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ESTIMATE FOR BILINEAR CALDERÓN-ZYGMUND OPERATOR AND ITS COMMUTATOR ON PRODUCT OF VARIABLE EXPONENT SPACES

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Abstract. The goal of this paper is to establish the boundedness of bilinear Calderón-Zygmund operator BT and its commutator $[b_1, b_2, BT]$ which is generated by $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ (or $\dot{\Lambda}_\alpha(\mathbb{R}^n)$) and the BT on generalized variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$. Under assumption that the functions φ_1 and φ_2 satisfy certain conditions, the authors proved that the BT is bounded from product of spaces $\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ into space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$. Furthermore, the boundedness of commutator $[b_1, b_2, BT]$ on spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and on spaces $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is also established.

1. Introduction

It is well known that Calderón-Zygmund operators regard as an important class of integral operators in harmonic analysis, they not only play a key role in the harmonic analysis (see [5, 17, 38, 42, 49]), but also their use is best justified by the variety of applications in which they appear; for example, see [2, 33]. In 1975, Cofiman and Meyer [6] first introduced the theory of multilinear Calderón-Zygmund integral operators. And its theory was further investigated by Grafakos and Torres in [12]. Since then, the properties of multilinear Calderón-Zygmund integral operators on various of function spaces are widely focused. For example, in [3, 4], Chen and Fan obtained the boundedeness of bilinear singular integral operators on product of Lebesgue spaces. Xu in [48] showed that the multilinear Calderón-Zygmund operator is bounded from product of spaces $L^{p_1}(\mu) \times L^{p_2}(\mu)$ into space $L^p(\mu)$. Lin [28] established the boundedness of multilinear Calderón-Zygmund operators on product of BMO spaces, product of LMO spaces and product of λ-central BMO spaces. The

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more researches on the multilinear Calderón-Zygmund integral operators on different kinds of function spaces can be seen [29, 31, 32, 46, 47] and their references therein.

On the other hand, in recent years, the theory of variable exponent function spaces is widely focused. They are important not only in the theory as generalizations of classical function spaces, but also for their applications in the fields of fluid dynamics and PDEs (see [18,24]). In 1931, Orlicz [37] first obtained the definition of variable exponent Lebesgue spaces. Since then, the development of variable exponent Lebesgue spaces becomes very slower. Until 1991, Kováčik and Rákosnik [27] systematically studied the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$. Later, the researches associated with the variable exponent function spaces are widely discussed; for example, see [7, 8, 26, 41, 44]. To research the local behaviour of solutions for the second order elliptic partial differential equations, Morrey in [35] introduced the definition of Morrey spaces. On the basis of this, the various of definitions for generalized Morrey spaces on different kinds of function spaces are established (see [13,34,36]). Moreover, because of the results on Morrey spaces and generalized Morrey spaces are comprehensive, many researchers passed to the variable exponent Morrey spaces and generalized variable exponent Morrey spaces; for example, Almeida *et al.* in [1] obtained the boundedness of Hardy-Littlewood maximal operators and potential operators are bounded on variable Morrey spaces defined over a bounded open set. In 2010, Guliyev *et al.* [15] showed that the Hardy-Littlewood maximal operators, potential operators and singular integral operators are bounded on generalized variable exponent Morrey spaces over bounded domains. The more researches about the integral operators on variable exponent spaces can be seen [9, 11, 14, 16, 19, 20, 30, 39, 40, 43].

Motivated by these results, in this paper, we mainly consider the boundedness of bilinear Calderón-Zygmund operator and its commutator associated with BMO functions on generalized variable exponent Morrey spaces. Before stating the organization of this article, we first recall some definitions and notations.

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function, and set

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) = p_- > 0, \qquad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p_+ < \infty.$$

We denote $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 \le p_- \le p(x) \le p_+ < \infty$, and denote $\mathcal{P}_0(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $0 < p_- \le p(x) \le p_+ < \infty$, $x \in \mathbb{R}^n$.

For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ (see [7]) denotes all real-valued measurable functions f defined on $\mathbb{R}^n$ such that, for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x < \infty.$$

This becomes a Banach function space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \leq 1\right\}. \tag{1.1}$$

Now we recall some classes of variable exponent functions. Let f be a locally integrable function on $\mathbb{R}^n$. Then the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y, \tag{1.2}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on the space $L^{p(\cdot)}(\mathbb{R}^n)$. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the class of globally log-Hölder continuous function $p(\cdot) \in LH(\mathbb{R}^n)$ regarding as an important subset of $\mathcal{B}(\mathbb{R}^n)$, satisfies the following two conditions

$$|p(x) - p(y)| \leq -\frac{C}{\log(|x-y|)}, \quad |x-y| \leq \frac{1}{2},$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}, \quad |y| \geq |x|.$$

Now we recall the definition of bilinear Calderón-Zygmund operator in [45] as follows.

Definition 1.1. A kernel $K(\cdot,\cdot,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\setminus\{(x,x,x) : x \in \mathbb{R}^n\})$ is called a bilinear Calderón-Zygmund kernel if it satisfies the following conditions:

(1) for all $(x, y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with $x \neq y_i$ for $i \in \{1, 2\}$, there exists a positive constant C such that

$$|K(x, y_1, y_2)| \leq \frac{C}{(|x-y_1| + |x-y_2|)^{2n}}; \tag{1.3}$$

(2) there exist constants $\delta > 0$ and $C > 0$ such that, for all $x, x', y_1, y_2 \in \mathbb{R}^n$ with satisfying $|x - x'| \leq \frac{1}{2}\max\{|x-y_1|, |x-y_2|\}$,

$$|K(x, y_1, y_2) - K(x', y_1, y_2)| \leq C\frac{|x-x'|^\delta}{(|x-y_1| + |x-y_2|)^{2n+\delta}}; \tag{1.4}$$

(3) there exist constants $\delta > 0$ and $C > 0$ such that, for all $x, y_1, y_1', y_2 \in \mathbb{R}^n$ with satisfying $|y_1 - y_1'| \leq \frac{1}{2}\max\{|x-y_1|, |x-y_2|\}$,

$$|K(x, y_1, y_2) - K(x, y_1', y_2)| \leq C\frac{|y_1-y_1'|^\delta}{(|x-y_1| + |x-y_2|)^{2n+\delta}}; \tag{1.5}$$

