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HOPF HYPERSURFACES OF THE HOMOGENEOUS NEARLY KÄHLER $\mathbb{S}^3 \times \mathbb{S}^3$ SATISFYING CERTAIN COMMUTING CONDITIONS

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Abstract. In this article, we first introduce the notion of commuting Ricci tensor and pseudo-anti commuting Ricci tensor for Hopf hypersurfaces in the homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ and prove that the mean curvature of hypersurface is constant under certain assumptions. Next, we prove the nonexistence of Ricci soliton on Hopf hypersurface with potential Reeb vector field, which improves a result of Hu et al. on the nonexistence of Einstein Hopf hypersurfaces in the homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$.

1. Introduction

Recall that a nearly Kähler (abbrev. NK) manifold is an almost Hermitian manifold $(\widetilde{M}, \widetilde{g}, J)$ such that the covariant derivative of the almost complex structure J is skew-symmetry, i.e., $(\widetilde{\nabla}_X J)X = 0$ for all $X \in T\widetilde{M}$. The six-dimensional nearly Kähler manifolds are particularly important according to the result of Nagy's classification [19] that all complete simply connected nearly Kähler manifolds are products of twistor spaces of quaternionic Kähler manifolds of positive scalar curvature, homogeneous spaces and six-dimensional nearly Kähler manifolds. Butruille [6] showed that the only homogeneous six-dimensional NK manifolds are the six-sphere $\mathbb{S}^6$, the $\mathbb{S}^3 \times \mathbb{S}^3$, the complex projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$. Moreover, Foscolo and Haskins [11] constructed inhomogeneous NK structures on both $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$. In order to avoid confusion, from now on, when we say NK $\mathbb{S}^3 \times \mathbb{S}^3$, we always mean the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$.

Due to the research of Bolton et al. [7], people became more interested in the study of submanifolds in homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$, and very rich results were obtained, referring to Lagrangian, CR submanifolds [1–3,9,17] and almost complex surfaces [7,16]. In particular, there are many results for the classification of

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real hypersurfaces under certain assumptions, such as the commutativity condition, the anticommutative structure tensors, $\mathcal{P}$-isotropic normal and so on, see [10,12–15]. On the other hand, we notice that some of the above mentioned assumptions have been applied in the real hypersurfaces of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and complex quadrics. For example, Berndt-Suh [4] studied the real hypersurfaces of $G_2(\mathbb{C}^{m+2})$ with commuting shape operator and showed that the hypersurface is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In 2006, Suh [20] proved that there do not exist any anti-commuting real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature. More results can be found in references [5,18,21–23].

The Kähler structure J on NK $\mathbb{S}^3 \times \mathbb{S}^3$ induces a structure vector field U called *Reeb vector field* on M denoted by $U := -J\xi$, where ξ is the local unit normal vector field of M in $\mathbb{S}^3 \times \mathbb{S}^3$. If the Reeb vector field U is invariant under the shape operator, i.e., $AU = \alpha U$ for a certain function α, M is said to be a *Hopf hypersurface*. For an Einstein Hopf hypersurface in NK $\mathbb{S}^3 \times \mathbb{S}^3$, the following nonexistence was proved.

Theorem 1.1 ([12, Theorem 1.2]). *The homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ admits no Einstein Hopf hypersurface.*

Since the above nonexistence, we intend to study the characteristics of Hopf hypersurfaces of NK $\mathbb{S}^3 \times \mathbb{S}^3$ by weakening the Einstein condition. We investigate the classification of hypersurfaces in the complex quadric and complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ in the series of articles by Suh and Jeong [18,21,24,25], where they considered the so-called commuting Ricci tensor and pseudo-anti commuting Ricci tensor to weaken the Einstein condition. Noticing the nonexistence of Einstein Hopf hypersurfaces of NK $\mathbb{S}^3 \times \mathbb{S}^3$, in the present paper we shall introduce the notions of commuting Ricci tensor and pseudo-anti commuting Ricci tensor. The Ricci tensor is *commuting* if the following formula holds:

