i y^j} y^i = 2[F^2]_{x^j}, \tag{1.1}$$

while it is *projectively flat* if and only if

$$F_{x^i y^j} y^i = F_{x^j}, \tag{1.2}$$

where $x = (x^1, \ldots, x^n) \in \mathbb{U}$ and $y = y^j \frac{\partial}{\partial x^j}|_x \in T_x\mathbb{U}$.

The notion of dual flatness was first introduced by Amari-Nagaoka when they studied the information geometry on Riemannian spaces. Later on, Shen extended the notion of dual flatness to Finsler spaces and introduced the equation (1.1) in [15]. In [16], Xia gave a characterization of dually flat (α, β)-metrics. Li found a method to construct dually flat Finsler metrics in [9]. Liu-Mo explicitly constructed all dually flat Randers metrics by using bijection between Randers metrics and their navigation representation. Wang-Liu-Li gave the equivalent conditions for those metrics to be dually flat and got a group of new dually flat metrics. Some new examples of dually flat spherically symmetric Finsler metrics were constructed by Huang-Mo in [7]. Yu gave properties of dual flatness under some suitable conditions. Many non-trivial explicit examples were constructed by a new kind of deformation technique. Moreover, the relationship of dual flatness and projective flatness of such metrics were shown by Yu in [18].

To study and characterize locally projectively flat Finsler metrics is in fact the Hibert's 4th problem in the regular case. Hamel obtained the equation (1.2) in 1903. Beltrami theorem shows that a Riemannian metric is locally projectively flat if and only if it has constant sectional curvature. The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. It is known that every locally projectively flat Finsler metric is of scalar curvature. Shen discussed the classification problem on locally projective flat Finsler metrics of constant flag curvature [14]. Li proved that the locally projectively flat Finsler metrics with constant flag curvature K are totally determined by their behaviors at the origin by solving some PDEs. The classifications when $K = 0, K = 1, K = -1$ are given in an algebraic way [10]. Cheng-Shen characterized locally projectively flat Finsler metrics with almost isotropic S-curvature [3]. Lately, by solving the equivalent PDEs, Cretu constructed a new group of projectively flat Finsler metric in [4]. Recently, Liu-Mo studied locally projectively flat Finsler metrics of constant flag curvature. They found equations that characterize these metrics by warped product. Using the obtained equations, they manufactured new locally projectively flat Finsler warped product metrics with vanishing flag curvature [12].

In this paper, we will study dually flat or projectively flat Finsler warped product metrics.

Consider the product domain $M := I \times \check{M}$, where I is an interval of $\mathbb{R}$ and $\check{M} \subset \mathbb{R}^{n-1}$ is an $(n-1)$-dimensional domain equipped with a Riemannian metric $\check{\alpha}$. A point $u \in M$ shall be denoted by $(u^1, \check{u})$ where $u^1 \in I$ and $\check{u} \in \check{M}$, and a vector $v \in T_u M$ can be decomposed into $v = v^1 \frac{\partial}{\partial u^1} + \check{v}$, where $\check{v} \in T_{\check{u}} \check{M}$. Finsler metrics on M, given in the form

$$F(u,v) = \check{\alpha}(\check{u},\check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u},\check{v})}\right), \tag{1.3}$$

where ϕ is a suitable function defined on a domain of $\mathbb{R}^2$, are called *Finsler warped product metrics* [2]. We shall consider the dually flat or projectively flat warped product metrics in the domain $M := I \times \check{M} \subset \mathbb{R} \times \mathbb{R}^{n-1}$. It must be pointed out that as we say F is dually flat or projectively flat in this paper, we means it is dually flat or projectively flat in the natural coordinate of $\mathbb{R} \times \mathbb{R}^{n-1}$.

Theorem 1.1. *On the n-dimensional product manifold $M = I \times \check{M}$ with $n \geq 3$, a Finsler warped product metric $F(u,v) = \check{\alpha}(\check{u},\check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u},\check{v})}\right)$ is dually flat if and only if*

(i) *$\check{\alpha}$ is dually flat and $\phi^2(r,s) = f(r)s^2 + const.$, where $f(r)$ is an arbitrary positive differentiable function such that F is a regular Finsler metric.*
(ii) *$\check{\alpha}$ is Euclidean and $\phi^2(r,s) = f(r)s^2 + g(s)$, where $f(r) > 0$ and $g(s)$ are arbitrary differentiable functions such that F is a regular Finsler metric.*

Remark 1. The trivial case "$\check{\alpha}$ is Euclidean and g is constant" is the intersection of (i) and (ii).