(4) there exist constants $\delta > 0$ and $C > 0$ such that, for all $x, y_1, y_2, y_2' \in \mathbb{R}^n$ with satisfying $|y_2 - y_2'| \leq \frac{1}{2}\max\{|x-y_1|, |x-y_2|\}$,

$$|K(x, y_1, y_2) - K(x, y_1, y_2')| \leq C\frac{|y_2-y_2'|^\delta}{(|x-y_1| + |x-y_2|)^{2n+\delta}}. \tag{1.6}$$

Let $L_c^\infty(\mathbb{R}^n)$ be the space of all $L^\infty(\mathbb{R}^n)$ functions with compact support. A bilinear operator BT is called a bilinear Calderón-Zygmund operator with kernel K satisfying (1.3), (1.4), (1.5) and (1.6) if, for all $f_1, f_2 \in L_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n \setminus \Big(\mathrm{supp}(f_1) \bigcap \mathrm{supp}(f_2)\Big)$,

$$BT(f_1,f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2. \tag{1.7}$$

Given $b_1, b_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the commutator $[b_1, b_2, BT]$ generated by the BT and b_1, b_2

$$\begin{aligned}[b_1,b_2,BT](f_1,f_2)(x) &= b_1(x)b_2(x)BT(f_1,f_2)(x) - b_1(x)BT(f_1,b_2f_2)(x)\\ &\quad - b_2(x)BT(b_1f_1,f_2)(x) + BT(b_1f_1,b_2f_2)(x).\end{aligned} \tag{1.8}$$

Also, the commutators $[b_1, BT]$ and $[b_2, BT]$ are, respectively, defined by

$$[b_1,BT](f_1,f_2)(x) = b_1(x)BT(f_1,f_2)(x) - BT(b_1f_1,f_2)(x), \tag{1.9}$$

$$[b_2,BT](f_1,f_2)(x) = b_2(x)BT(f_1,f_2)(x) - BT(f_1,b_2f_2)(x). \tag{1.10}$$

Next, we need to recall the following inequality introduced in [9], that is, for any $x \in \mathbb{R}^n$ and $p(\cdot) \in LH(\mathbb{R}^n)\bigcap \mathcal{P}(\mathbb{R}^n)$, there exists some positive constant C such that

$$\|\chi_{B(x,r)}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le Cr^{\theta_p(x,r)}, \tag{1.11}$$

where

$$\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \le 1,\\ \frac{n}{p(\infty)}, & r \ge 1.\end{cases}$$

The following definition of generalized variable exponent Morrey space is from [15].

Definition 1.2. Let $p(\cdot) \in \mathcal{P}_1(\mathbb{R}^n)$ and φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Then the generalized variable exponent Morrey space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)} = \Big\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)} < \infty \Big\},$$

where

$$\begin{aligned}\|f\|_{\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)}\|f\|_{L^{p(\cdot)}(B(x,r))}\\ &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)}\|\chi_{B(x,r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{1.12}$$

Remark 1.3. (1) If we take $\varphi(x,r) = r^{-\theta_p(x,r)}$ in (1.12), then the generalized variable exponent Morrey space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is just the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

(2) If we take $\varphi(x,r)=r^{\frac{\lambda-n}{p(x)}}$ with $0<\lambda<n$, then the generalized variable exponent Morrey space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is just the variable exponent Morrey space $M^{p(\cdot),\lambda}(\mathbb{R}^n)$ introduced by Almeida *et al.* in [1].

(3) If we take $p(\cdot)\equiv const$ and $\varphi(x,r)=r^{\frac{\lambda-n}{const}}$ in (1.12), then the generalized variable exponent Morrey space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is just the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ (see [35]).

(4) If we take $p(\cdot)\equiv const$ in (1.12), then the generalized variable exponent Morrey space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is just the generalized Morrey space $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ introduced in [36].

It is position to state the organization of this paper as follows. Section 2 provides some lemmas about the Hölder inequality and the space BMO on variable exponent spaces. Under assumption that the functions φ_i $(i=1,2)$ satisfy certain conditions, the authors prove that the BT is bounded from the product of spaces $\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)\times\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ into space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ in Section 3. In Section 4, the authors prove that the commutator $[b_1,b_2,BT]$ generated by $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$ and the BT is bounded from the product of spaces $\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)\times\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ into space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$, where $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$ and $\varphi=\prod_{i=1}^{2}\varphi_i$. In Section 5, the boundedness of commutator $[b_1,b_2,BT]$ generated by $b_1,b_2\in\dot{\Lambda}(\mathbb{R}^n)$ and BT on the space $L^{p(\cdot)}(\mathbb{R}^n)$ and on the space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is also established.

Finally, we make some conventions on notation. Throughout the paper, C represents a positive constant being independent of the main parameters involved, but it may be different from line to line. For a μ-measurable set E, χ_E denotes its characteristic function. For any variable exponent $p(\cdot)$, we denote by $p'(\cdot)$ its conjugate index, i.e., $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$.

2. Preliminaries

To prove the main results of this paper, in this section, we need to recall some necessary lemmas, see [7, 22, 23, 27], respectively.

Lemma 2.1. *If $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, there exists a constant $C>0$ such that, for all balls $B\subset\mathbb{R}^n$,*

$$C^{-1}|B|\le\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C|B|.$$

Lemma 2.2. *If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, then, for all $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, the following Hölder inequality*

$$\int_{\mathbb{R}^n}|f(x)g(x)|\mathrm{d}x\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

holds.

Lemma 2.3. *Let $p(\cdot),p_1(\cdot),p_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with satisfying $\frac{1}{p(x)}=\frac{1}{p_1(x)}+\frac{1}{p_2(x)}$. Then there exists a positive constant C being independent of functions f and g*

such that, for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ *and* $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$*,*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.4. *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$*, then there exists a positive constant* C *such that, for all* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $i, j \in \mathbb{Z}$ *with* $j > i$*,*

$$\begin{aligned} C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} &\le \sup_{B:\ \text{ball}} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}, \end{aligned} \tag{2.1}$$

and

$$\|(b-b_{B_i})\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{2.2}$$

where B_i *represents a ball with the same center to* B *and radius* 2^i *times of* B*.*

3. BT on space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$

The main theorem of this section is stated as follows.

