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ON WEAKLY S-PRIME SUBMODULES

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Abstract. Let R be a commutative ring with a non-zero identity, S be a multiplicatively closed subset of R and M be a unital R-module. In this paper, we define a submodule N of M with $(N :_R M) \cap S = \emptyset$ to be weakly S-prime if there exists $s \in S$ such that whenever $a \in R$ and $m \in M$ with $0 \neq am \in N$, then either $sa \in (N :_R M)$ or $sm \in N$. Many properties, examples and characterizations of weakly S-prime submodules are introduced, especially in multiplication modules. Moreover, we investigate the behavior of this structure under module homomorphisms, localizations, quotient modules, cartesian product and idealizations. Finally, we define two kinds of submodules of the amalgamation module along an ideal and investigate conditions under which they are weakly S-prime.

1. Introduction

Throughout this paper, unless otherwise stated, R denotes a commutative ring with non-zero identity and M is a unital R-module. It is well-known that a proper submodule N of M is called prime if $rm \in N$ for $r \in R$ and $m \in M$ implies $r \in (N :_R M)$ or $m \in N$ where $(N :_R M) = \{r \in R : rM \subseteq N\}$. Since prime ideals and submodules have a vital role in ring and module theory, several generalizations of these concepts have been studied extensively by many authors (see, for example, [3, 5, 13, 16, 18, 19]).

In 2007, Atani and Farzalipour introduced the concept of weakly prime submodules as a generalization of prime submodules. Following [7], a proper submodule N of M is said to be weakly prime if for $r \in R$ and $m \in M$, whenever $0 \neq rm \in N$, then $r \in (N :_R M)$ or $m \in N$. In 2019 a new kind of generalizations of prime submodules has been introduced and studied by Şengelen Sevim et al. [18]. For a multiplicatively closed subset S of R, they called a proper submodule N of an R-module M with $(N :_R M) \cap S = \emptyset$ an S-prime if there exists $s \in S$ such that for $r \in R$ and $m \in M$, whenever $rm \in N$, then either $sr \in (N :_R M)$ or $sm \in N$. In particular, an ideal I of R is called an S-prime ideal if I is an S-prime submodule of an R-module R,

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[13]. Recently, Almahdi et al. generalized S-prime ideals by defining the notion of weakly S-prime ideals. A proper ideal I of R disjoint with S is said to be weakly S-prime if there exists $s \in S$ such that for $a, b \in R$ and $0 \neq ab \in I$, then either $sa \in I$ or $sb \in I$, [3].

Our objective in this paper is to define and study the concept of weakly S-prime submodules as an extension of the above concepts. Let S be a multiplicatively closed subset of R. We call a submodule N of an R-module M with $(N :_R M) \cap S = \emptyset$ a weakly S-prime submodule if there exists $s \in S$ such that for $a \in R$ and $m \in M$, whenever $0 \neq am \in N$, then either $sa \in (N :_R M)$ or $sm \in N$. In Section 2, we obtain many equivalent statements to characterize this class of submodules (see Theorems 1 and 2), particularly in multiplication modules (Theorem 4). Moreover, various properties of weakly S-prime submodules are considered and many examples are given for supporting the results (see for example Theorem 3, Propositions 1, 2, and Examples 1, 3). We investigate the behavior of this structure under module homomorphisms, localizations, quotient modules, cartesian product of modules (see Propositions 4, 8, Theorem 5 and Corollary 3). Let S be a multiplicatively closed subset of R, M be an R-module and consider the idealization ring $R \ltimes M$. For any submodule K of M, the set $S \ltimes K = \{(s, k) : s \in S,\ k \in K\}$ is a multiplicatively closed subset of $R \ltimes M$. In Theorem 7, we justify the relation among weakly S-prime ideals of R, weakly S-prime submodules of M and weakly $S \ltimes K$-prime ideals of the idealization ring $R \ltimes M$.

Let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring $R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1,\ j \in J\}$ of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f, [10]. The amalgamation of M_1 and M_2 along J with respect to φ is defined in [14] as

$$M_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module with the scaler product defined as

$$(r, f(r) + j)(m_1, \varphi(m_1) + m_2) = (rm_1, \varphi(rm_1) + f(r)m_2 + j\varphi(m_1) + jm_2).$$

For submodules N_1 and N_2 of M_1 and M_2, respectively, the sets

$$N_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : m_1 \in N_1\}$$

and

$$\overline{N_2}^\varphi = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : \varphi(m_1) + m_2 \in N_2\}$$

are submodules of $M_1 \bowtie^\varphi JM_2$. Section 3 is devoted for studying several conditions under which the submodules $N_1 \bowtie^\varphi JM_2$ and $\overline{N_2}^\varphi$ of $M_1 \bowtie^\varphi JM_2$ are (weakly) S-prime submodules, (see Theorems 8, 10). Furthermore, we conclude some particular results for the duplication of a module along an ideal (see Corollaries 4-6, 7-9 and Theorem 9).

For the sake of completeness, we start with some definitions and notations which will be used in the sequel. A non-empty subset S of a ring R is said to be a multiplicatively closed set if S is a subsemigroup of R under multiplication. An R-module M is called multiplication provided for each submodule N of M, there exists an ideal I of R such that $N = IM$. In this case, I is said to be a presentation ideal of N. In particular, for every submodule N of a multiplication module M, $ann(M/N) = (N :_R M)$ is a presentation for N. The product of two submodules N and K of a multiplication module M is defined as $NK = (N :_R M)(K :_R M)M$. For $m_1, m_2 \in M$, by m_1m_2, we mean the product of Rm_1 and Rm_2 which is equal to IJM for presentation ideals I and J of m_1 and m_2, respectively, [4]. Let N be a proper submodule of an R-module M. The radical of N (denoted by M-$rad(N)$) is defined in [11] to be the intersection of all prime submodules of M containing N. If M is multiplication, then $M\text{-}rad(N) = \{m \in M : m^k \subseteq N \text{ for some } k \geq 0\}$. As usual, $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ denotes the ring of integers, the ring of integers modulo n and the field of rational numbers, respectively. For more details and terminology, one may refer to [1, 2, 8, 12, 15].

2. Characterizations of weakly S-prime submodules

We begin with the definitions and relationships of the main concepts of the paper.

Definition 1. Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M with $(N :_R M) \cap S = \emptyset$. We call N a weakly S-prime submodule if there exists (a fixed) $s \in S$ such that for $a \in R$ and $m \in M$, whenever $0 \neq am \in N$, then either $sa \in (N :_R M)$ or $sm \in N$. The fixed element $s \in S$ is said to be a weakly S-element of N.

It is clear that every S-prime submodule is a weakly S-prime submodule. Since the zero submodule is (by definition) a weakly S-prime submodule of any R-module, then the converse is not true in general. For a less trivial example, let M be a non-zero local multiplication R-module with the unique maximal submodule K such that $(K :_R M)K = 0$. If we consider $S = \{1_R\}$, then every proper submodule of M is weakly S-prime, [2]. Hence, there is a weakly S-prime submodule in M that is not S-prime.

Also, every weakly prime submodule N of an R-module M satisfying $(N :_R M) \cap S = \emptyset$ is a weakly S-prime submodule of M and the two concepts coincide if $S \subseteq U(R)$ where $U(R)$ denotes the set of units in R. The following example shows that the converse need not be true.

Example 1. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_6$ and let $N = 2\mathbb{Z} \times \langle \bar{3} \rangle$. Then N is a (weakly) S-prime submodule of M where $S = \{2^n : n \in \mathbb{N} \cup \{0\}\}$. Indeed, let $(0, \bar{0}) \neq r \cdot (r', m) \in N$ for $r, r' \in \mathbb{Z}$ and $m \in \mathbb{Z}_6$ such that $2r \notin (N : M) = 6\mathbb{Z}$. Then $r \cdot m \in \langle \bar{3} \rangle$ with $r \notin 3\mathbb{Z}$ and so $m \in \langle \bar{3} \rangle$. Thus, $2 \cdot (r', m) \in N$ as needed. On the other hand, N is not a weakly prime submodule since $(0,\bar{0}) \neq 2 \cdot (1,\bar{0}) \in N$ but $2 \notin (N : M)$ and $(1,\bar{0}) \notin N$.

Let N be a submodule of an R-module M and I be an ideal of R. The residual of N by I is the set $(N :_M I) = \{m \in M : Im \subseteq N\}$. It is clear that $(N :_M I)$ is a submodule of M containing N. More generally, for any subset $A \subseteq R$, $(N :_M A)$ is a submodule of M containing N.

Theorem 1. *Let S be a multiplicatively closed subset of a ring R. Then for a submodule N of an R-module M with $(N :_R M) \cap S = \emptyset$, the following conditions are equivalent.*

(1) *N is a weakly S-prime submodule of M.*
(2) *There exists $s \in S$ such that $(N :_M a) = (0 :_M a)$ or $(N :_M a) \subseteq (N :_M s)$ for each $a \notin (N :_R sM)$.*
(3) *There exists $s \in S$ such that for any $a \in R$ and for any submodule K of M, if $0 \neq aK \subseteq N$, then $sa \subseteq (N :_R M)$ or $sK \subseteq N$.*
(4) *There exists $s \in S$ such that for any ideal I of R and a submodule K of M, if $0 \neq IK \subseteq N$, then $sI \subseteq (N :_R M)$ or $sK \subseteq N$.*

Proof. (1)⇒(2). Let $s \in S$ be a weakly S-element of N and $a \notin (N :_M sM)$. Let $m \in (N :_M a)$. If $am = 0$, then clearly $m \in (0 :_M a)$. If $0 \neq am \in N$, then, we conclude $sm \in N$ as $sa \notin (N :_R M)$ and N is weakly S-prime in M. Thus, $m \in (N :_M s)$ and so $(N :_M a) \subseteq (0 :_M a) \cup (N :_M s)$. Therefore, $(N :_M a) \subseteq (0 :_M a)$ (which implies $(N :_M a) = (0 :_M a)$) or $(N :_M a) \subseteq (N :_M s)$.