$$\phi S = S\phi, \tag{1}$$

where ϕ is an almost contact structure induced by the Kähler structure J on NK $\mathbb{S}^3 \times \mathbb{S}^3$ and S is the Ricci operator of M defined by $g(SX, Y) = Ric(X, Y)$ for all $X, Y \in TM$, and the Ricci tensor is said to be *pseudo-anti commuting* if

$$\phi S + S\phi = 2\kappa\phi, \tag{2}$$

where $\kappa \neq 0$ is a constant. We obtain the following theorems.

Theorem 1.2. *Let M be a Hopf hypersurface of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ with commuting Ricci tensor. Then the mean curvature $H = 2\alpha$. In particular, if $P\{U\}^\perp = \{U\}^\perp$, the mean curvature is constant, where P is the almost product structure of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$.*

Theorem 1.3. *Let M be a Hopf hypersurfaces of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ with pseudo-anti commuting Ricci tensor. Then the mean curvature $H = \alpha$ is constant if $\kappa > \frac{19}{12}$.*

Furthermore, another well-known generalization of Einstein metric is so called *Ricci soliton* defined by

$$\frac{1}{2}\mathcal{L}_V g + Ric = \rho g, \quad \rho = constant,$$

which is a solution of the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2Ric(g(t))$. V is called the *potential vector field*. If the potential vector field V vanishes or is Killing, i.e., $\mathcal{L}_V g = 0$, a Ricci soliton becomes an Einstein metric. Also, we notice that when V is the Reeb vector field U, it satisfies the pseudo-anti commuting condition $\phi S + S\phi = 2\kappa\phi$ with $\kappa = \rho$ (by Eqs. (13) and (8) in Section 2), but the reverse is not true.

For the Ricci soliton on Hopf hypersurfaces, we have the following nonexistence.

Theorem 1.4. *There does not exist Ricci soliton on Hopf hypersurfaces of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ with the potential Reeb field U.*

Remark 1.5. Since the Ricci soliton is a generalization of Einstein metric, our result can be viewed as the improvement of Theorem 1.1.

This article is organized as follows: In Section 2, in order to prove these conclusions, we need some notations and formulas for real hypersurfaces in homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$. The proof of Theorem 1.2, Theorem 1.3 and Theorem 1.4 are given in Section 3, Section 4 and Section 5, respectively.

2. Preliminaries

2.1. The homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$

In this section we first recall some notions and results from [7]. Denote by $\mathbb{S}^3$ the 3-sphere of $\mathbb{R}^4$ as the set of all unitary quaternions. For any $(p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$, by the natural identification $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$ we can write a tangent vector $Z(p, q) = (V_{p,q}, W_{p,q})$ or simply $Z = (V, W)$. On $\mathbb{S}^3 \times \mathbb{S}^3$, there is an almost complex structure J defined by

$$JZ(p, q) = \frac{1}{\sqrt{3}}(2pq^{-1}W - V, -2qp^{-1}V + W).$$

We can define a Hermitian metric $\widetilde{g}$ compatible with J as

$$\begin{aligned}\widetilde{g}(Z, Z') &= \frac{1}{2}(\langle Z, Z'\rangle + \langle JZ, JZ'\rangle)\\ &= \frac{4}{3}(\langle V, V'\rangle + \langle W, W'\rangle) - \frac{2}{3}(\langle p^{-1}V, q^{-1}W'\rangle + \langle p^{-1}V', q^{-1}W\rangle),\end{aligned}$$

where $Z = (V, W), Z' = (V', W')$ are tangent vector fields, and $\langle\cdot,\cdot\rangle$ is the standard product metric on $\mathbb{S}^3 \times \mathbb{S}^3$. Thus $(\widetilde{g}, J)$ gives the homogeneous nearly Kähler structure on $\mathbb{S}^3 \times \mathbb{S}^3$.