Theorem 3.1. *Let* $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$*,* $p(\cdot) \in \mathcal{P}_0$ *with* $0 < p_0 < p_-$ *such that* $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$*,* $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$ *and* $\theta_p(\cdot,\cdot) = \theta_{p_1}(\cdot,\cdot) + \theta_{p_2}(\cdot,\cdot)$*. Suppose that the bilinear Calderón-Zygmund operator* BT *is defined as in* (1.7)*, and* $\varphi_i : \mathbb{R}^n \times (0,\infty)$ *are positive measurable functions with* $i = \{1,2\}$*. If there exists some constant* $C > 0$ *such that, for any* $x \in \mathbb{R}^n$ *and* $r > 0$*,*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_i(x,s)s^{\theta_{p_i}(x,s)}}{t^{\theta_{p_i}(x,t)}}\frac{\mathrm{d}t}{t} \le C\varphi_i(x,r), \quad i = 1,2, \tag{3.1}$$

and denote $\varphi(x,r) = \varphi_1(x,r)\varphi_2(x,r)$*, then*

$$\|BT(f_1,f_2)\|_{\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)} \le C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)},$$

where $f_i \in \mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$*,* $i = 1,2$*.*

To prove the above theorem, we need to recall the following two lemmas, see [15, 21], respectively.

Lemma 3.2. *Let* $p(\cdot) \in LH(\mathbb{R}^n)\bigcap\mathcal{P}(\mathbb{R}^n)$*. Suppose that* $\sup\nu(x) < \infty$ *and* $\inf[n+\nu(x)p(x)] > 0$*. Then*

$$\||x-\cdot|^{\nu(x)}\chi_{B(x,r)}(\cdot)\|_{L^{p(\cdot)}} \le Cr^{\nu(x)+\theta_p(x,r)}, \quad x \in \mathbb{R}^n \text{ and } r > 0. \tag{3.2}$$

Lemma 3.3. *Let* $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$*,* $p(\cdot) \in \mathcal{P}_0$ *with* $0 < p_* < p_-$ *such that* $(p(\cdot)/p_*)' \in \mathcal{B}(\mathbb{R}^n)$ *and satisfy* $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$*. Suppose that* BT *is defined as in* (1.7)*. Then there exists a constant* $C > 0$ *such that, for all* $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$*,* $i = 1,2$*,*

$$\|BT(f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},$$

where C *does not depend on* x *and* r*.*

Proof of Theorem 3.1. Without loss of generality, we set $B = B(x,r)$ be a ball centered at x and radius r. And decompose functions f_i as

$$f_i = f_i^1 + f_i^\infty = f_i\chi_{B(x,2r)} + f_i\chi_{\mathbb{R}^n\setminus B(x,2r)}, \qquad i = 1,2.$$

Then, via (1.7), (1.12) and the Minkowski inequality, write

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)} BT(f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\le &\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)} BT(f_1^1,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&+ \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)} BT(f_1^1,f_2^\infty)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&+ \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)} BT(f_1^\infty,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&+ \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)} BT(f_1^\infty,f_2^\infty)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
=: &\,\mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4.
\end{aligned}$$

Notice that $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $p(\cdot) \in \mathcal{P}_0$ satisfies $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$ for some $0 < p_0 < p_-$. Then, by applying Lemma 3.3 and (1.12), we have

$$\begin{aligned}
\mathrm{D}_1 &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)} BT(f_1^1,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)} f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B(x,r)} f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\le C \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\theta_{p_1}(x,r)}}{\varphi_1(x,r)} \|\chi_{B(x,r)} f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{r^{-\theta_{p_2}(x,r)}}{\varphi_2(x,r)} \|\chi_{B(x,r)} f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}$$

To estimate D_2, we first consider $|BT(f_1^1, f_2^\infty)(y)|$ with $y \in B(x,r)$. By (1.3), Lemma 2.2 and (3.2), we can deduce that

$$\begin{aligned}
&|BT(f_1^1,f_2^\infty)(y)| \\
\le &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(y,z_1,z_2)||f_1^1(z_1)||f_2^\infty(z_2)| \mathrm{d}z_1 \mathrm{d}z_2 \\
\le &\, C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_1^1(z_1)||f_2^\infty(z_2)|}{\left(\sum\limits_{i=1}^2 |x-y_i|\right)^{2n}} \mathrm{d}z_1 \mathrm{d}z_2 \\
\le &\, C\int_{B(x,2r)} |f_1(z_1)|\mathrm{d}z_1 \int_{\mathbb{R}^n\setminus B(x,2r)} \frac{|f_2(z_2)|}{|x-z_2|^{2n}} \mathrm{d}z_2 \\
\le &\, C\|\chi_{B(x,2r)} f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}^n\setminus B(x,2r)} |x-z_2|^{-2n+\beta}|f_2(z_2)|\bigg(\int_{|x-z_2|}^{\infty} t^{-\beta-1}\mathrm{d}t\bigg)\mathrm{d}z_2\\
\leq{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\times \int_{2r}^{\infty} t^{-\beta-1}\bigg(\int_{\{z_2:2r<|x-z_2|<t\}} |x-z_2|^{-2n+\beta}|f_2(z_2)|\mathrm{d}z_2\bigg)\mathrm{d}t\\
\leq{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\times \frac{1}{|B(x,2r)|}\int_{2r}^{\infty} t^{-\beta-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2}(\mathbb{R}^n)}\||x-\cdot|^{-n+\beta}\chi_{B(x,t)}(\cdot)\|_{L^{p_2'}(\mathbb{R}^n)}\mathrm{d}t\\
\leq{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\times \frac{1}{|B(x,2r)|}\int_{2r}^{\infty} t^{-\beta-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2}(\mathbb{R}^n)}t^{-n+\beta+\frac{n}{p_2'(x)}}\mathrm{d}t\\
\leq{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\times \frac{1}{|B(x,2r)|}\int_{2r}^{\infty} t^{-\theta_{p_2}(x,t)-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\mathrm{d}t,
\end{aligned}
$$

further, from (1.11), (1.12), Lemma 2.1, Lemma 2.3 and (3.1), it then follows that