(2)⇒(3). Choose $s \in S$ as in (2) and suppose $0 \neq aK \subseteq N$ and $sa \notin (N :_R M)$ for some $a \in R$ and a submodule K of M. Then $K \subseteq (N :_M a) \backslash (0 :_M a)$ and by (2) we get $K \subseteq (N :_M a) \subseteq (N :_M s)$. Thus, $sK \subseteq N$ as required.

(3)⇒(4). Choose $s \in S$ as in (3) and suppose $0 \neq IK \subseteq N$ and $sI \nsubseteq (N :_R M)$ for some ideal I of R and a submodule K of M. Then there exists $a \in I$ with $sa \notin (N :_R M)$. If $aK \neq 0$, then by (3), we have $sK \subseteq N$ as needed. Assume that $aK = 0$. Since $IK \neq 0$, there is some $b \in I$ with $bK \neq 0$. If $sb \notin (N :_R M)$, then from (3), we have $sK \subseteq N$. Now, assume that $sb \in (N :_R M)$. Since $sa \notin (N :_R M)$, we have $s(a+b) \notin (N :_R M)$. Hence, $0 \neq (a+b)K \subseteq N$ implies $sK \subseteq I$ again by (3) and we are done.

(4)⇒(1). Let $a \in R$, $m \in M$ with $0 \neq am \in N$. The result follows directly by taking $I = aR$ and $K = \langle m \rangle$ in (4). □

Theorem 2. *Let M be a faithful multiplication R-module and S be a multiplicatively closed subset of R. Then the following are equivalent.*

(1) *N is a weakly S-prime submodule of M.*
(2) *$N \cap SM = \emptyset$ and there exists $s \in S$ such that whenever K, L are submodules of M and $0 \neq KL \subseteq N$, then $sK \subseteq N$ or $sL \subseteq N$.*

Proof. Clearly, we have $N \cap SM = \emptyset$ if and only if $(N :_R M) \cap S = \emptyset$.

(1)⇒(2). Let I be a presentation ideal of K and s be a weakly S-element of N. Then $0 \neq IL \subseteq N$ yields that either $sI \subseteq (N :_R M)$ or $sL \subseteq N$ by Theorem 1. Hence, $sK = sIM \subseteq N$ or $sL \subseteq N$, as needed.

(2)⇒(1). Let $s \in S$ be as in (2) and suppose $0 \neq IL \subseteq N$ for some ideal I of R and submodule L of M. Put $K = IM$ and assume that $sL \nsubseteq N$. Then $0 \neq KL \subseteq N$ which implies $sK \subseteq N$. Therefore, $sI \subseteq (N :_R M)$ and the result follows by Theorem 1. □

Let I be a proper ideal of a ring R. In the following proposition, the notation $Z_I(R)$ denotes the set $\{r \in R : rs \in I \text{ for some } s \in R \backslash I\}$.

Theorem 3. *Let N be a submodule of an R-module M and S be a multiplicatively closed subset of R. Then the following statements hold.*

1. *If N is a weakly S-prime submodule of M, then for every submodule K with $(N :_R K) \cap S = \emptyset$ and $Ann(K) = 0$, $(N :_R K)$ is a weakly S-prime ideal of R. In particular, if M is faithful, then $(N :_R M)$ is a weakly S-prime ideal of R.*
2. *If M is multiplication and $(N :_R M)$ is a weakly S-prime ideal of R, then N is a weakly S-prime submodule of M.*
3. *If M is faithful multiplication and I is an ideal of R, then I is weakly S-prime in R if and only if IM is a weakly S-prime submodule of M.*
4. *If N is a weakly S-prime submodule of M and A is a subset of R such that $(0 :_M A) = 0$ and $Z_{(N:_RM)}(R) \cap A = \emptyset$, then $(N :_M A)$ is a weakly S-prime submodule of M.*

Proof. (1) Suppose $s \in S$ is a weakly S-element of N and let $a, b \in R$ with $0 \neq ab \in (N :_R K)$. Since $Ann(K) = 0$, we have $0 \neq abK \subseteq N$ which implies $sa \in (N :_R M)$ or $sbK \subseteq N$ by Theorem 1. Hence, $sa \in (N :_R K)$ or $sb \in (N :_R K)$. Thus, $(N :_R K)$ is a weakly S-prime ideal associated with the same $s \in S$. The "in particular" part is clear.

(2) Suppose M is multiplication and $(N :_R M)$ is a weakly S-prime ideal of R. Let I be an ideal of R and K be a submodule of M with $0 \neq IK \subseteq N$. Since M is multiplication, we may write $K = JM$ for some ideal J of R. Thus, $0 \neq IJ \subseteq (N :_R M)$, and by [13, Theorem 1], there exists an $s \in S$ such that $sI \subseteq (N :_R M)$ or $sJ \subseteq (N :_R M)$. Thus, $sI \subseteq (N :_R M)$ or $sK = sJM \subseteq (N :_R M)M = N$. Therefore, N is a weakly S-prime submodule of M by Theorem 1(4).

(3) Suppose M is faithful multiplication and I is an ideal of R. Since $(IM :_R M) = I$, the result follows from (1) and (2).

(4) Let $s \in S$ be a weakly S-element of N. We firstly note that $((N :_M A) :_R M) \cap S = \emptyset$. Indeed, if $t \in ((N :_M A) :_R M) \cap S$, then $tA \subseteq (N :_R M)$ and so $t \in (N :_R M)$ as $Z_{(N:_RM)}(R) \cap A = \emptyset$, a contradiction. Let $r \in R$ and $m \in M$ such that $0 \neq rm \in (N :_M A)$. Then $0 \neq Arm \subseteq N$ since $(0 :_M A) = 0$. By assumption, either $sr \in (N :_R M)$ or $sAm \subseteq N$. Thus, $sr \in ((N :_M A) :_R M)$ or $sm \in (N :_M A)$ as needed. □

We show by the following example that the condition "faithful module" in Theorem 3(1) is crucial.

Example 2. Let p_1, p_2 and p_3 be distinct prime integers. Consider the non-faithful $\mathbb{Z}$-module $M = \mathbb{Z}_{p_1p_2} \times \mathbb{Z}_{p_1p_2}$ and the multiplicatively closed subset $S = \{p_3^n : n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. While $N = \overline{0} \times \overline{0}$ is a weakly S-prime submodule of M, we have clearly $(N :_{\mathbb{Z}} M) = \langle p_1p_2 \rangle$ is not a weakly S-prime ideal of $\mathbb{Z}$.

Let N be a proper submodule of an R-module M. Then N is said to be a maximal weakly S-prime submodule if there is no weakly S-prime submodule which contains N properly. In the following corollary, by $Z(M)$, we denote the set $\{r \in R : rm = 0 \text{ for some } m \in M\backslash\{0_M\}\}$.

Corollary 1. *Let N be a submodule of M such that $Z_{(N:_RM)}(R) \cup Z(M) \subseteq (N :_R M)$. If N is a maximal weakly S-prime submodule of M, then N is an S-prime submodule of M.*

Proof. Let $s \in S$ be a weakly S-element of N. Suppose that $am \in N$ and $sa \notin (N :_R M)$ for some $a \in R$ and $m \in M$. Since $a \notin (N :_R M)$, then by assumption, $a \notin Z_{(N:_RM)}(R)$ and $(0 :_M a) = 0$. It follows by Theorem 3(4) that $(N :_M a)$ is a weakly S-prime submodule of M. Therefore, $sm \in (N :_M a) = N$ by the maximality of N and so N is an S-prime submodule of M. □

As $N = (N : M)M$ for any submodule N of a multiplication R-module M, we have the following consequence of Theorem 3.

Theorem 4. *Let M be a faithful multiplication R-module and N be a submodule of M. Then the following are equivalent.*

(1) *N is a weakly S-prime submodule of M.*
(2) *$(N :_R M)$ is a weakly S-prime ideal of R.*
(3) *$N = IM$ for some weakly S-prime ideal I of R.*

For a next result, we need to recall the following lemma.

Lemma 1 ([1])**.** *For an ideal I of a ring R and a submodule N of a finitely generated faithful multiplication R-module M, the following hold.*

(1) *$(IN :_R M) = I(N :_R M)$.*
(2) *If I is finitely generated faithful multiplication, then*
    (a) *$(IN :_M I) = N$.*
    (b) *Whenever $N \subseteq IM$, then $(JN :_M I) = J(N :_M I)$ for any ideal J of R.*

Proposition 1. *Let I be a finitely generated faithful multiplication ideal of a ring R, S a multiplicatively closed subset of R and N a submodule of a finitely generated faithful multiplication R-module M. Then*

(1) *If IN is a weakly S-prime submodule of M and $(N :_R M) \cap S = \emptyset$, then either I is a weakly S-prime ideal of R or N is a weakly S-prime submodule of M.*

(2) *N is a weakly S-prime submodule of IM if and only if $(N :_M I)$ is a weakly S-prime submodule of M.*

Proof. (1) Let $s \in S$ be a weakly S-element of IN. Suppose $N = M$. In this case, $I = I(N :_R M) = (IN :_R M)$ is a weakly S-prime ideal of R by Theorem 4. Now, suppose that N is proper. Hence, Lemma 1 implies $N = (IN :_M I)$ and so we conclude that $(N :_R M) = ((IN :_M I) :_R M) = (I(N :_R M) :_M I)$. Suppose $a \in R$, $m \in M$ such that $0 \neq am \in N$ and $sa \notin (N :_R M)$. Since I is faithful, then $(0 :_M I) = Ann_R(I)M = 0$, [1] and so $0 \neq Iam \subseteq IN$. Since clearly $sa \notin (IN :_R M)$ and IN is a weakly S-prime submodule, $sIm \subseteq IN$ by Theorem 1. By Lemma 1(2), we have $sm \in (IN :_M I) = N$, and thus N is a weakly S-prime submodule of M.