Remark 2. To make the Finsler metrics regular, the functions appearing in the above Theorem must satisfy the following conditions:

$$\omega = f(r)s^2 + g(s) > 0,$$

$$2g(s) - sg'(s) > 0,$$

$$2f(r)(s^2 g''(s) - 2sg'(s) + 2g(s)) + 2g(s)g''(s) - g'(s)^2 > 0.$$

See the proof of Theorems 1.1 for details.

Theorem 1.2. *On the n-dimensional product manifold $M = I \times \check{M}$ with $n \geq 3$, a Finsler warped product metric $F(u,v) = \check{\alpha}(\check{u},\check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u},\check{v})}\right)$ is projectively flat if and only if $\check{\alpha}$ is Euclidean and $\phi(r,s) = f(r)s + g(s)$, where $f(r)$ is an arbitrary positive differentiable function and $g(s)$ is a non-constant function such that F is a regular Finsler metric.*

Remark 3. To make the Finsler metrics regular, the function appearing in the above theorem must satisfy the followings:

$$f(r)s + g(s) > 0,\ g(s) - sg'(s) > 0,\ g''(s) > 0.$$

See the proof of Theorems 1.2 for details.

2. Fundamental lemmas

Throughout this paper, we shall use the abbreviations $\omega = \phi^2$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$, and hence we can write

$$F = \breve{\alpha}\phi(r,s), \quad F^2 = \breve{\alpha}^2\omega(r,s).$$

The index conventions are as follows

$$1 \le A, B, C, \ldots \le n, \quad 2 \le i, j, k, \ldots \le n.$$

Let us recall the strong convexity of the Finsler warped product metric, which will be used to discuss strong convexity of metrics appearing in the proofs of Theorem 1.1 and Theorem 1.2.

Let M be an n-dimensional smooth manifold, and $F : TM \to [0, +\infty]$ be a non-negative function on its tangent space. If F satisfies the following conditions:

(i) Regularity: F is C^∞ on the entire slit tangent bundle $TM\backslash\{0\}$.

(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

(iii) Strong convexity: for non-zero vectors y,

$$g_{ij}(x,y) = \frac{1}{2}[F^2]_{y^i y^j}$$

constitute positive definite matrix.

Lemma 2.1 ([2]). *The Finsler warped product metric $F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi(r,s)$ is strongly convex if and only if ϕ or ω satisfies*

$$2\omega - s\omega_s > 0, \ 2\omega\omega_{ss} - \omega_s^2 > 0,$$

or equivalently,

$$\phi - s\phi_s > 0, \ \phi_{ss} > 0.$$

The following lemma is fundamental in the proofs.

Lemma 2.2. *Suppose that $\breve{\alpha}^2\sigma = \breve{\alpha}_{\breve{u}^i}\breve{v}^i\psi$, where $\sigma = \sigma(r,s)$ and $\psi = \psi(r,s)$ are any differentiable functions. Then $\breve{\alpha}^2\sigma_s = \breve{\alpha}_{\breve{u}^i}\breve{v}^i\psi_s$.*

Proof. Differentiating $\breve{\alpha}^2\sigma = \breve{\alpha}_{\breve{u}^i}\breve{v}^i\psi$ with respect to the variable $\breve{v}^j$ yields

$$[\breve{\alpha}^2]_{\breve{v}^j}\sigma - \breve{\alpha}\breve{\alpha}_{\breve{v}^j}s\sigma_s = (\breve{\alpha}_{\breve{u}^i\breve{v}^j}\breve{v}^i + \breve{\alpha}_{\breve{u}^j})\psi - \frac{1}{\breve{\alpha}}\breve{\alpha}_{\breve{u}^i}\breve{v}^i\breve{\alpha}_{\breve{v}^j}s\psi_s. \tag{2.1}$$

Contracting (2.1) with $\breve{v}^j$ yields

$$2\breve{\alpha}^2\sigma - \breve{\alpha}^2 s\sigma_s = 2\breve{\alpha}_{\breve{u}^i}\breve{v}^i\psi - \breve{\alpha}_{\breve{u}^i}\breve{v}^i s\psi_s.$$

Plugging $\breve{\alpha}^2\sigma = \breve{\alpha}_{\breve{u}^i}\breve{v}^i\psi$ into the above equation, we have

$$\breve{\alpha}^2 s\sigma_s = \breve{\alpha}_{\breve{u}^i}\breve{v}^i s\psi_s,$$

which means $\breve{\alpha}^2\sigma_s = \breve{\alpha}_{\breve{u}^i}\breve{v}^i\psi_s$. □

By a direct calculation, we have the following results.