$$
\begin{aligned}
\mathrm{D}_2 ={}& \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}BT(f_1^1,f_2^\infty)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
\leq{}& C\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\quad\times \|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{1}{|B(x,2r)|}\int_{2r}^{\infty} t^{-\theta_{p_2}(x,t)-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
\leq{}& C\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\times r^{\theta_{p_1}(x,r)}\int_{2r}^{\infty} t^{-\theta_{p_1}(x,t)-1}\|\chi_{B(x,t)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\quad\times \frac{1}{|B(x,2r)|}\int_{2r}^{\infty} t^{-\theta_{p_2}(x,t)-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
\leq{}& C\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\times r^{\theta_{p_1}(x,r)}\int_{2r}^{\infty} \frac{\varphi_1(x,t)}{\varphi_1(x,t)}t^{-\theta_{p_1}(x,t)-1}\|\chi_{B(x,t)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\quad\times \frac{1}{|B(x,2r)|}\int_{2r}^{\infty} \frac{\varphi_1(x,t)}{\varphi_2(x,t)}t^{-\theta_{p_2}(x,t)-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
\leq{}& C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{1}{|B(x,2r)|}\\
&\quad\times r^{\theta_{p_1}(x,r)}\int_r^\infty \frac{\varphi_1(x,t)}{t}\mathrm{d}t\int_r^\infty \frac{\varphi_1(x,t)}{t}\mathrm{d}t\\
\leq{}& C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\\
&\times\frac{1}{|B(x,2r)|}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\|\chi_{B(x,r)}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}r^{\theta_{p_1}(x,r)}\varphi_1(x,r)\varphi_2(x,r)\\
\leq{}& C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)},
\end{aligned}
$$

where we have used the following fact (see [15])

$$
\|\chi_{B(x,r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq r^{\theta_p(x,r)}\int_{2r}^\infty t^{-\theta_p(x,t)-1}\|\chi_{B(x,t)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\mathrm{d}t. \tag{3.3}
$$

With an argument similar to that used in the estimate of D_2, it is easy to obtain that

$$
\mathrm{D}_3\leq C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
$$

Now let us estimate D_4. For any $y\in B(x,r)$, by applying (1.3), (1.7), (1.12), Lemma 2.2 and (3.2), we obtain that

$$
\begin{aligned}
&|BT(f_1^\infty,f_2^\infty)(y)|\\
\leq{}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|K(y,z_1,z_2)||f_1^\infty(z_1)||f_2^\infty(z_2)|\mathrm{d}z_1\mathrm{d}z_2\\
\leq{}& C\prod_{i=1}^2\int_{\mathbb{R}^n\setminus B(x,2r)}|x-z_i|^{-n+\beta}|f_i(z_i)|\bigg(\int_{|x-z_i|}^\infty t^{-\beta-1}\mathrm{d}t\bigg)\mathrm{d}z_i\\
\leq{}& C\prod_{i=1}^2\int_{2r}^\infty\bigg(\int_{\{z_i:2r<|x-z_i|<t\}}|x-z_i|^{-n+\beta}|f_i(z_i)|\mathrm{d}z_i\bigg)t^{-\beta-1}\mathrm{d}t\\
\leq{}& C\prod_{i=1}^2\int_{2r}^\infty\bigg(\int_{B(x,t)}|x-z_i|^{-n+\beta}|f_i(z_i)|\mathrm{d}z_i\bigg)t^{-\beta-1}\mathrm{d}t\\
\leq{}& C\prod_{i=1}^2\int_{2r}^\infty t^{-\beta-1}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\||x-\cdot|^{-n+\beta}\|_{L^{p_i'(\cdot)}(B(x,t))}\mathrm{d}t\\
\leq{}& C\prod_{i=1}^2\int_{2r}^\infty t^{-\beta-1}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}t^{-n+\beta+\theta_{p_i'}(x,t)}\mathrm{d}t\\
\leq{}& C\prod_{i=1}^2\int_{2r}^\infty t^{-\theta_{p_i}(x,t)-1}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\mathrm{d}t
\end{aligned}
$$

$$\leq C\prod_{i=1}^{2}\int_{2r}^{\infty}\frac{1}{\varphi_i(x,t)}t^{-\theta_{p_i}(x,t)}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\frac{\varphi_i(x,t)}{t}\mathrm{d}t$$
$$\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\varphi_i(x,r),$$

further, from (1.11), (1.12) and (3.1), we have

$$\begin{aligned}\mathrm{D}_4 &= \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}BT(f_1^\infty,f_2^\infty)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,r)\varphi_2(x,r)r^{-\theta_p(x,r)}}{\varphi(x,r)}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\varphi_1(x,r)\varphi_2(x,r)r^{\theta_p(x,r)}\\ &\leq C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.\end{aligned}$$

Which, combing the estimates of D_1, D_2 and D_3, the proof of Theorem 3.1 is finished. □

4. $[b_1, b_2, BT]$ associated with BMO function on space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$

Before stating the main theorem of this section, we first recall the definition of bounded mean oscillation spaces (= BMO), see [10] or [25].

Definition 4.1. A real value function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be the *space* $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{x\in B}\frac{1}{|B|}\int_B |f(x)-f_B|\mathrm{d}x, \tag{4.1}$$

where the supremum is taken over all balls in $\mathbb{R}^n$, and f_B represents the mean value of f on ball B, that is,

$$f_B := \frac{1}{|B|}\int_B f(y)\mathrm{d}y.$$

It is now position to state the main theorem as follows.

Theorem 4.2. *Let* $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}_0$ *with* $0 < p_0 < p_-$ *such that* $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$ *and* $\theta_p(\cdot,\cdot) = \theta_{p_1}(\cdot,\cdot) + \theta_{p_2}(\cdot,\cdot)$. *Suppose that the bilinear Calderón-Zygmund operator BT is defined as in* (1.7), *and* $\varphi_i : \mathbb{R}^n \times (0,\infty)$ *is a positive measurable function with* $i = \{1,2\}$. *If there exists a constant* $C > 0$ *such that, for any* $x \in \mathbb{R}^n$ *and* $r > 0$, *the following inequalities*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_i(x,s)s^{\theta_{p_i}(x,s)}}{t^{\theta_{p_i}(x,t)}}\frac{\mathrm{d}t}{t} \leq C\varphi_i(x,r) \tag{4.2}$$

hold, and $\varphi(x,r)=\varphi_1(x,r)\varphi_2(x,r)$, *then*

$$\|[b_1,b_2,BT](f_1,f_2)\|_{\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)}\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)},$$

where $f_i\in\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i=1,2$.