(2) Suppose N is a weakly S-prime submodule of IM with a weakly S-element $s' \in S$. Then $((N :_M I) :_R M) \cap S = (N :_R IM) \cap S = \emptyset$. Let $a \in R$ and $m \in M$ with $0 \neq am \in (N :_M I)$ and $s'a \notin ((N :_M I) :_R M) = (N :_R IM)$. If $amI = 0$, then $am \in (0_M : I) = Ann_R(I)M = 0$, a contradiction. Thus, $0 \neq amI \subseteq N$. Since N is a weakly S-prime submodule of IM, Theorem 1 yields that $s'mI \subseteq N$, and so $s'm \in (N :_M I)$, as required.

Conversely, suppose $(N :_M I)$ is a weakly S-prime submodule of M with a weakly S-element $s' \in S$. Then $(N :_R IM) \cap S = ((N :_M I) :_R M) \cap S = \emptyset$. Now, let $a \in R$ and $m' \in IM$ such that $0 \neq am' \in N$ and $s'a \notin (N :_R IM) = ((N :_M I) :_R M)$. Then $a(\langle m' \rangle :_M I) = (\langle am' \rangle :_M I) \subseteq (N :_M I)$. If $a(\langle m' \rangle :_M I) = 0$, then by (2) of Lemma 1, we have $am' \in a(Im' :_M I) \subseteq a(\langle m' \rangle :_M I) = 0$, a contradiction. Thus, $0 \neq a(\langle m' \rangle :_M I) \subseteq (N :_M I)$ and so $s'(\langle m' \rangle :_M I) \subseteq (N :_M I)$ as $s'a \notin ((N :_M I) :_R M)$. Again, by Lemma 1, we conclude that $s'm' \in (I \langle s'm' \rangle :_M I) = Is'(\langle m' \rangle :_M I) \subseteq I(N :_M I) = (IN :_M I) = N$. Therefore, N is a weakly S-prime submodule of IM. □

Proposition 2. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M such that $(N :_R M) \cap S = \emptyset$. If $(N :_M s)$ is a weakly prime submodule of M for some $s \in S$, then N is a weakly S-prime submodule of M. The converse holds for non-zero submodules N if $S \cap Z(M) = \emptyset$.*

Proof. Suppose $(N :_M s)$ is a weakly prime submodule of M for some $s \in S$ and let $a \in R$, $m \in M$ such that $0 \neq am \in N \subseteq (N :_M s)$. Then either $a \in ((N :_M s) :_R M) = ((N :_R M) :_R s)$ or $m \in (N :_M s)$ and so either $sa \in (N :_R M)$ or $sm \in N$ as required. Conversely, suppose $N \neq 0_M$ is a weakly S-prime submodule of M with weakly S-element $s \in S$. Let $a \in R$ and $m \in M$ such that $0 \neq am \in (N :_M s)$. Since $S \cap Z(M) = \emptyset$, we have $0 \neq sam \in N$ which implies either $s^2a \in (N :_R M)$ or $sm \in N$. If $sm \in N$, then $m \in (N :_M s)$ and we are done. Suppose $s^2a \in (N :_R M)$. If $s^2aM = 0$, then $s^2 \in S \cap Z(M)$, a contradiction. Hence, $0 \neq s^2aM \subseteq N$ implies either $s^3 \in (N :_R M)$ or $saM \subseteq N$. But $(N :_R M) \cap S = \emptyset$ implies $saM \subseteq N$ and so $a \in (N :_R sM) = ((N :_M s) :_R M)$. Therefore, $(N :_M s)$ is a weakly prime submodule of M. □

If $S \cap Z(M) \neq \emptyset$, then the converse of Proposition 2 need not be true as we can see in the following example.

Example 3. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_6$ and let $N = \langle 0 \rangle \times \langle \bar{0} \rangle$. Then N is a weakly S-prime submodule of M for $S = \{3^n : n \in \mathbb{N}\}$. Now, for each $n \in \mathbb{N}$, we have clearly $(N :_M 3^n) = \langle 0 \rangle \times \langle \bar{2} \rangle$ which is not a weakly prime submodule of M. Indeed, $2 \cdot (0, \bar{1}) \in (N :_M 3^n)$ but $2 \notin ((N :_M 3^n) :_R M) = \langle 0 \rangle$ and $(0, \bar{1}) \notin (N :_M 3^n)$. We note that $S \cap Z(M) = S \neq \emptyset$.

Proposition 3. *Let M be a faithful multiplication R-module and S be a multiplicatively closed subset of R. Then*

1. *If N is a weakly S-prime submodule of M that is not S-prime, then $s\sqrt{0_R}N = 0_M$ for some $s \in S$.*
2. *If N and K are two weakly S-prime submodules of M that are not S-prime, then $sNK = 0_M$ for some $s \in S$.*

Proof. (1) Let N be a weakly S-prime submodule of M which is not S-prime. Then by (1) of Theorem 3 and [18, Proposition 2.9(ii)], $(N :_R M)$ is a weakly S-prime ideal of R that is not S-prime. Hence, we get $s(N :_R M)\sqrt{0_R} = 0_R$ by [3, Proposition 9] and thus, $sN\sqrt{0_R} = s(N :_R M)M\sqrt{0_R} = 0_R M = 0_M$.

(2) Since N and K are two weakly S-prime submodules that are not S-prime, $(N :_R M)$ and $(K :_R M)$ are weakly S-prime ideals of R that are not S-prime by Theorem 3 and [18, Proposition 2.9(ii)]. Hence, there exists some $s \in S$ such that $s(N :_R M)(K :_R M) = 0_R$ by [3, Corollary 11] and so $sNK = 0$. □

Corollary 2. *Let M be a faithful multiplication R-module, S be a multiplicatively closed subset of a ring R. If N is a weakly S-prime submodule of M, then either $N \subseteq \sqrt{0_R}M$ or $s\sqrt{0_R}M \subseteq N$ for some $s \in S$. Additionally, if R is a reduced ring, then $N = 0_M$ or N is S-prime.*

Proof. Suppose that N is a weakly S-prime submodule of M. Then from Theorem 3(1), $(N :_R M)$ is a weakly S-prime ideal of R and by [3, Corollary 6], we conclude either $(N :_R M) \subseteq \sqrt{0_R}$ or $s\sqrt{0_R} \subseteq (N :_R M)$. Since $N = (N :_R M)M$, we are done. □

Proposition 4. *Let N be a submodule of an R-module M and S be a multiplicatively closed subset of R with $Z(M) \cap S = \emptyset$. Then*

1. *If N is a weakly S-prime submodule of M, then $S^{-1}N$ is a weakly prime submodule of $S^{-1}M$ and there exists an $s \in S$ such that $(N :_M t) \subseteq (N :_M s)$ for all $t \in S$.*
2. *If M is finitely generated, then the converse of (1) holds.*

Proof. (1) Suppose $s \in S$ is a weakly S-element of N. In proving that $S^{-1}N$ is a weakly prime submodule of $S^{-1}M$ we do not need the assumption $Z(M) \cap S =$

$\emptyset$. Let $0_{S^{-1}M} \neq \frac{r}{s_1}\frac{m}{s_2} \in S^{-1}N$ for some $\frac{r}{s_1} \in S^{-1}R$ and $\frac{m}{s_2} \in S^{-1}M$. Then $urm \in N$ for some $u \in S$. If $urm = 0$, then $\frac{rm}{s_1 s_2} = \frac{urm}{us_1 s_2} = 0_{S^{-1}M}$, a contradiction. Hence, $0 \neq urm \in N$ yields either $sur \in (N :_R M)$ or $sm \in N$. Thus, $\frac{r}{s_1} = \frac{sur}{sus_1} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $\frac{m}{s_2} = \frac{sm}{ss_2} \in S^{-1}N$ and so $S^{-1}N$ is a weakly prime submodule of $S^{-1}M$. Now, let $t \in S$ and $m \in (N :_M t)$. Then $0 \neq tm \in N$ as $Z(M) \cap S = \emptyset$ and so $st \in (N :_M M) \cap S$ or $sm \in N$. Since the first one gives a contradiction, we have $m \in (N :_M s)$. Thus, $(N :_M t) \subseteq (N :_M s)$ for all $t \in S$.

(2) Suppose M is finitely generated. Choose $s \in S$ as in (1). If $(N :_R M) \cap S \neq \emptyset$, then clearly $S^{-1}N = S^{-1}M$, a contradiction. Let $0 \neq am \in N$ for some $a \in R$ and $m \in M$. Since $Z(M) \cap S = \emptyset$, we have $0 \neq \frac{a}{1}\frac{m}{1} \in S^{-1}N$. By assumption, either $\frac{a}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ as M is finitely generated or $\frac{m}{1} \in S^{-1}N$. Hence, $va \in (N :_R M)$ for some $v \in S$ or $wm \in N$ for some $w \in S$. If $va \in (N :_R M)$, then our hypothesis implies $aM \subseteq (N :_M v) \subseteq (N :_M s)$ and so $sa \in (N :_R M)$. If $wm \in N$, then again $m \in (N :_M w) \subseteq (N :_M s)$, and so $sm \in N$. Therefore, N is a weakly S-prime submodule of M. □

However, $S^{-1}N$ being a weakly prime submodule of $S^{-1}M$ does not imply that N is a weakly prime submodule of M. For example, it was shown in [18, Example 2.4] that $N = \mathbb{Z} \times \{0\}$ is not a (weakly) S-prime submodule of the $\mathbb{Z}$-module $\mathbb{Q} \times \mathbb{Q}$ where $S = \mathbb{Z} \backslash \{0\}$. But $S^{-1}N$ is a weakly prime submodule of the vector space (over $S^{-1}\mathbb{Z} = \mathbb{Q}$) $S^{-1}(\mathbb{Q} \times \mathbb{Q})$.