Lemma 2.3. *The Finsler warped product metric* (1.3) *satisfies*

$$
\begin{aligned}
&[F^2]_{rv^1} = \breve{\alpha}\omega_{rs},\\
&[F^2]_{\breve{u}^i v^1} = \frac{1}{2\breve{\alpha}}[\breve{\alpha}^2]_{\breve{u}^i}(\omega_s - s\omega_{ss}),\\
&[F^2]_{r\breve{v}^i} = [\breve{\alpha}^2]_{\breve{v}^i}(\omega_r - \frac{1}{2}s\omega_{rs}),\\
&[F^2]_{\breve{u}^j\breve{v}^i} = [\breve{\alpha}^2]_{\breve{u}^j\breve{v}^i}(\omega - \frac{1}{2}s\omega_s) - \frac{1}{2\breve{\alpha}}\breve{\alpha}_{\breve{v}^i}[\breve{\alpha}^2]_{\breve{u}^j}s(\omega_s - s\omega_{ss}),
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
&F_{rv^1} = \phi_{rs},\\
&F_{\breve{u}^i v^1} = -\frac{1}{\breve{\alpha}}\breve{\alpha}_{\breve{u}^i}s\phi_{ss},\\
&F_{r\breve{v}^i} = \breve{\alpha}_{\breve{v}^i}(\phi_r - s\phi_{rs}),\\
&F_{\breve{u}^j\breve{v}^i} = \breve{\alpha}_{\breve{u}^j\breve{v}^i}(\phi - s\phi_s) + \frac{1}{\breve{\alpha}}\breve{\alpha}_{\breve{u}^j}\breve{\alpha}_{\breve{v}^i}s^2\phi_{ss}.
\end{aligned}
\tag{2.3}
$$

3. The dually flat Finsler warped product metrics

In this section, we first characterize dually flat Finsler warped product metric.

Lemma 3.1. *On the n-dimensional product manifold $M = I \times \breve{M}$ with $n \geq 3$, a Finsler warped product metric $F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\left(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\right)$ is dually flat if and only if $\breve{\alpha}$ is dually flat and ω satisfies*

$$
\breve{\alpha}^2(2\omega_r - s\omega_{rs}) = \breve{\alpha}_{\breve{u}^i}\breve{v}^i(\omega_s - s\omega_{ss}).
\tag{3.1}
$$

Proof. A Finsler warped product metric is dually flat if and only if it satisfies

$$
[F^2]_{u^A v^B}v^A - 2[F^2]_{u^B} = 0,
$$

which can be written as

$$
\begin{cases}
[F^2]_{rv^1}v^1 + [F^2]_{\breve{u}^i v^1}\breve{v}^i - 2[F^2]_r = 0,\\
[F^2]_{r\breve{v}^i}v^1 + [F^2]_{\breve{u}^j\breve{v}^i}\breve{v}^j - 2[F^2]_{\breve{u}^i} = 0.
\end{cases}
$$

By Lemma 2.3, the above equations can be rewritten as

$$
\begin{aligned}
&\breve{\alpha}^2\breve{\alpha}_{\breve{v}^i}s(2\omega_r - s\omega_{rs}) - \breve{\alpha}_{\breve{u}^j}\breve{v}^j\breve{\alpha}_{\breve{v}^i}s(\omega_s - s\omega_{ss})\\
&= \frac{1}{2}(2[\breve{\alpha}^2]_{\breve{u}^i} - [\breve{\alpha}^2]_{\breve{u}^j\breve{v}^i}\breve{v}^j)(2\omega - s\omega_s)
\end{aligned}
\tag{3.2}
$$

and

$$
\breve{\alpha}^2(2\omega_r - s\omega_{rs}) = \breve{\alpha}_{\breve{u}^i}\breve{v}^i(\omega_s - s\omega_{ss}).
\tag{3.3}
$$