To prove the above theorem, we need the following lemma introduced in [21].

Lemma 4.3. *Let* $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$, $p_1(\cdot),p_2(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}_0$ *with* $0<p_*<p_-$ *such that* $(p(\cdot)/p_*)'\in\mathcal{B}(\mathbb{R}^n)$ *and satisfy* $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$. *Suppose that* BT *is defined as in* (1.7)*. Then there exists a constant* $C>0$ *such that, for all* $f_i\in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i=1,2$,

$$\|[b_1,b_2,BT](f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)},$$

where C *does not depend on* x *and* r.

Proof of Theorem 4.2. Let $B=B(x,r)$ be a ball centered at x and radius r. And decompose f_i as

$$f_i=f_i^1+f_i^\infty=f_i\chi_{B(x,2r)}+f_i\chi_{\mathbb{R}^n\setminus B(x,2r)},\quad i=1,2.$$

Then, by applying (1.8), (1.12) and the Monkowski inequality, we have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
\le&\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1^1,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&+\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1^1,f_2^\infty)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&+\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1^\infty,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&+\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1^\infty,f_2^\infty)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
=:&\ \mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4.
\end{aligned}$$

From (1.11), (1.12) and Lemma 4.3, it then follows that

$$\begin{aligned}
\mathrm{E}_1=&\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1^1,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
\le&\ C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\\
&\times\|\chi_{B(x,r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,r)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)}\varphi_1(x,r)\varphi_2(x,r)$$
$$\times r^{\theta_p(x,r)} \frac{1}{\varphi_1(x,r)} r^{-\theta_{p_1}(x,r)} \|\chi_{B(x,r)} f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}$$
$$\times \frac{1}{\varphi_2(x,r)} r^{-\theta_{p_2}(x,r)} \|\chi_{B(x,r)} f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

To estimate E_2, we first consider $|[b_1, b_2, BT](f_1^1, f_2^\infty)(y)|$ with $y \in B(x,r)$. By applying (1.3), (1.12), Lemma 2.2, (2.1), Fubini's theorem, Lemma 3.2, (3.3) and (4.2), we obtain

$$|[b_1, b_2, BT](f_1^1, f_2^\infty)(y)|$$
$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(y,z_1,z_2)||b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||f_1^1(z_1)||f_2^\infty(z_2)|\mathrm{d}z_1\mathrm{d}z_2$$
$$\leq C\int_{B(x,2r)}\int_{\mathbb{R}^n\setminus B(x,2r)} \frac{|b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||f_1(z_1)||f_2(z_2)|}{\left[\sum_{i=1}^{2}|y-z_i|\right]^{2n}}\mathrm{d}z_1\mathrm{d}z_2$$
$$\leq C\int_{B(x,2r)} |b_1(y)-b_1(z_1)||f_1(z_1)|\mathrm{d}z_1 \int_{\mathbb{R}^n\setminus B(x,2r)} \frac{|b_2(y)-b_2(z_2)||f_2(z_2)|}{|y-z_2|^{2n}}\mathrm{d}z_2$$
$$\leq C\Bigg(|b_1(y)-(b_1)_{2B}|\int_{B(x,2r)}|f_1(z_1)|\mathrm{d}z_1 + \int_{B(x,2r)}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|\mathrm{d}z_1\Bigg)$$
$$\times \frac{1}{|B(x,2r)|}\int_{\mathbb{R}^n\setminus B(x,2r)} \frac{|b_2(y)-b_2(z_2)||f_2(z_2)|}{|x-z_2|^n}\mathrm{d}z_2$$
$$\leq C\Bigg(|b_1(y)-(b_1)_{2B}|\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}$$
$$+ \|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}(b_1(\cdot)-(b_1)_{2B})\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Bigg)$$
$$\times \frac{1}{|B(x,2r)|}\Bigg(|b_2(y)-(b_1)_{2B}|\int_{\mathbb{R}^n\setminus B(x,2r)}\frac{|f_2(z_2)|}{|x-z_2|^n}\mathrm{d}z_2 + \int_{\mathbb{R}^n\setminus B(x,2r)}\frac{|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|}{|x-z_2|^n}\mathrm{d}z_2\Bigg)$$
$$\leq C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Big(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} + |b_1(y)-(b_1)_{2B}|\Big)$$