Remark 1. Let M be an R-module and S, T be two multiplicatively closed subsets of R with $S \subseteq T$. If N is a weakly S-prime submodule of M and $(N :_R M) \cap T = \emptyset$, then N is a weakly T-prime submodule of M.

Let S be a multiplicatively closed subset of a ring R. The saturation of S is the set $S^* = \{x \in R : xy \in S \text{ for some } y \in R\}$, see [12]. It is clear that S^* is a multiplicatively closed subset of R and that $S \subseteq S^*$.

Proposition 5. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M such that $(N :_R M) \cap S = \emptyset$. Then N is a weakly S-prime submodule of M if and only if N is a weakly S^*-prime submodule of M.*

Proof. Let N be a weakly S^*-prime submodule of M with a weakly S-element $s^* \in S^*$. Choose $r \in R$ such that $s = s^*r \in S$. Suppose $0 \neq am \in N$ for some $a \in R$ and $m \in M$. Then either $s^*a \in (N :_R M)$ or $s^*m \in N$. Thus, $sa \in (N :_R M)$ or $sm \in N$ and we are done. Conversely, suppose N is weakly S^*-prime. By using Remark 1, it is enough to prove that $(N :_R M) \cap S^* = \emptyset$. Suppose there exists $s^* \in (N :_R M) \cap S^*$. Then there is $r \in R$ such that $s = s^*r \in (N :_R M) \cap S$, a contradiction. □

Lemma 2. *Let S be a multiplicatively closed subset of a ring R. If I is a weakly S-prime ideal of R and $\{0_R\}$ is an S-prime ideal of R, then $\sqrt{I}$ is an S-prime ideal of R.*

Proof. Suppose I is weakly S-prime associated to s_1 and $\{0_R\}$ is S-prime associated with s_2. Since $I \cap S = \emptyset$, we have $\sqrt{I} \cap S = \emptyset$. Let $a, b \in R$ with $ab \in \sqrt{I}$. Then $a^n b^n \in I$ for some positive integer n. If $a^n b^n \neq 0$, then we have $s_1 a^n \in I$ or $s_1 b^n \in I$ that is $s_1 a \in \sqrt{I}$ or $s_1 b \in \sqrt{I}$. If $a^n b^n = 0$, then by assumption, either $s_2 a^n = 0$ or $s_2 b^n = 0$ and so $s_2 a \in \sqrt{I}$ or $s_2 b \in \sqrt{I}$. Thus, $\sqrt{I}$ is an S-prime ideal of R associated with $s = s_1 s_2$. □

Proposition 6. *Let M be a finitely generated faithful multiplication R-module and S be a multiplicatively closed subset of R. If N is a weakly S-prime submodule of M and $\{0_R\}$ is an S-prime ideal of R, then M-rad(N) is an S-prime submodule of R.*

Proof. By [16, Lemma 2.4], we have $(M\text{-}rad(N) : M) = \sqrt{(N :_R M)}$. Since N is a weakly S-prime submodule of M, $(N :_R M)$ is so by Theorem 3. By Lemma 2, $\sqrt{(N :_R M)}$ is an S-prime ideal of R. Thus, the claim follows from [18, Proposition 2.9(ii)]. □

Proposition 7. *Let S be a multiplicatively closed subset of a ring R. If N is a weakly S-prime submodule of an R-module M, then for any submodule K of M with $(K :_R M) \cap S \neq \emptyset$, $N \cap K$ is a weakly S-prime submodule of M. Additionally, if M is multiplication, then NK is a weakly S-prime submodule of M.*

Proof. Note that $(N \cap K :_R M) \cap S = \emptyset$ as $(N :_R M) \cap S = \emptyset$. Let $s \in S$ be a weakly S-element of N and let $0 \neq am \in N \cap K \subseteq N$. Then $sa \in (N :_R M)$ or $sm \in N$. Choose $s' \in (K :_R M) \cap S$. Then $ss'a \in (N :_R M) \cap (K :_R M) = (N \cap K :_R M)$ or $ss'm \in N \cap (K :_R M)M = N \cap K$. Thus, $N \cap K$ is a weakly S-prime submodule of M with a weakly S-element $t = ss'$. Putting in mind that $NK = (N :_R M)(K :_R M)M$, the rest of the proof is very similar. □

Notice that if N is weakly prime and K is as above, then $N \cap K$ need not be weakly prime. For instance, consider the $\mathbb{Z}_{12}$-module $\mathbb{Z}_{12}$, $S = \{\bar{1}, \bar{3}, 9\}$, $N = \langle \bar{2} \rangle$ and $K = \langle \bar{3} \rangle$. Then $N \cap K = \langle \bar{6} \rangle$ is not a weakly prime submodule of $\mathbb{Z}_{12}$.

Proposition 8. *Let $f : M_1 \to M_2$ be a module homomorphism where M_1 and M_2 are two R-modules and S be a multiplicatively closed subset of R. Then the following statements hold.*

1. *If f is an epimorphism and N is a weakly S-prime submodule of M_1 containing $Ker(f)$, then $f(N)$ is a weakly S-prime submodule of M_2.*
2. *If f is a monomorphism and K is a weakly S-prime submodule of M_2 with $(f^{-1}(K) :_R M_1) \cap S = \emptyset$, then $f^{-1}(K)$ is a weakly S-prime submodule of M_1.*

Proof. (1) First, observe that $(f(N) :_{R_2} M_2) \cap S = \emptyset$. Indeed, assume that $t \in (f(N) :_{R_2} M_2) \cap S$. Then $f(tM_1) = tf(M_1) = tM_2 \subseteq f(N)$, and so $tM_1 \subseteq N$ as $Ker(f) \subseteq N$. It follows that $t \in (N : M_1) \cap S$, a contradiction. Let s be a weakly S-element of N and $a \in R$, $m_2 \in M_2$ with $0 \neq am_2 \in f(N)$. Then $m_2 = f(m_1)$ for some $m_1 \in M_1$ and $0 \neq af(m_1) = f(am_1) \in f(N)$ and since $Ker(f) \subseteq N$, we have $0 \neq am_1 \in N$. This yields either $sa \in (N :_R M_1)$ or $sm_1 \in N$. Thus, clearly we have either $sa \in (f(N) :_R M_2)$ or $sm_2 = f(sm_1) \in f(N)$ as required.

(2) Let s be a weakly S-element of K and let $a \in R$, $m \in M_1$ with $0 \neq am \in f^{-1}(K)$. Then $0 \neq f(am) = af(m) \in K$ as f is a monomorphism. Since K is a weakly S-prime submodule of M_2, we have $sa \in (K :_R M_2)$ or $sf(m) \in K$. Thus, clearly we have $sa \in (f^{-1}(K) :_R M_1)$ or $sm \in f^{-1}(K)$ as needed. □

Corollary 3. *Let S be a multiplicatively closed subset of a ring R and N, K are two submodules of an R-module M with $K \subseteq N$. Then*

(1) *If N is a weakly S-prime submodule of M, then N/K is a weakly S-prime submodule of M/K.*
(2) *If K' is a weakly S-prime submodule of M with $(K' :_R N) \cap S = \emptyset$, then $K' \cap N$ is a weakly S-prime submodule of N.*
(3) *If N/K is a weakly S-prime submodule of M/K and K is an S-prime submodule of M, then N is an S-prime submodule of M.*
(4) *If N/K is a weakly S-prime submodule of M/K and K is a weakly S-prime submodule of M, then N is a weakly S-prime submodule of M.*

Proof. Note that $(N/K :_R M/K) \cap S = \emptyset$ if and only if $(N :_R M) \cap S = \emptyset$.

(1) Consider the canonical epimorphism $\pi : M \to M/K$ defined by $\pi(m) = m + K$. Then $\pi(N) = N/K$ is a weakly S-prime submodule of M/K by (1) of Proposition 8.

(2) Let K' be a weakly S-prime submodule of M and consider the natural injection $i : N \to M$ defined by $i(m) = m$ for all $m \in N$. Then $(i^{-1}(K') :_R N) \cap S = \emptyset$. Indeed, if $s \in (i^{-1}(K') :_R N) \cap S$, then $sN \subseteq i^{-1}(K') = K' \cap N \subseteq K'$ and so $s \in (K' :_R N) \cap S$, a contradiction. Thus $i^{-1}(K') = K' \cap N$ is a weakly S-prime submodule of M by (2) of Proposition 8.

(3) Let s_1 be a weakly S-element of N/K and suppose K is an S-prime submodule of M associated with $s_2 \in S$. Let $a \in R$ and $m \in M$ such that $am \in N$. If $am \in K$, then $s_2 a \in (K :_R M) \subseteq (N :_R M)$ or $s_2 m \in K \subseteq N$. If $am \notin K$, then $K \neq a(m + K) \in N/K$ which implies either $s_1 a \in (N/K :_R M/K)$ or $s_1(m + K) \in N/K$. Thus, $s_1 a \in (N :_R M)$ or $s_1 m \in N$. It follows that N is an S-prime submodule of M associated with $s = s_1 s_2 \in S$.

(4) Similar to (3). □

The next example shows that the converse of Corollary 3(1) is not valid in general.

Example 4. Consider the submodules $N = K = \langle 6 \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}$ and the multiplicatively closed subset $S = \{5^n : n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. It is clear that N/K is a weakly S-prime submodule of $\mathbb{Z}/K$ but N is not a weakly S-prime submodule of $\mathbb{Z}$ as $0 \neq 2 \cdot 3 \in N$ but neither $2s \in (N :_{\mathbb{Z}} \mathbb{Z})$ nor $3s \in N$ for all $s \in S$.