Plugging (3.3) into (3.2) yields

$$
([\breve{\alpha}^2]_{\breve{u}^j\breve{v}^i}\breve{v}^j - 2[\breve{\alpha}^2]_{\breve{u}^i})(2\omega - s\omega_s) = 0.
\tag{3.4}
$$

If $2\omega - s\omega_s = 0$, we have

$$\omega = f(r)s^2,$$

where $f(r)$ is an arbitrary function. Thus $F = \breve{\alpha}\sqrt{\omega} = \sqrt{f(r)}|v^1|$, which is impossible since the dimension $n > 1$. Hence $2\omega - s\omega_s \neq 0$. Then by (3.4) we have

$$[\breve{\alpha}^2]_{\breve{u}^j \breve{v}^i} \breve{v}^j = 2[\breve{\alpha}^2]_{\breve{u}^i}, \tag{3.5}$$

which means $\breve{\alpha}$ is dually flat.

Conversely, if $\breve{\alpha}$ is dually flat and (3.1) holds, it is easy to see that F is dually flat according to the proof. □

Lemma 3.2. *A Riemannian metric on the domain $\mathbb{U}$ is both projectively flat and dually flat if and only if it is Euclidean.*

Proof. It is known that a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is dually flat if and only if it can be expressed as

$$a_{ij}(x) = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x), \tag{3.6}$$

where $\psi = \psi(x)$ is a scalar function [15].

By a direct calculation, we have

$$\alpha_{x^m} = \frac{1}{2\alpha}\frac{\partial a_{ij}}{\partial x^m} y^i y^j \tag{3.7}$$

and

$$\alpha_{x^m y^k} = \frac{1}{\alpha}\frac{\partial a_{kj}}{\partial x^m} y^j - \frac{1}{2\alpha^3} a_{kl} \frac{\partial a_{ij}}{\partial x^m} y^i y^j y^l. \tag{3.8}$$

For α is projectively flat, by (1.2) and (3.6)-(3.8), we obtain

$$\alpha^2 \frac{\partial a_{ij}}{\partial x^k} y^i y^j = a_{kl} \frac{\partial a_{ij}}{\partial x^m} y^i y^j y^l y^m.$$

Thus, $\frac{\partial a_{ij}}{\partial x^m} y^i y^j y^m$ is divided by α^2, which implies

$$\frac{\partial a_{ij}}{\partial x^k} y^i y^j = \eta a_{km} y^m,$$

where $\eta = \eta_i(x)y^i$ is a 1-form.

Differentiating the above equation with respect to y^j and y^l yields

$$2\frac{\partial a_{jl}}{\partial x^k} = \eta_l a_{kj} + \eta_j a_{kl}. \tag{3.9}$$

Exchanging the index j and k in (3.9), we get

$$2\frac{\partial a_{kl}}{\partial x^j} = \eta_l a_{kj} + \eta_k a_{jl}. \tag{3.10}$$

By (3.6), (3.9) and (3.10), we obtain

$$\eta_k a_{jl} - \eta_j a_{kl} = 0.$$

Contracting the above equation with a^{jl} yields

$$(n-1)\eta_k = 0.$$

Thus $\eta_k = 0$, i.e., $\frac{\partial a_{ij}}{\partial x^k} = 0$. So α is Euclidean. $\square$

Proof of Theorem 1.1. Differentiating (3.1) with respect to the variable $\check{v}^k$ yields

$$\begin{aligned}[\check{\alpha}^2]_{\check{v}^k}(2\omega_r - s\omega_{rs}) - \check{\alpha}\check{\alpha}_{\check{v}^k}s(\omega_{rs} - s\omega_{rss}) \\ = (\check{\alpha}_{\check{u}^i\check{v}^k}\check{v}^i + \check{\alpha}_{\check{u}^k})(\omega_s - s\omega_{ss}) + \check{\alpha}^{-1}\check{\alpha}_{\check{u}^i}\check{v}^i\check{\alpha}_{\check{v}^k}s^2\omega_{sss}.\end{aligned} \tag{3.11}$$

Contracting (3.11) with $\check{v}^k$ yields

$$2\check{\alpha}^2(2\omega_r - s\omega_{rs}) - \check{\alpha}^2 s(\omega_{rs} - s\omega_{rss}) = 2\check{\alpha}_{\check{u}^i}\check{v}^i(\omega_s - s\omega_{ss}) + \check{\alpha}_{\check{u}^i}\check{v}^i s^2\omega_{sss}.$$