$$
\begin{aligned}
&\times \frac{1}{|B(x,2r)|}\Bigg[|b_2(y)-(b_2)_{2B}|\int_{\mathbb{R}^n\setminus B(x,2r)}|x-z_2|^{-n}|f_2(z_2)|\mathrm{d}z_2\\
&+\int_{2r}^{\infty}\Bigg(\int_{\{z_2:2r<|x-z_2|<t\}}|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|\mathrm{d}z_2\Bigg)\frac{1}{t^{n+1}}\mathrm{d}t\Bigg]\\
\le{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Bigg(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Bigg)\\
&\times \frac{1}{|B(x,2r)|}\Bigg[|b_2(y)-(b_2)_{2B}|\int_{\mathbb{R}^n\setminus B(x,2r)}|x-z_2|^{-n}|f_2(z_2)|\mathrm{d}z_2\\
&+\int_{2r}^{\infty}|(b_2)_{2B}-(b_2)_{B(x,t)}|\Bigg(\int_{B(x,t)}|f_2(z_2)|\mathrm{d}z_2\Bigg)\frac{1}{t^{n+1}}\mathrm{d}t\\
&+\int_{2r}^{\infty}\Bigg(\int_{B(x,t)}|b_2(z_2)-(b_2)_{B(x,t)}||f_2(z_2)|\mathrm{d}z_2\Bigg)\frac{1}{t^{n+1}}\mathrm{d}t\Bigg]\\
\le{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Bigg(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Bigg)\\
&\times \frac{1}{|B(x,2r)|}\Bigg[|b_2(y)-(b_2)_{2B}|\int_{2r}^{\infty}t^{-\beta-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\times \||x-\cdot|^{-n+\beta}\chi_{B(x,t)}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\int_{2r}^{\infty}\ln\frac{t}{r}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,t)}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\frac{1}{t^{n+1}}\mathrm{d}t\\
&+\int_{2r}^{\infty}\|\chi_{B(x,t)}(b_2(\cdot)-(b_2)_{B(x,t)})\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\frac{1}{t^{n+1}}\mathrm{d}t\Bigg]\\
\le{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Bigg(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Bigg)\\
&\times \frac{1}{|B(x,2r)|}\Bigg[|b_2(y)-(b_2)_{2B}|\int_{2r}^{\infty}t^{-\theta_{p_2}(x,t)-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\int_{2r}^{\infty}\ln\frac{t}{r}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,t)}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\frac{1}{t^{n+1}}\mathrm{d}t\\
&+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\int_{2r}^{\infty}\|\chi_{B(x,t)}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\frac{1}{t^{n+1}}\mathrm{d}t\Bigg]\\
\le{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Bigg(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Bigg)\\
&\times \frac{1}{|B(x,2r)|}\Bigg[|b_2(y)-(b_2)_{2B}|\int_{2r}^{\infty}t^{-\theta_{p_2}(x,t)-1}\|\chi_{B(x,t)}f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\int_{2r}^{\infty}\ln\frac{t}{r}\|\chi_{B(x,t)}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\frac{t^{\theta_{p_2}(x,t)}\varphi_2(x,t)}{t^{n+1}}\mathrm{d}t\\
&+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\int_{2r}^{\infty}\|\chi_{B(x,t)}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\frac{t^{\theta_{p_2}(x,t)}\varphi_2(x,t)}{t^{n+1}}\mathrm{d}t\Bigg]\\
\le{}& C\|\chi_{B(x,2r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Big(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Big)\\
&\times\frac{1}{|B(x,2r)|}\Bigg[|b_2(y)-(b_2)_{2B}|\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\int_{2r}^{\infty}\frac{\varphi_2(x,t)}{t}\mathrm{d}t\\
&+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\varphi_2(x,t)}{t}\mathrm{d}t\Bigg]\\
\le{}& C\varphi_2(x,r)\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\frac{1}{|B(x,2r)|}\\
&\times r^{\theta_p(x,r)}\int_{2r}^{\infty}t^{-\theta_p(x,t)-1}\|\chi_{B(x,t)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\times\Big(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Big)\Big(|b_2(y)-(b_2)_{2B}|+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big)\\
\le{}& C\varphi_2(x,r)\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\frac{1}{|B(x,2r)|}\\
&\times r^{\theta_{p_1}(x,r)}\int_{2r}^{\infty}t^{-\theta_p(x,t)-1}\|\chi_{B(x,t)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\times\Big(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Big)\Big(|b_2(y)-(b_2)_{2B}|+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big)\\
\le{}& C\varphi_2(x,r)\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\frac{1}{|B(x,2r)|}\\
&\times r^{\theta_{p_1}(x,r)}\int_{2r}^{\infty}\frac{\varphi_1(x,t)}{t}\mathrm{d}t\\
&\times\Big(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Big)\Big(|b_2(y)-(b_2)_{2B}|+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big)\\
\le{}& C\varphi_1(x,r)\varphi_2(x,r)\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|}\\
&\times\Big(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}+|b_1(y)-(b_1)_{2B}|\Big)\Big(|b_2(y)-(b_2)_{2B}|+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big),
\end{aligned}
$$

where we have used the following fact in [25]

$$
|b_{B(x,t)}-b_{B(x,r)}|\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\ln\frac{t}{r}\qquad\text{for } 0<2r<t<\infty.
$$

Further, via (1.1), (1.11), Lemmas 2.1, 2.2, 2.3 and 2.4, we obtain that

$$\begin{aligned}
\mathrm{E}_2 = & \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,r)}[b_1,b_2,BT](f_1^1,f_2^\infty)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\le & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \\
& \times \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
& \times \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}(b_2(\cdot)-(b_2)_{2B})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& + C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \\
& \times \frac{r^{-\theta_p(x,r)}}{\varphi(x,r)} \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& + C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\theta_p(x,r)}}{\varphi(x,r)} \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \\
& \times \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \varphi_1(x,r)\varphi_2(x,r) \\
& \times \|\chi_{B(x,r)}(b_1(\cdot)-(b_1)_{2B})(b_2(\cdot)-(b_2)_{2B})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& + C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \\
& \times r^{-\theta_p(x,r)} \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
& \times \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}(b_1(\cdot)-(b_1)_{2B})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\le & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \\
& \times \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
& \times \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& + C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \\
& \times \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
& \times \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& + C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{\|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\varphi(x,r)} r^{-\theta_p(x,r)} \\
& \times \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}(b_1(\cdot)-(b_1)_{2B})\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
& \times \|\chi_{B(x,r)}(b_2(\cdot)-(b_2)_{2B})\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
& + C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \\
& \times r^{-\theta_p(x,r)} \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\le & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} r^{-\theta_p(x,r)} \\
& \times \|\chi_{B(x,2r)}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \frac{r^{\theta_{p_1}(x,r)}}{|B(x,2r)|} \|\chi_{B(x,r)}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,r)}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