Proposition 9. *Let S be a multiplicatively closed subset of a ring R and N, K be two weakly S-prime submodules of an R-module M such that $((N+K) :_R M) \cap S = \emptyset$. Then $N + K$ is a weakly S-prime submodule of M.*

Proof. Suppose N and K are two weakly S-prime submodules of M. By Corollary 3(1), $N/(N \cap K)$ is a weakly S-prime submodule of $M/(N \cap K)$. Now, from the module isomorphism $N/(N \cap K) \cong (N + K)/K$, we conclude that $(N+K)/K$ is a weakly S-prime submodule of M/K. Thus, $N+K$ is a weakly S-prime submodule of M by Corollary 3(4). □

Theorem 5. *Let S_1, S_2 be multiplicatively closed subsets of rings R_1, R_2 respectively and N_1, N_2 be non-zero submodules of an R_1-module M_1 and an R_2-module M_2, respectively. Consider $M = M_1 \times M_2$ as an $(R_1 \times R_2)$-module, $S = S_1 \times S_2$ and $N = N_1 \times N_2$. Then the following are equivalent.*

1. *N is a weakly S-prime submodule of M.*
2. *N_1 is an S_1-prime submodule of M_1 and $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ or N_2 is an S_2-prime submodule of M_2 and $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$.*
3. *N is an S-prime submodule of M.*

Proof. (1)⇒(2). Suppose N is a weakly S-prime submodule of M with a weakly S-element $s = (s_1, s_2) \in S$. Assume that $(N_1 :_{R_1} M_1) \cap S_1$ and $(N_2 :_{R_2} M_2) \cap S_2$ are both empty. Choose $0 \neq m \in N_1$. Then $(0_{M_1}, 0_{M_2}) \neq (1, 0)(m, 1_{M_2}) \in N$ which implies $(s_1, s_2)(1, 0) \in (N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ or $(s_1, s_2)(m, 1_{M_2}) \in N_1 \times N_2$. Hence, we have either $s_1 \in (N_1 :_{R_1} M_1) \cap S_1$ or $s_2 \in N_2 \cap S_2 \subseteq (N_2 :_{R_2} M_2) \cap S_2$, a contradiction. Now, we may assume that $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and we show that N_2 is an S_2-prime submodule of M_2. Suppose $am' \in N_2$ for some $a \in R_2$ and $m' \in M_2$. Then $(0_{M_1}, 0_{M_2}) \neq (1_{R_1}, a)(m, m') \in N$ implies either $(s_1, s_2)(1_{R_1}, a) \in (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ or $(s_1, s_2)(m, m') \in N_1 \times N_2$. Thus, $s_2 a \in (N_2 :_{R_2} M_2)$ or $s_2 m' \in N_2$ and so N_2 is an S_2-prime submodule of M_2.

(2)⇒(3). It follows from [18, Theorem 2.14].

(3)⇒(1). It is straightforward. □

Theorem 6. *Let $M = M_1 \times M_2 \times \cdots \times M_n$ be an $R_1 \times R_2 \times \cdots \times R_n$-module and $S = S_1 \times S_2 \times \cdots \times S_n$ where R_i's are rings, S_i is a multiplicatively closed subset of R_i and N_i is a non-zero submodule of M_i for each $i = 1, 2, \ldots, n$. Then the following assertions are equivalent.*

1. *$N = N_1 \times N_2 \times \cdots \times N_n$ is a weakly S-prime submodule of M.*
2. *For $i = 1, 2, \ldots, n$, N_i is an S-prime submodule of M_i and $(N_j :_{R_j} M_j) \cap S_j \neq \emptyset$ for all $j \neq i$.*

Proof. We prove the claim by using mathematical induction on n. The claim follows by Theorem 5 for $n = 2$. Now, we assume that the claim holds for all $k < n$ and prove it for $k = n$. Suppose $N = N_1 \times N_2 \times \cdots \times N_n$ is a weakly S-prime submodule of M. Then Theorem 5 implies that $N = N' \times N_n$ where, say, $N' = N_1 \times N_2 \times \cdots \times N_{n-1}$ is a weakly S-prime submodule of $M' = M_1 \times M_2 \times \cdots \times M_{n-1}$ and $S_n \cap (N_n :_{R_n} M_n) \neq \emptyset$. Thus, the result follows by the induction hypothesis. □

Let M be an R-module and S be a multiplicatively closed subset of R with $S \cap Ann_R(M) = \emptyset$. Following [18], M is called S-torsion-free if there is $s \in S$ such that whenever $rm = 0$ for $r \in R$ and $m \in M$, then $sr = 0$ or $sm = 0$. Compare with [18, Proposition 2.24], we have the following result.

Proposition 10. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an S-torsion-free R-module M. If $\eta : R \to R/(N :_R M)$ is the canonical homomorphism, then N is weakly S-prime in M if and only if M/N is an $\eta(S)$-torsion-free $R/(N :_R M)$-module.*

Proof. First, we clearly note that $s \in S \cap (N :_R M)$ if and only if $\bar{s} \in \eta(S) \cap Ann_{R/(N:_RM)}(M/N)$.

($\Rightarrow$) Suppose N is a weakly S-prime in M with a weakly S-element $s_1 \in S$. Let $\bar{r} \in R/(N :_R M)$, $\bar{m} \in M/N$ such that $\bar{r}\bar{m} = \bar{0}$. Then $rm \in N$ and we have two cases. If $rm = 0$, then by assumption there is $s_2 \in S$ such that $s_2 r = 0$ or $s_2 m = 0$. Thus $\bar{s}_2\bar{r} = \bar{0}$ or $\bar{s}_2\bar{m} = \bar{0}$ where $\bar{s}_2 \in \eta(S)$ as needed. If $rm \neq 0$, then $s_1 r \in (N :_R M)$ or $s_1 m \in N$ and so $\bar{s}_1\bar{r} = \bar{0}_{R/(N:_RM)}$ or $\bar{s}_1\bar{m} = \bar{0}_{M/N}$ where $\bar{s}_1 \in \eta(S)$. Therefore, M/N is an $\eta(S)$-torsion-free $R/(N :_R M)$-module associated to $\bar{s}_1\bar{s}_2 \in \eta(S)$.

($\Leftarrow$) Follows directly by [18, Proposition 2.24]. □

Let R be a ring and M be an R-module. Recall that the idealization of M in R denoted by $R \ltimes M$ is the commutative ring $R \oplus M$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ [17]. For an ideal I of R and a submodule N of M, the set $I \ltimes N = I \oplus N$ is not always an ideal of $R \ltimes M$ and it is an ideal if and only if $IM \subseteq N$ [6, Theorem 3.1]. Among many other properties of an ideal $I \ltimes N$ of $R \ltimes M$, we have $\sqrt{I \ltimes N} = \sqrt{I} \ltimes M$ and in particular, $\sqrt{0 \ltimes 0} = \sqrt{0} \ltimes M$, [6, Theorem 3.2]. It is clear that if S is a multiplicatively closed subset of R and K a submodule of M, then $S \ltimes K = \{(s, k) : s \in S,\, k \in K\}$ is a multiplicatively closed subset of $R \ltimes M$. In [3, Proposition 27], it is proved that if $I \ltimes M$ is a weakly $S \ltimes M$-prime (or weakly $S \ltimes 0$-prime) ideal of $R \ltimes M$ where I is an ideal of R disjoint with S, then I is a weakly S-prime ideal of R. In general, we have:

Theorem 7. *Let S be a multiplicatively closed subset of a ring R, I be an ideal of R and $K \subseteq N$ be submodules of an R-module M with $IM \subseteq N$. Let $I \ltimes N$ be a weakly $S \ltimes K$-prime ideal of $R \ltimes M$. Then*

(1) *I is a weakly S-prime ideal of R and N is a weakly S-prime submodule of M whenever $(N :_R M) \cap S = \emptyset$.*
(2) *There exists $s \in S$ such that for all $a, b \in R$, $ab = 0$, $sa \notin I$, $sb \notin I$ implies $a, b \in ann(N)$ and for all $c \in R$, $m \in M$, $cm = 0$, $sc \notin (N :_R M)$, $sm \notin N$ implies $c \in ann(I)$ and $m \in (0 :_M I)$.*
(3) *If $I \ltimes N$ is not $S \ltimes K$-prime, then $(s,k)(I \ltimes N) = (sI \ltimes 0) \oplus (0 \ltimes sN + Ik)$ for some $(s,k) \in S \ltimes K$.*
(4) *If $I \ltimes N$ is $S \ltimes K$-prime, then $sM \subseteq N$ for some $s \in S$.*

Proof. Let $(s,k) \in S \ltimes K$ be a weakly $S \ltimes K$-element of $I \ltimes N$.

(1) Note that clearly $(S \ltimes K) \cap (I \ltimes N) = \emptyset$ if and only if $I \cap S = \emptyset$. Suppose that $a, b \in R$ with $0 \neq ab \in I$. Then $(0,0) \neq (a,0)(b,0) \in I \ltimes N$ implies that either $(s,k)(a,0) \in I \ltimes N$ or $(s,k)(b,0) \in I \ltimes N$. Thus, either $sa \in I$ or $sb \in I$ and I is weakly S-prime in R. Now, let $0 \neq rm \in N$ for $r \in R$, $m \in M$. Then $(0,0) \neq (r,0)(0,m) \in I \ltimes N$ and so $(sr, rk) = (s,k)(r,0) \in I \ltimes N$ or $(0, sm) = (s,k)(0,m) \in I \ltimes N$. In the first case, we get $sr \in I \subseteq (N :_R M)$ and the second case implies $sm \in N$. Therefore, N is a weakly S-prime submodule of M.

(2) Let $a, b \in R$ such that $ab = 0$ and $sa \notin I$, $sb \notin I$. Suppose $a \notin ann(N)$ so that there exists $n \in N$ such that $an \neq 0$. Thus, $(0,0) \neq (a,0)(b,n) = (0, an) \in I \ltimes N$ and so either $(s,k)(a,0) \in I \ltimes N$ or $(s,k)(b,n) \in I \ltimes N$. Hence, $sa \in I$ or $sb \in I$, a contradiction. Similarly, if $b \notin ann(N)$, then we get a contradiction. Therefore, $a, b \in ann(N)$ as needed. Next, we assume $cm = 0$ for $c \in R$, $m \in M$ and $sc \notin (N :_R M)$, $sm \notin N$. We have two cases.