Plugging (3.1) into the above equation, we have

$$\check{\alpha}_{\check{u}^i}\check{v}^i s^2\omega_{sss} + \check{\alpha}^2 s(\omega_{rs} - s\omega_{rss}) = 0. \tag{3.12}$$

Substituting (3.12) into (3.11) yields

$$(\check{\alpha}_{\check{u}^i\check{v}^k}\check{v}^i + \check{\alpha}_{\check{u}^k})(\omega_s - s\omega_{ss}) = [\check{\alpha}^2]_{\check{v}^k}(2\omega_r - s\omega_{rs}). \tag{3.13}$$

From (3.1) and (3.13), one can see that

$$\check{\alpha}(\check{\alpha}_{\check{u}^i\check{v}^k}\check{v}^i + \check{\alpha}_{\check{u}^k})(\omega_s - s\omega_{ss}) = 2\check{\alpha}_{\check{v}^k}\check{\alpha}_{\check{u}^i}\check{v}^i(\omega_s - s\omega_{ss}).$$

Case 1. $\omega_s - s\omega_{ss} \neq 0$. Then

$$\check{\alpha}(\check{\alpha}_{\check{u}^i\check{v}^j}\check{v}^i + \check{\alpha}_{\check{u}^j}) = 2\check{\alpha}_{\check{v}^j}\check{\alpha}_{\check{u}^i}\check{v}^i. \tag{3.14}$$

For $\check{\alpha}$ is dually flat, differentiating (3.5) with respect to the variable $\check{v}^j$, we obtain

$$[\check{\alpha}^2]_{\check{u}^k\check{v}^i\check{v}^j}\check{v}^k + [\check{\alpha}^2]_{\check{u}^j\check{v}^i} = 2[\check{\alpha}^2]_{\check{u}^i\check{v}^j}. \tag{3.15}$$

Exchanging the index i and j in (3.15) yields

$$[\check{\alpha}^2]_{\check{u}^k\check{v}^j\check{v}^i}\check{v}^k + [\check{\alpha}^2]_{\check{u}^i\check{v}^j} = 2[\check{\alpha}^2]_{\check{u}^j\check{v}^i}. \tag{3.16}$$

By (3.15) and (3.16), we have

$$[\check{\alpha}^2]_{\check{u}^i\check{v}^j} = [\check{\alpha}^2]_{\check{u}^j\check{v}^i}.$$

Contracting the above equation with $\check{v}^j$ yields

$$\check{\alpha}_{\check{v}^i}\check{\alpha}_{\check{u}^j}\check{v}^j + \check{\alpha}\check{\alpha}_{\check{u}^j\check{v}^i}\check{v}^j = 2\check{\alpha}\check{\alpha}_{\check{u}^i}.$$

Substituting (3.14) into the above equation, we get

$$\check{\alpha}_{\check{u}^j\check{v}^i}\check{v}^j = \check{\alpha}_{\check{u}^i}, \tag{3.17}$$

which means $\check{\alpha}$ is projectively flat. By Lemma 3.1 and Lemma 3.2, $\check{\alpha}$ is a Euclidean metric. Thus by (3.1), we have $2\omega_r - s\omega_{rs} = 0$. Solving it yields

$$\omega = f(r)s^2 + g(s),$$

where $f(r) > 0$ and $g(s)$ are any differentiable functions.

Case 2. $\omega_s - s\omega_{ss} = 0$. By (3.1), we have $2\omega_r - s\omega_{rs} = 0$. Solving them yields

$$\omega = f(r)s^2 + c, \tag{3.18}$$

where c is a constant. Then $F^2 = \breve{\alpha}^2\phi^2 = f(r)v^1v^1 + c\breve{\alpha}^2$.

Conversely, it is easy to check that F is locally dually flat if it satisfies the conditions in Theorem 1.1.

Let us discuss strong convexity of the metrics. For $\omega = f(r)s^2 + g(s)$, we have

$$2\omega - s\omega_s = 2g(s) - sg'(s),$$

$$2\omega\omega_{ss} - \omega_s^2 = 2f(r)(s^2g''(s) - 2sg'(s) + 2g(s)) + 2g(s)g''(s) - g'(s)^2.$$

Hence, by Lemma 2.1, we get that F is a Finsler metric if and only if $\omega = f(r)s^2 + g(s) > 0$, $2g(s) - sg'(s) > 0$ and $2f(r)(s^2g''(s) - 2sg'(s) + 2g(s)) + 2g(s)g''(s) - g'(s)^2 > 0$. $\square$

At the end of this section, let us construct a dually flat Finsler warped product metric by Theorem 1.1.