With an argument similar to that used in the estimate of E_2, it is easy to see that

$$\mathrm{E}_3 \leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

Now let us estimate E_4. For any $y \in B(x,r)$, by applying (1.3), (1.11), (1.12), Lemma 2.2, Lemma 2.4, (3.2) and (4.2), we have

$$\begin{aligned}
&|[b_1,b_2,BT](f_1^\infty,f_2^\infty(y)|\\
\leq\ & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|K(y,z_1,z_2)||b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||f_1^\infty(z_1)||f_2^\infty(z_2)|\mathrm{d}z_1\mathrm{d}z_2\\
\leq\ & C\prod_{i=1}^{2}\int_{\mathbb{R}^n\setminus 2B(x,r)}\frac{|b_i(y)-b_i(z_i)|}{|y-z_i|^n}|f_i(z_i)|\mathrm{d}z_i\\
\leq\ & C\prod_{i=1}^{2}|b_i(y)-(b_i)_{B(x,r)}|\int_{\mathbb{R}^n\setminus 2B(x,r)}\frac{|f_i(z_i)|}{|x-z_i|^n}\mathrm{d}z_i\\
&+C\prod_{i=1}^{2}\int_{\mathbb{R}^n\setminus 2B(x,r)}\frac{|b_i(z_i)-(b_i)_{B(x,r)}|}{|x-z_i|^n}|f_i(z_i)|\mathrm{d}z_i\\
\leq\ & C\prod_{i=1}^{2}|b_i(y)-(b_i)_{B(x,r)}|\int_{\mathbb{R}^n\setminus 2B(x,r)}|x-z_i|^{-n+\beta}|f_i(z_i)|\left(\int_{|x-z_i|}^{\infty}\frac{1}{t^{\beta+1}}\mathrm{d}t\right)\mathrm{d}z_i\\
&+C\prod_{i=1}^{2}\int_{\mathbb{R}^n\setminus 2B(x,r)}|b_i(z_i)-(b_i)_{B(x,r)}||f_i(z_i)|\left(\int_{|x-z_i|}^{\infty}\frac{1}{t^{n+1}}\mathrm{d}t\right)\mathrm{d}z_i\\
\leq\ & C\prod_{i=1}^{2}|b_i(y)-(b_i)_{B(x,r)}|\int_{2r}^{\infty}\left(\int_{\{z_i:2r<|x-z_i|<t\}}|x-z_i|^{-n+\beta}|f_i(z_i)|\mathrm{d}z_i\right)\frac{\mathrm{d}t}{t^{\beta+1}}\\
&+C\prod_{i=1}^{2}\int_{2r}^{\infty}\left(\int_{\{z_i:2r<|x-z_i|<t\}}|b_i(z_i)-(b_i)_{B(x,r)}||f_i(z_i)|\mathrm{d}z_i\right)\frac{1}{t^{n+1}}\mathrm{d}t\\
\leq\ & C\prod_{i=1}^{2}|b_i(y)-(b_i)_{B(x,r)}|\int_{2r}^{\infty}t^{-\beta-1}\int_{B(x,t)}|x-z_i|^{-n+\beta}|f_i(z_i)|\mathrm{d}z_i\mathrm{d}t\\
&+C\prod_{i=1}^{2}\int_{2r}^{\infty}\int_{B(x,t)}|b_i(z_i)-(b_i)_{B(x,r)}||f_i(z_i)|\mathrm{d}z_i\frac{1}{t^{n+1}}\mathrm{d}t\\
\leq\ & C\prod_{i=1}^{2}|b_i(y)-(b_i)_{B(x,r)}|\int_{2r}^{\infty}t^{-\beta-1}\|\chi_{B(x,t)}|x-z_i|^{-n+\beta}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\\
&\times\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&+C\prod_{i=1}^{2}\int_{2r}^{\infty}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,t)}(b_i(\cdot)-(b_i)_{B(x,r)})\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\frac{\mathrm{d}t}{t^{n+1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\prod_{i=1}^{2}|b_i(y)-(b_i)_{B(x,r)}|\int_{2r}^{\infty}t^{-\theta_{p_i}(x,t)-1}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\quad+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\int_{2r}^{\infty}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,t)}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\frac{1}{t^{n+1}}\mathrm{d}t\\
&\quad+C\prod_{i=1}^{2}\int_{2r}^{\infty}|(b_i)_{B(x,t)}-(b_i)_{B(x,r)}|\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,t)}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\frac{\mathrm{d}t}{t^{n+1}}\\
&\leq C\prod_{i=1}^{2}|b_i(y)-(b_i)_{B(x,r)}|\int_{2r}^{\infty}t^{-\theta_{p_i}(x,t)-1}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\quad+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\int_{2r}^{\infty}t^{-\theta_{p_i}(x,t)-1}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\quad+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\int_{2r}^{\infty}t^{-\theta_{p_i}(x,t)-1}\ln\frac{t}{r}\|\chi_{B(x,t)}f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\mathrm{d}t\\
&\leq C\prod_{i=1}^{2}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}|b_i(y)-(b_i)_{B(x,r)}|\int_{2r}^{\infty}\frac{\varphi_i(x,t)}{t}\mathrm{d}t\\
&\quad+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\varphi_i(x,t)}{t}\mathrm{d}t\\
&\leq C\prod_{i=1}^{2}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}|b_i(y)-(b_i)_{B(x,r)}|\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\varphi_i(x,t)}{t}\mathrm{d}t\\
&\quad+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\varphi_i(x,t)}{t}\mathrm{d}t\\
&\leq C\prod_{i=1}^{2}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}|b_i(y)-(b_i)_{B(x,r)}|\varphi_i(x,r)\\
&\quad+C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\varphi_i(x,r),
\end{aligned}
$$

further, from (1.12), (1.13), Lemmas 2.3 and 2.4, it then follows that

$$
\begin{aligned}
\mathrm{E}_4&=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1^{\infty},f_2^{\infty})\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\varphi_1(x,r)\\
&\quad\times\varphi_2(x,r)\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}
\end{aligned}
$$

$$+ C\prod_{i=1}^{2}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\varphi_1(x,r)\varphi_2(x,r)$$
$$\times \|\chi_{B(x,r)}(b_i(y)-(b_i)_{B(x,r)})\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}$$
$$\times \varphi_1(x,r)\varphi_2(x,r)\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)},$$

which, combing the estimates of E_1, E_2 and E_3, the proof of Theorem 4.2 is completed. □

5. $[b_1, b_2, BT]$ with Lipschitz function on space $L^{p(\cdot)}(\mathbb{R}^n)$ and on space $\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)$

Before stating the main theorems of this section, we first recall the definition of Lipschitz space introduced in [50] as follows.

Definition 5.1. Let $0<\alpha<1$. A function b is said to be the Lipschitz space $\dot{\Lambda}_\alpha(\mathbb{R}^n)$, denoted by $b\in\dot{\Lambda}_\alpha(\mathbb{R}^n)$, if there exists a constant $C>0$ such that

$$|b(x)-b(y)|\leq C|x-y|^\alpha \quad \text{for all } x,y\in\mathbb{R}^n. \tag{5.1}$$

Then the smallest constant C satisfying (5.1) is denoted by $\|b\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}$.