Case 1. If $c \notin ann(I)$, then there exists $a \in I$ such that $ca \neq 0$. Hence, $(0,0) \neq (c,0)(a,m) = (ca, 0) \in I \ltimes N$ and so $(s,k)(c,0) \in I \ltimes N$ or $(s,k)(a,m) \in I \ltimes N$. Therefore, $sc \in I \subseteq (N :_R M)$ or $sm + ka \in N$ (and so $sm \in N$ as $K \subseteq N$) which contradicts the assumption.

Case 2. If $m \notin (0 :_M I)$, then there exists $a \in I$ such that $am \neq 0$. Thus, $(0,0) \neq (a,m)(c,m) = (ac, am) \in I \ltimes N$ implies either $(s,k)(a,m) \in I \ltimes N$ or $(s,k)(c,m) \in I \ltimes N$. It follows that either $sc \in I \subseteq (N :_R M)$ or $sm \in N$ which is also a contradiction.

(3) If $I \ltimes N$ is not $S \ltimes K$-prime, then $(s,k)(I \ltimes N)(\sqrt{0} \ltimes M) = (0,0)$ for some $(s,k) \in S \ltimes K$ by [3, Proposition 9]. Thus, by [6, Theorem 3.3] $s\sqrt{0}I \ltimes (sIM + s\sqrt{0}N + \sqrt{0}Ik) = (0,0)$. Then clearly $sIM = 0$ and so $sI \ltimes 0$ is an ideal of $R \ltimes M$. Now, $(s,k)(I \ltimes N) = sI \ltimes (sN + Ik) = (sI \ltimes 0) \oplus (0 \ltimes sN + Ik)$ as required.

(4) If $I \ltimes N$ is $S \ltimes K$-prime in $R \ltimes M$, then $(s,k)(\sqrt{0} \ltimes M) \subseteq (I \ltimes N)$ for some $(s,k) \in S \ltimes K$ by [3, Corollary 6]. Thus, $s\sqrt{0} \ltimes (sM + \sqrt{0}k) \subseteq (I \ltimes N)$ and so clearly, $sM \subseteq N$ as needed. □

In general if I is a (weakly) S-prime ideal of a ring R and N a (weakly) S-prime submodule of an R-module M, then $I \ltimes N$ need not be a (weakly) $S \ltimes K$-prime ideal of $R \ltimes M$.

Example 5. Consider the multiplicatively closed subset $S = \{3^n : n \in \mathbb{N}\}$ of $\mathbb{Z}$. While clearly 0 is (weakly) S-prime in $\mathbb{Z}$ and $\langle\bar{2}\rangle$ is (weakly) S-prime in the $\mathbb{Z}$-module $\mathbb{Z}_6$, the ideal $0 \ltimes \langle\bar{2}\rangle$ is not (weakly) $S \ltimes 0$-prime in $\mathbb{Z} \ltimes \mathbb{Z}_6$. Indeed, $(0,0) \neq (0,\bar{1})(2,\bar{1}) = (0,\bar{2}) \in 0 \ltimes \langle\bar{2}\rangle$ but $(s,\bar{0})(0,\bar{1}) \notin 0 \ltimes \langle\bar{2}\rangle$ and $(s,\bar{0})(2,\bar{1}) \notin 0 \ltimes \langle\bar{2}\rangle$ for all $s \in S$.

3. (Weakly) S-prime submodules of amalgamation modules

Let R be a ring, J an ideal of R and M an R-module. We recall that the set

$$R \bowtie J = \{(r, r+j) : r \in R,\ j \in J\}$$

is a subring of $R \times R$ called the amalgamated duplication of R along J, see [10]. Recently, in [9], the duplication of the R-module M along the ideal J denoted by $M \bowtie J$ is defined as

$$M \bowtie J = \{(m, m') \in M \times M : m - m' \in JM\}$$

which is an $(R \bowtie J)$-module with scalar multiplication defined by $(r, r+j).(m, m') = (rm, (r+j)m')$ for $r \in R$, $j \in J$ and $(m, m') \in M \bowtie J$. Many properties and results concerning this kind of modules can be found in [9].

Let N be a submodule of an R-module M and J be an ideal of R. Then clearly

$$N \bowtie J = \{(n, m) \in N \times M : n - m \in JM\}$$

and

$$\bar{N} = \{(m, n) \in M \times N : m - n \in JM\}$$

are submodules of $M \bowtie J$. If S is a multiplicatively closed subset of R, then obviously, the sets

$$S \bowtie J = \{(s, s+j) : s \in S,\ j \in J\} \text{ and } \bar{S} = \{(r, r+j) : r + j \in S\}$$

are multiplicatively closed subsets of $R \bowtie J$.

In general, let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1,\ j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [14], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module with the scalar product defined as

$$(r, f(r) + j)(m_1, \varphi(m_1) + m_2) = (rm_1, \varphi(rm_1) + f(r)m_2 + j\varphi(m_1) + jm_2).$$

For submodules N_1 and N_2 of M_1 and M_2, respectively, clearly the sets

$$N_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : m_1 \in N_1\}$$

and

$$\overline{N_2}^{\varphi} = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2 : \varphi(m_1) + m_2 \in N_2\}$$

are submodules of $M_1 \bowtie^{\varphi} JM_2$. Moreover if S_1 and S_2 are multiplicatively closed subsets of R_1 and R_2, respectively, then

$$S_1 \bowtie^f J = \{(s_1, f(s_1) + j) : s \in S_1,\ j \in J\}$$

and

$$\overline{S_2}^{\varphi} = \{(r, f(r) + j) : r \in R_1,\ f(r) + j \in S_2\}$$

are clearly multiplicatively closed subsets of $M_1 \bowtie^{\varphi} JM_2$.

Note that if $R = R_1 = R_2$, $M = M_1 = M_2$, $f = Id_R$ and $\varphi = Id_M$, then the amalgamation of M_1 and M_2 along J with respect to φ is exactly the duplication of the R-module M along the ideal J. Moreover, in this case, we have $N_1 \bowtie^{\varphi} JM_2 = N \bowtie J$, $\overline{N_2}^{\varphi} = \bar{N}$, $S_1 \bowtie^f J = S \bowtie J$ and $\overline{S_2}^{\varphi} = \bar{S}$.

Theorem 8. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as above. Let S be a multiplicatively closed subset of R_1 and N_1 be a submodule of M_1. Then*

(1) *$N_1 \bowtie^{\varphi} JM_2$ is an $S \bowtie^f J$-prime submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_1 is an S-prime submodule of M_1.*

(2) *$N_1 \bowtie^{\varphi} JM_2$ is a weakly $S \bowtie^f J$-prime submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_1 is a weakly S-prime submodule of M_1 and for $r_1 \in R_1$, $m_1 \in M_1$ with $r_1m_1 = 0$ but $s_1r_1 \notin (N_1 :_{R_1} M_1)$ and $s_1m_1 \notin N_1$ for all $s_1 \in S$, then $f(r_1)m_2 + j\phi(m_1) + jm_2 = 0$ for every $j \in J$ and $m_2 \in JM_2$.*

Proof. We clearly note that $(N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2) \cap S \bowtie^f J = \emptyset$ if and only if $(N_1 :_{R_1} M_1) \cap S = \emptyset$.

(1) Suppose $(s, f(s) + j)$ is an $S \bowtie^f J$-element of $N_1 \bowtie^{\varphi} JM_2$ and let $r_1m_1 \in N_1$ for $r_1 \in R_1$ and $m_1 \in M_1$. Then $(r_1, f(r_1)) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$ with $(r_1, f(r_1))(m_1, \varphi(m_1)) = (r_1m_1, \varphi(r_1m_1)) \in N_1 \bowtie^{\varphi} JM_2$. Thus, either

$$(s, f(s) + j)(r_1, f(r_1)) \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$$

or

$$(s, f(s) + j)(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2.$$

In the first case, for all $m \in M_1$, $(s, f(s)+j)(r_1, f(r_1))(m, \varphi(m)) \in N_1 \bowtie^{\varphi} JM_2$ and so $sr_1M_1 \subseteq N_1$. In the second case, $sm_1 \in N_1$ and so N_1 is an S-prime submodule of M_1. Conversely, let s be an S-element of N_1. Let $(r_1, f(r_1)+j_1) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2$ such that

$$\begin{aligned}&(r_1m_1, \varphi(r_1m_1) + f(r_1)m_2 + j_1\varphi(m_1) + j_1m_2)\\ &= (r_1, f(r_1) + j_1)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2.\end{aligned}$$

Then $r_1m_1 \in N_1$ and hence either $sr_1M_1 \subseteq N_1$ or $sm_1 \in N_1$. If $sr_1M_1 \subseteq N_1$, then clearly $(s, f(s))(r_1, f(r_1) + j_1) \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$

and if $sm_1 \in N_1$, then $(s, f(s))(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$. Therefore, $N_1 \bowtie^{\varphi} JM_2$ is an $S \bowtie^f J$-prime submodule of $M_1 \bowtie^{\varphi} JM_2$ associated to $(s, f(s)) \in S \bowtie^f J$.