Example 3.1. Let $\omega = \omega(r, s)$ be a positive function defined by

$$\omega(r, s) = (r^2 + 3)s^2 + \exp(-s \arctan s).$$

We have

$$2\omega - s\omega_s = \exp(-s\arctan s)[2 + s\arctan s + \frac{s^2}{1+s^2}] > 0,$$

and

$$\begin{aligned} & 2\omega\omega_{ss} - \omega_s^2 \\ = {} & 2(r^2+3)\exp(-s\arctan s)[\frac{s^4+1}{(1+s^2)^2} + (-s\arctan s + \frac{-s^2}{1+s^2} - 1)^2] \\ & + \exp(-2s\arctan s)[(-\arctan s - \frac{s}{1+s^2})^2 + \frac{-4}{(1+s^2)^2}] \\ > {} & \exp(-2s\arctan s)\frac{6s^4+2}{(1+s^2)^2} > 0. \end{aligned}$$

By Theorem 1.1, the Finsler warped product metric

$$F = \breve{\alpha}\sqrt{(r^2+3)s^2 + \exp(-s\arctan s)},$$

is dually flat, where $\breve{\alpha}$ is Euclidean.

4. The projectively flat Finsler warped product metrics

In this section, we first characterize projectively flat Finsler warped product metrics by an equivalent equation.

Lemma 4.1. *On the n-dimensional product manifold $M = I \times \check{M}$ with $n \geq 3$, a Finsler warped product metric $F(u,v) = \check{\alpha}(\check{u},\check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u},\check{v})}\right)$ is projectively flat if and only if $\check{\alpha}$ is projectively flat and the following equation holds*

$$\check{\alpha}^2(\phi_r - s\phi_{rs}) + \check{\alpha}_{\check{u}^i}\check{v}^i s\phi_{ss} = 0. \tag{4.1}$$

Proof. A Finsler warped product metric is projectively flat if and only if it satisfies the following equations

$$F_{u^A v^B} v^A = F_{u^B},$$

which can be represented as

$$\begin{cases} F_{rv^1}v^1 + F_{\check{u}^i v^1}\check{v}^i = F_r, \\ F_{r\check{v}^i}v^1 + F_{\check{u}^j\check{v}^i}\check{v}^j = F_{\check{u}^i}. \end{cases}$$

Plugging (2.3) into the above equations yields

$$\check{\alpha}^2(\phi_r - s\phi_{rs}) + \check{\alpha}_{\check{u}^i}\check{v}^i s\phi_{ss} = 0 \tag{4.2}$$

and

$$\check{\alpha}\check{\alpha}_{\check{v}^i}s(\phi_r - s\phi_{rs}) + \check{\alpha}^{-1}\check{\alpha}_{\check{u}^j}\check{v}^j\check{\alpha}_{\check{v}^i}s^2\phi_{ss} = (\check{\alpha}_{\check{u}^i} - \check{\alpha}_{\check{u}^j\check{v}^i}\check{v}^j)(\phi - s\phi_s). \tag{4.3}$$

Substituting (4.2) into (4.3) yields

$$(\check{\alpha}_{\check{u}^i} - \check{\alpha}_{\check{u}^j\check{v}^i}\check{v}^j)(\phi - s\phi_s) = 0. \tag{4.4}$$

Case 1. $\phi - s\phi_s = 0$. We have $\phi = f(r)s$, where $f(r)$ is an any function. So $F = \check{\alpha}\phi = f(r)v^1$, which contradicts to the positivity of F.
Case 2. $\check{\alpha}_{\check{u}^i} = \check{\alpha}_{\check{u}^j\check{v}^i}\check{v}^j$, which means $\check{\alpha}$ is projectively flat.