It is now position to state the main theorems as follows.

Theorem 5.2. *Let $b_1,b_2\in\dot{\Lambda}_\alpha(\mathbb{R}^n)$, $p_1(\cdot),p_2(\cdot),q_1(\cdot),q_2(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}_0$ with $0<p_*<p_-$ such that $(p(\cdot)/p_*)'\in\mathcal{B}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}$, $\frac{1}{q_1(\cdot)}=\frac{1}{p_1(\cdot)}-\frac{\alpha}{n}$ and $\frac{1}{q_2(\cdot)}=\frac{1}{p_2(\cdot)}-\frac{\alpha}{n}$. Suppose that BT is defined as in* (1.7)*. Then there exists a positive constant C being independent of x and r such that, for all $f_i\in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i=1,2$,*

$$\|[b_1,b_2,BT](f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq C\prod_{i=1}^{2}\|b_i\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.$$

Theorem 5.3. *Let $b_1,b_2\in\dot{\Lambda}_\alpha(\mathbb{R}^n)$, $p_1(\cdot),p_2(\cdot),q_1(\cdot),q_2(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}_0$ with $0<p_0<p_-$ such that $p(\cdot)/p_0\in\mathcal{B}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}$, $\frac{1}{q_1(\cdot)}=\frac{1}{p_1(\cdot)}-\frac{\alpha}{n}$ and $\frac{1}{q_2(\cdot)}=\frac{1}{p_2(\cdot)}-\frac{\alpha}{n}$. Suppose that BT is defined as in* (1.7) *and $\varphi_i:\mathbb{R}^n\times(0,\infty)$ is a positive measurable function for $i\in\{1,2\}$. If there exists some constant $C>0$ such that, for any $x\in\mathbb{R}^n$ and $r>0$,*

$$\int_r^\infty t^{\alpha-1}\varphi_i(x,t)\mathrm{d}t\leq Cr^{-\frac{\alpha p_i(x)}{q_i(x)-p_i(x)}}, \tag{5.2}$$

and denote $\varphi(x,r)=\varphi_1(x,r)\varphi_2(x,r)$. *Then there exists a constant* $C>0$ *such that, for all* $f_i\in\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i=1,2$,

$$\|[b_1,b_2,BT](f_1,f_2)\|_{\mathcal{L}^{p(\cdot),\varphi}(\mathbb{R}^n)}\le C\prod_{i=1}^{2}\|b_i\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|f_i\|_{\mathcal{L}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}.$$

Proof of Theorem 5.2. For any $x\in\mathbb{R}^n$, by applying (1.3) and (5.1), we have

$$\begin{aligned}
&|[b_1,b_2,BT](f_1,f_2)(x)|\\
&\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|b_1(x)-b_1(y_1)||b_2(x)-b_2(y_2)||f_1(y_1)||f_2(y_2)|}{\left[\sum_{i=1}^{2}|x-y_i|^n\right]^2}\mathrm{d}y_1\mathrm{d}y_2\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|x-y_1|^\alpha|x-y_2|^\alpha}{|x-y_1|^n|x-y_2|^n}\\
&\quad\times|f_1(y_1)||f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_1(y_1)||f_2(y_2)|}{|x-y_1|^{n-\alpha}|x-y_2|^{n-\alpha}}\mathrm{d}y_1\mathrm{d}y_2\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}I_\alpha(|f_1|)(x)I_\alpha(|f_2|)(x),
\end{aligned}$$

where I_α represents the fractional integral operator defined by, for all $x\in\mathbb{R}^n$,

$$I_\alpha(f)(x):=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y,\quad 0<\alpha<n.$$

Further, from Lemma 2.3 and the $(L^{p(\cdot)},L^{q(\cdot)})$-boundedness of I_α (see [8]), it then follows that

$$\begin{aligned}
&\|[b_1,b_2,BT](f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|I_\alpha(|f_1|)I_\alpha(|f_2|)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|I_\alpha(|f_1|)\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|I_\alpha(|f_2|)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{p(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}$, $\frac{1}{q_1(\cdot)}=\frac{1}{p_1(\cdot)}-\frac{\alpha}{n}$ and $\frac{1}{q_2(\cdot)}=\frac{1}{p_2(\cdot)}-\frac{\alpha}{n}$ with $1<(p_1)_+,(p_2)_+<\frac{n}{\alpha}$. □

Proof of Theorem 5.3. By applying (1.12), Lemma 2.3, Theorem 5.2 and the $(L^{p(\cdot)},L^{q(\cdot)})$-boundedness of I_α, we have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\|\chi_{B(x,r)}[b_1,b_2,BT](f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}\\
&\quad\times\|\chi_{B(x,r)}I_\alpha(|f_1|)I_\alpha(|f_2|)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}r^{-\theta_p(x,r)}
\end{aligned}$$

$$
\begin{aligned}
&\times \|\chi_{B(x,r)} I_\alpha(|f_1|)\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B(x,r)} I_\alpha(|f_2|)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} \|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} r^{-\theta_p(x,r)} [\varphi(x,r)]^{\frac{p(x)}{q_1(x)}+\frac{p(x)}{q_2(x)}} \\
&\times r^{\theta_{q_1}(x,r)+\theta_{q_2}(x,r)} \frac{r^{-\theta_{q_1}(x,r)}}{[\varphi(x,r)]^{\frac{p(x)}{q_1(x)}}} \|\chi_{B(x,r)} I_\alpha(|f_1|)\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\times \frac{1}{[\varphi(x,r)]^{\frac{p(x)}{q_2(x)}}} r^{-\theta_{q_2}(x,r)} \|\chi_{B(x,r)} I_\alpha(|f_2|)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b_1\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} \|b_2\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} \|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)},
\end{aligned}
$$

where we have used the following fact (see [15])

$$
\|I_\alpha(f)\|_{\mathcal{L}^{p(\cdot),\varphi^{\frac{q(\cdot)}{p(\cdot)}}}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{L}^{q(\cdot),\varphi}(\mathbb{R}^n)}.
$$

Hence, the proof of Theorem 5.3 is completed. □

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