(2) Suppose $(s, f(s) + j)$ is a weakly $S \bowtie^f J$-element of $N_1 \bowtie^{\varphi} JM_2$. Let $r_1 \in R_1$ and $m_1 \in M_1$ such that $0 \neq r_1m_1 \in N_1$ so that $(0,0) \neq (r_1, f(r_1))(m_1, \varphi(m_1)) = (r_1m_1, \varphi(r_1m_1)) \in N_1 \bowtie^{\varphi} JM_2$. By assumption, either $(s, f(s)+j)(r_1, f(r_1)) \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(s, f(s) + j)(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$ and so N_1 is S-prime in M_1 as in the proof of (1). Now, we use the contrapositive to prove the other part. Let $r_1 \in R_1$, $m_1 \in M_1$ with $r_1m_1 = 0$ and $f(r_1)m_2 + j\phi(m_1) + jm_2 \neq 0$ for some $j \in J$ and some $m_2 \in JM_2$. Then

$$
\begin{aligned}
(0,0) &\neq (r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \\
&= (0, f(r_1)m_2 + j\varphi(m_1) + jm_2) \in N_1 \bowtie^{\varphi} JM_2.
\end{aligned}
$$

By assumption, either $(s, f(s)+j)(r_1, f(r_1)+j) \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(s, f(s) + j)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$ and so again $sr_1 \in (N_1 :_{R_1} M_1)$ or $sm_1 \in N_1$ as needed. Conversely, let s be a weakly S-element of N_1 and let $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2$ such that

$$
\begin{aligned}
(0,0) &\neq (r_1m_1, \varphi(r_1m_1) + f(r_1)m_2 + j\varphi(m_1) + jm_2) \\
&= (r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2.
\end{aligned}
$$

If $0 \neq r_1m_1$, then the proof is similar to that of (1). Suppose $r_1m_1 = 0$. Then $f(r_1)m_2 + j\varphi(m_1) + jm_2 \neq 0$ and so by assumption there exists $s' \in S$ such that either $s'r_1 \in (N_1 :_{R_1} M_1)$ or $s'm_1 \in N_1$. Thus, $(s', f(s'))(r_1, f(r_1) + j) \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(s', f(s'))(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$. Therefore, $N_1 \bowtie^{\varphi} JM_2$ is a weakly $S \bowtie^f J$-prime submodule of $M_1 \bowtie^{\varphi} JM_2$ associated to $(ss', f(ss')) \in S \bowtie^f J$. □

In particular, if S is a multiplicatively closed subset of R_1, then $S \times f(S)$ is a multiplicatively closed subset of $R_1 \bowtie^f J$. Moreover, one can similarly prove Theorem 8 if we consider $S \times f(S)$ instead of $S \bowtie^f J$.

Corollary 4. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as in Theorem 8 and let N_1 be a submodule of M_1. Then*

(1) *$N_1 \bowtie^{\varphi} JM_2$ is a prime submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_1 is a prime submodule of M_1.*
(2) *$N_1 \bowtie^{\varphi} JM_2$ is a weakly prime submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_1 is a weakly prime submodule of M_1 and for $r_1 \in R_1$, $m_1 \in M_1$ with $r_1m_1 = 0$ but $r_1 \notin (N_1 :_{R_1} M_1)$ and $m_1 \notin N_1$, then $f(r_1)m_2 + j\phi(m_1) + jm_2 = 0$ for every $j \in J$ and $m_2 \in JM_2$.*

Proof. We just take $S = \{1_{R_1}\}$ (and so $S \times f(S) = \{(1_{R_1}, 1_{R_2})\}$) and use Theorem 8. □

Theorem 9. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as in Theorem 8 where f and φ are epimorphisms. Let S be a multiplicatively closed subset of R_2 and N_2 be a submodule of M_2. Then*

(1) *N_2 is an S-prime submodule of M_2 if and only if $\overline{N_2}^\varphi$ is an $\overline{S}^\varphi$-prime submodule of $M_1 \bowtie^\varphi JM_2$.*

(2) *If $\overline{N_2}^\varphi$ is an $\overline{S}^\varphi$-prime submodule of $M_1 \bowtie^\varphi JM_2$, and $(N_2 :_{R_2} JM_2) \cap S = \emptyset$, then $(N_2 :_{M_2} J)$ is an S-prime submodule of M_2.*

Proof. (1) We note that $(\overline{N_2}^\varphi :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2) \cap \overline{S}^\varphi = \emptyset$ if and only if $(N_2 :_{R_2} M_2) \cap S = \emptyset$. Indeed if $(t, f(t)+j) = (t, s) \in \overline{S}^\varphi$ such that $(t, s)(M_1 \bowtie^\varphi JM_2) \subseteq \overline{N_2}^\varphi$, then for each $m_2 = \varphi(m_1) \in M_2$, we have $(t, s)(m_1, m_2) \in \overline{N_2}^\varphi$. Therefore, $sm_2 \in N_2$ and $s \in (N_2 :_{R_2} M_2)$. The converse is similar.

Suppose N_2 is an S-prime submodule of M_2 associated to $s = f(t) \in S$. Let $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie JM_2$ such that

$$(r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^\varphi.$$

Then $(f(r_1) + j)(\varphi(m_1) + m_2) \in N_2$ and so $s(f(r_1) + j) \in (N_2 :_{R_2} M_2)$ or $s(\varphi(m_1) + m_2) \in N_2$. If $s(f(r_1) + j) \in (N_2 :_{R_2} M_2)$, then for all $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2$, clearly $(t, s)(r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^\varphi$ and so $(t, s)(r_1, f(r_1) + j) \in (\overline{N_2}^\varphi :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2)$. If $s(\varphi(m_1) + m_2) \in N_2$, then $(t, s)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^\varphi$ and the result follows. Conversely, suppose $\overline{N_2}^\varphi$ is an $\overline{S}^\varphi$-prime submodule of $M_1 \bowtie^\varphi JM_2$ associated to $(t, f(t)+j) = (t, s) \in \overline{S}^\varphi$. Let $r_2 = f(r_1) \in R_2$ and $m_2 = \varphi(m_1) \in M_2$ such that $r_2 m_2 \in N_2$. Then $(r_1, r_2) \in R_1 \bowtie^f J$ and $(m_1, m_2) \in M_1 \bowtie^\varphi JM_2$ with $(r_1, r_2)(m_1, m_2) \in \overline{N_2}^\varphi$. Thus, $(t, s)(r_1, r_2)(M_1 \bowtie^\varphi JM_2) \subseteq \overline{N_2}^\varphi$ or $(t, s)(m_1, m_2) \in \overline{N_2}^\varphi$. If $(t, s)(r_1, r_2)(M_1 \bowtie^\varphi JM_2) \subseteq \overline{N_2}^\varphi$, then for all $m = \varphi(m') \in M_2$, we have $(t, s)(r_1, r_2)(m', m) \in \overline{N_2}^\varphi$ and so $sr_2 M_2 \subseteq N_2$. If $(t, s)(m_1, m_2) \in \overline{N_2}^\varphi$, then $sm_2 \in N_2$ and we are done.

(2) Suppose $\overline{N_2}^\varphi$ is an $\overline{S}^\varphi$-prime submodule of $M_1 \bowtie^\varphi JM_2$ associated to $(t, f(t)+j') = (t, s) \in \overline{S}^\varphi$. Let $r_2 \in R_2$, $m_2 \in M_2$ such that $r_2 m_2 \in (N_2 :_{M_2} J)$. Then $r_2 J m_2 \subseteq N_2$ and so for all $j \in J$, we have $(r_1, f(r_1))(0, jm_2) \in \overline{N_2}^\varphi$ where $f(r_1) = r_2$. By assumption, $(t, s)(r_1, r_2) \in (\overline{N_2}^\varphi :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2)$ or $(t, s)(0, jm_2) \in \overline{N_2}^\varphi$. If $(t, s)(r_1, r_2) \in (\overline{N_2}^\varphi :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2)$, then for all $m_2 \in M_2$ and all $j \in J$, we have $(t, s)(r_1, r_2)(0, jm_2) \in \overline{N_2}^\varphi$ and so $sr_2 jm_2 \in N_2$. Thus, $sr_2 \in (N_2 :_{R_2} JM_2) = ((N_2 :_{M_2} J) :_{R_2} M_2)$. If $(t, s)(r_1, r_2) \notin (\overline{N_2}^\varphi :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2)$, then $(t, s)(0, jm_2) \in \overline{N_2}^\varphi$ for all $j \in J$ and so $sm_2 \in (N_2 :_{M_2} J)$ as required. □

In particular, if we consider $S = \{1_{R_2}\}$ and take $T = \{(1_{R_1}, 1_{R_2})\}$ instead of $\overline{S}^\varphi$ in Theorem 9, then we get the following corollary.

Corollary 5. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as in Theorem 8 where f and φ are epimorphisms and let N_2 be a submodule of M_2. Then*

(1) *N_2 is a prime submodule of M_2 if and only if $\overline{N_2}^{\varphi}$ is a prime submodule of $M_1 \bowtie^{\varphi} JM_2$.*

(2) *If $\overline{N_2}^{\varphi}$ is a prime submodule of $M_1 \bowtie^{\varphi} JM_2$ and $J \nsubseteq (N_2 :_{R_2} M_2)$, then $(N_2 :_{M_2} J)$ is a prime submodule of M_2.*

Theorem 10. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as in Theorem 8 where f and φ are epimorphisms. Let S be a multiplicatively closed subset of R_2 and N_2 be a submodule of M_2. Then*

(1) *$\overline{N_2}^{\varphi}$ is a weakly $\overline{S}^{\varphi}$-prime submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_2 is a weakly S-prime submodule of M_2 and for $r_1 \in R_1$, $m_1 \in M_1$, $m_2 \in JM_2$, $j \in J$ with $(f(r_1)+j)(\varphi(m_1)+m_2) = 0$ but $s(f(r_1)+j) \notin (N_2 :_{R_2} M_2)$ and $s(\varphi(m_1)+m_2) \notin N_2$ for all $s \in S$, then $r_1m_1 = 0$.*

(2) *If $\overline{N_2}^{\varphi}$ is a weakly $\overline{S}^{\varphi}$-prime submodule of $M_1 \bowtie^{\varphi} JM_2$, $(N_2 :_{R_2} JM_2) \cap S = \emptyset$ and $Z_{R_2}(M_2) \cap J = \{0\}$, then $(N_2 :_{M_2} J)$ is a weakly S-prime submodule of M_2.*