Conversely, if $\check{\alpha}$ is projectively flat and (4.1) holds, it is easy to see that F is projectively flat. □

Proof of Theorem 1.2. Letting $h = s\phi_s - \phi$, (4.1) can be rewritten as follows

$$\check{\alpha}^2 h_r = \check{\alpha}_{\check{u}^i}\check{v}^i h_s. \tag{4.5}$$

Since $\check{\alpha}$ is projectively flat, differentiating (4.5) with respect to the variable $\check{v}^j$ yields

$$[\check{\alpha}^2]_{\check{v}^j}h_r - \check{\alpha}\check{\alpha}_{\check{v}^j}sh_{rs} = 2\check{\alpha}_{\check{u}^j}h_s - \check{\alpha}^{-1}\check{\alpha}_{\check{u}^i}\check{v}^i\check{\alpha}_{\check{v}^j}sh_{ss}. \tag{4.6}$$

By Lemma 2.2 and (4.5), we have

$$[\check{\alpha}^2]_{\check{v}^j}h_r = 2\check{\alpha}_{\check{u}^j}h_s. \tag{4.7}$$

Together with (4.5), it yields

$$2\check{\alpha}(\check{\alpha}\check{\alpha}_{\check{u}^j} - \check{\alpha}_{\check{v}^j}\check{\alpha}_{\check{u}^i}\check{v}^i)h_r h_s = 0. \tag{4.8}$$

If $h_r h_s \neq 0$, then

$$\check{\alpha}\check{\alpha}_{\check{u}^j} = \check{\alpha}_{\check{v}^j}\check{\alpha}_{\check{u}^i}\check{v}^i.$$

For $\breve{\alpha}$ is projectively flat, we have

$$\breve{\alpha}\breve{\alpha}_{\breve{u}^j} = \breve{\alpha}_{\breve{v}^j}\breve{\alpha}_{\breve{u}^i}\breve{v}^i = \breve{\alpha}\breve{\alpha}_{\breve{u}^i\breve{v}^j}\breve{v}^i.$$

So

$$[\breve{\alpha}^2]_{\breve{u}^i\breve{v}^j}\breve{v}^i = 2[\breve{\alpha}^2]_{\breve{u}^j},$$

which means $\breve{\alpha}$ is dually flat.

By Lemma 3.2, we know that $\breve{\alpha}$ is Euclidean. Thus by (4.5), we have $\breve{\alpha}_{\breve{u}^i} = 0$ and $h_r = 0$, which contradicts to $h_r h_s \neq 0$. Thus $h_r h_s = 0$.

If $h_s = 0$, we have $h_r = 0$ by (4.5). Solving them, we get

$$\phi = f(r)s + c,$$

where $f(r)$ is an arbitrary function and c is a constant. While, Finsler metric $F = \breve{\alpha}\phi = \breve{\alpha}(f(r)s + c)$ is not strongly convex by Lemma 2.1.

Thus $h_s \neq 0$ and $h_r = 0$. By (4.5), we have

$$\breve{\alpha}_{\breve{u}^i}\breve{v}^i = 0.$$

Differentiating it with respect to the variable v^k yields

$$\breve{\alpha}_{\breve{u}^i\breve{v}^k}\breve{v}^i + \breve{\alpha}_{\breve{u}^k} = 0. \tag{4.9}$$

Since $\breve{\alpha}$ is projectively flat, we have $\breve{\alpha}_{\breve{u}^k} = 0$, which means $\breve{\alpha}$ is a Euclidean metric.

Meanwhile, the solution of $h_r = 0$ is

$$\phi = f(r)s + g(s),$$

where $f(r) > 0$ and $g(s)$ are any differentiable functions.

Conversely, it is easy to check that F is projectively flat if it satisfies the conditions in Theorem 1.2.

Finally, let us discuss the strong convexity of the metrics. For $\phi = f(r)s + g(s)$, we have

$$\phi - s\phi_s = g(s) - sg'(s),$$
$$\phi_{ss} = g''(s).$$

Hence, by Lemma 2.1, we get that F is a Finsler metric if and only if $g(s) - sg'(s) > 0$, $g''(s) > 0$ and $\breve{\alpha}$ is Euclidean. □

Example 4.1. Let $g(s)$ be a function defined by

$$g(s) = c_1 s + s\arctan s + c_2,$$

where c_1, c_2 are constants such that $c_2 \geq 1$, we have

$$g(s) - sg'(s) = \frac{(c_2 - 1)s^2 + c_2}{1 + s^2} > 0,\ g''(s) = \frac{2}{(1 + s^2)^2} > 0.$$

Let $\phi = \phi(r, s)$ be a function defined by

$$\phi(r, s) = -c_1 s + g(s).$$

By Theorem 1.2, the Finsler warped product metric

$$F = \breve{\alpha}(s \arctan s + c_2)$$

is projectively flat, where $\breve{\alpha}$ is Euclidean.

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