Proof. (1) Suppose $s = f(t) \in S$ is a weakly S-element of N_2. Let $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2$ such that

$$(0,0) \neq (r_1, f(r_1)+j)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^{\varphi}.$$

Then $(f(r_1)+j)(\varphi(m_1)+m_2) \in N_2$. If $(f(r_1)+j)(\varphi(m_1)+m_2) \neq 0$, then the result follows as in the proof of (1) in Theorem 9. Suppose $(f(r_1)+j)(\varphi(m_1)+m_2) = 0$ so that $r_1m_1 \neq 0$. Then by assumption, there exists $s' = f(t') \in S$ such that $s'(f(r_1)+j) \in (N_2 :_{R_2} M_2)$ or $s'(\varphi(m_1)+m_2) \in N_2$. It follows clearly that $(t', s')(r_1, f(r_1)+j) \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(t', s')(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^{\varphi}$. Hence, (tt', ss') is a weakly $\overline{S}^{\varphi}$-element of $\overline{N_2}^{\varphi}$. Conversely, let $(t, f(t)+j) = (t, s)$ be a weakly $\overline{S}^{\varphi}$-element of $\overline{N_2}^{\varphi}$. Let $r_2 = f(r_1) \in R_2$ and $m_2 = f(m_1) \in M_2$ such that $0 \neq r_2m_2 \in N_2$. Then $(r_1, r_2) \in R_1 \bowtie^f J$ and $(m_1, m_2) \in M_1 \bowtie^{\varphi} JM_2$ with $(0.0) \neq (r_1, r_2)(m_1, m_2) \in \overline{N_2}^{\varphi}$. Hence, either $(t, s)(r_1, r_2) \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(t, s)(m_1, m_2) \in \overline{N_2}^{\varphi}$. In the first case, for all $m = \varphi(m') \in M_2$, $(tr_1, sr_2)(m', m) \in \overline{N_2}^{\varphi}$. Hence, $sr_2m \in N_2$ and then $sr_2 \in (N_2 :_{R_2} M_2)$. In the second case, we have $sm_2 \in N_2$ and so s is a weakly S-element of N_2. Now, let $r_1 \in R_1$, $m_1 \in M_1$, $m_2 \in JM_2$, $j \in J$ with $(f(r_1)+j)(\varphi(m_1)+m_2) = 0$ and suppose $r_1m_1 \neq 0$. Then $(0,0) \neq (r_1, f(r_1)+j)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^{\varphi}$ and so $(t, s)(r_1, f(r_1)+j) \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie JM_2)$ or $(t, s)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^{\varphi}$. Hence, clearly, either $s(f(r_1)+j) \in (N_2 :_{R_2} M_2)$ or $s(\varphi(m_1)+m_2) \in N_1$ and the result follows by contrapositive.

(2) Suppose $(t, f(t)+j) = (t, s)$ is a weakly $\overline{S}^{\varphi}$-element of $\overline{N_2}^{\varphi}$. Let $r_2 = f(r_1) \in R_2$, $m_2 \in M_2$ such that $0 \neq r_2m_2 \in (N_2 :_{M_2} J)$. Then $r_2Jm_2 \subseteq N_2$ and so for all $j \in J$, we have $(r_1, r_2)(0, jm_2) \in \overline{N_2}^{\varphi}$. If $j \neq 0$ and $(r_1, r_2)(0, jm_2) = (0,0)$, then $r_2jm_2 = 0$ and so $r_2m_2 = 0$ as $Z_{R_2}(N_2) \cap J = \{0\}$, a contradiction. Thus, for all $j \neq 0$, $(r_1, r_2)(0, jm_2) \neq (0,0)$. By assumption and similar to the proof of (2) of Theorem 9, we have for all $j \neq 0$, either $sr_2jm_2 \in N_2$ or

$(t,s)(0,jm_2) \in \overline{N_2}^{\varphi}$ for all $m_2 \in M_2$. Thus, $sr_2 \in (N_2 :_{R_2} JM_2) = ((N_2 :_{M_2} J) :_{R_2} M_2)$ or $sm_2 \in (N_2 :_{M_2} J)$ and we are done. □

Corollary 6. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as in Theorem 8 where f and φ are epimorphisms. If N_2 is a submodule of M_2, then*

(1) *$\overline{N_2}^{\varphi}$ is a weakly prime submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_2 is a weakly prime submodule of M_2 and for $r_1 \in R_1$, $m_1 \in M_1$, $m_2 \in JM_2$, $j \in J$ with $(f(r_1)+j)(\varphi(m_1)+m_2) = 0$ but $(f(r_1)+j) \notin (N_2 :_{R_2} M_2)$ and $(\varphi(m_1)+m_2) \notin N_2$, then $r_1 m_1 = 0$.*

(2) *If $\overline{N_2}^{\varphi}$ is a weakly prime submodule of $M_1 \bowtie^{\varphi} JM_2$, $J \nsubseteq (N_2 :_{R_2} M_2)$ and $Z_{R_2}(N_2) \cap J = \{0\}$, then $(N_2 :_{M_2} J)$ is a weakly prime submodule of M_2.*

Corollary 7. *Let N be a submodule of an R-module M, J an ideal of R and S a multiplicatively closed subset of R. Then*

(1) *$N \bowtie J$ is an $(S \bowtie J)$-prime submodule of $M \bowtie J$ if and only if N is an S-prime submodule of M.*

(2) *$N \bowtie J$ is a weakly $(S \bowtie J)$-prime submodule of $M \bowtie J$ if and only if N is a weakly S-prime submodule of M and for $r \in R$, $m \in M$ with $rm = 0$ but $sr \notin (N :_{R_1} M)$ and $sm \notin N$ for all $s \in S$, then $(r+j)m' = 0$ for every $j \in J$ and $m' \in M$ where $(m, m') \in M \bowtie J$.*

Corollary 8. *Let N be a submodule of an R-module M, J an ideal of R and S a multiplicatively closed subset of R. Then*

(1) *N is an S-prime submodule of M if and only if $\overline{N}$ is an $\overline{S}$-prime submodule of $M \bowtie J$.*

(2) *If $\overline{N}$ is an $\overline{S}$-prime submodule of $M \bowtie J$ and $(N :_R JM) \cap S = \emptyset$, then $(N :_M J)$ is an S-prime submodule of M.*

Corollary 9. *Let N be a submodule of an R-module M, J an ideal of R and S a multiplicatively closed subset of R. Then*

(1) *$\overline{N}$ is a weakly $\overline{S}$-prime submodule of $M \bowtie J$ if and only if N is a weakly S-prime submodule of M and for $r \in R$, $m \in M$, $m' \in JM$, $j \in J$ with $(r+j)(m+m') = 0$ but $s(r+j) \notin (N :_R M)$ and $s(m+m') \notin N$ for all $s \in S$, then $rm = 0$.*

(2) *If $\overline{N}$ is a weakly $\overline{S}$-prime submodule of $M \bowtie J$, $(N :_M J) \cap S = \emptyset$ and $Z_R(N) \cap J = \{0\}$, then $(N :_M J)$ is a weakly S-prime submodule of M.*

In the following example, we show that in general N being a weakly S-prime submodule of M does not imply $N \bowtie J$ is a weakly $(S \bowtie J)$-prime submodule of $M \bowtie J$.

Example 6. Consider the $\mathbb{Z}$-submodule $N = 0 \times \langle \bar{0} \rangle$ of $M = \mathbb{Z} \times \mathbb{Z}_6$ and let $J = 2\mathbb{Z}$. Then N is a weakly prime submodule of M. Now

$$M \bowtie J = \{(m, m') \in M \times M : m - m' \in JM = 2\mathbb{Z} \times \langle \bar{2} \rangle\}$$

and

$$N \bowtie J = \{(n, m) \in N \times M : n - m \in 2\mathbb{Z} \times \langle \bar{2} \rangle\}.$$

If we consider $(2, 4) \in \mathbb{Z} \bowtie J$ and $((0, \bar{3}), (0, \bar{1})) \in M \bowtie J$, then we have $(2, 4) \cdot ((0, \bar{3}), (0, \bar{1})) = ((0, \bar{0}), (0, \bar{4})) \in N \bowtie J$. But we have $(2, 4) \notin ((N \bowtie J) :_{\mathbb{Z} \bowtie I} (M \bowtie J))$ as for example $(2, 4)((2, \bar{2}), (0, \bar{0})) \notin N \bowtie J$ and $((0, \bar{3}), (0, \bar{1})) \notin N \bowtie J$. Thus, $N \bowtie J$ is not a weakly prime submodule of $M \bowtie J$.

We note that the condition in the reverse implication of Corollary 7(2) does not hold in Example 6. For example, if we take $r = 2$ and $m = (0, \bar{3}) \in M$, then clearly, $rm = 0$, $r \notin (N :_R M) = 0$ and $m \notin N$ but for $m' = (0, \bar{2}) \in JM = 2\mathbb{Z} \times \langle \bar{2} \rangle$, we have $(r + 0)m' \neq 0$.

Also, if the condition in the reverse implication of Corollary 9(1) does not hold, then we may find a weakly S-prime submodule N of M such that $\overline{N}$ is not a weakly $\overline{S}$-prime submodule of $M \bowtie J$.

Example 7. Consider N, M and J as in Example 6. If we consider $(2, 4) \in \mathbb{Z} \bowtie J$ and $((0, \bar{1}), (0, \bar{3})) \in M \bowtie J$, then we have $(2, 4) \cdot ((0, \bar{1}), (0, \bar{3})) = \overline{N}$. But $(2, 4) \notin (\bar{N} :_{\mathbb{Z} \bowtie I} (M \bowtie J))$ and $((0, \bar{1}), (0, \bar{3})) \notin \overline{N}$. Thus, $\bar{N}$ is not a weakly prime submodule of $M \bowtie J$.

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