Let $\widetilde{\nabla}$ be the connection on NK $\mathbb{S}^3 \times \mathbb{S}^3$ with respect to $\widetilde{g}$ and we define a $(1,2)$ tensor $G(X,Y)$ by $G(X,Y) := (\widetilde{\nabla}_X J)Y$ for $X, Y \in T(\mathbb{S}^3 \times \mathbb{S}^3)$. For the tensor G the following relations hold:

$$G(X,Y) + G(Y,X) = 0, \tag{3}$$

$$G(X,JY) + JG(Y,X) = 0, \tag{4}$$

$$\widetilde{g}(G(X,Y),Z) + \widetilde{g}(G(X,Z),Y) = 0, \tag{5}$$

$$\begin{aligned}\widetilde{g}(G(X,Y),G(Z,W)) = \frac{1}{3}[&g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\\ &+ g(JX,Z)g(Y,JW) - g(JX,W)g(Y,JZ)].\end{aligned} \tag{6}$$

An almost product structure P on $\mathbb{S}^3 \times \mathbb{S}^3$ is introduced by

$$PZ = (pq^{-1}V, qp^{-1}W), \quad \forall\, Z = (V,W) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3).$$

One can check easily P satisfies the following relations:

$$\begin{aligned}&P^2 = Id, \quad JP = -PJ, \quad \widetilde{g}(PX,PY) = \widetilde{g}(X,Y),\\ &2(\widetilde{\nabla}_X P)Y = JG(X,PY) + JPG(X,Y),\\ &PG(X,Y) + G(PX,PY) = 0.\end{aligned}$$

The curvature $\widetilde{R}$ of the NK $\mathbb{S}^3 \times \mathbb{S}^3$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = {}&\frac{5}{12}[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y]\\ &+ \frac{1}{12}\{\widetilde{g}(JY,Z)JX - \widetilde{g}(JX,Z)JY - 2\widetilde{g}(JX,Y)JZ\}\\ &+ \frac{1}{3}\{\widetilde{g}(PY,Z)PX - \widetilde{g}(PX,Z)PY\\ &+ \widetilde{g}(JPY,Z)JPX - \widetilde{g}(JPX,Z)JPY\}.\end{aligned}$$

2.2. Hopf hypersurfaces of NK $\mathbb{S}^3 \times \mathbb{S}^3$

Let M be an immersed real hypersurface of the NK $\mathbb{S}^3 \times \mathbb{S}^3$ with induced metric g. There exists a local defined unit normal vector field ξ on M and we write $U := -J\xi$ as the structure vector field of M. There exist two induced one-forms η and μ, which are defined, respectively, by $\eta(\cdot) = \widetilde{g}(J\cdot,\xi)$ and $\mu(\cdot) = \widetilde{g}(P\cdot,\xi)$. For any vector field X on M, the tangent parts of JX and PX are, respectively, denoted by

$$\phi X = JX - \eta(X)\xi \quad \text{and} \quad TX = PX - \mu(X)\xi. \tag{7}$$

Moreover, the following identities hold:

$$\begin{aligned}&\phi^2 = -Id + \eta \otimes U, \quad \eta \circ \phi = 0, \quad \phi \circ U = 0, \quad \eta(U) = 1,\\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,U) = \eta(X),\\ &g(TX,Y) = g(X,TY),\end{aligned} \tag{8}$$

where $X, Y \in \mathfrak{X}(M)$. By these formulas, we know that (ϕ, η, U, g) is an almost contact metric structure on M.

Denote by ∇ and A the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weigarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi, \quad \widetilde{\nabla}_X \xi = -AX. \tag{9}$$

The curvature tensor R and Codazzi equation of M are, respectively, given as follows:

$$\begin{aligned} R(X,Y)Z = {} & \frac{5}{12}\Big[g(Y,Z)X - g(X,Z)Y\Big] \\ & + \frac{1}{12}\Big[g(JY,Z)\phi X - g(JX,Z)\phi Y + 2g(X,JY)\phi Z\Big] \\ & + \frac{1}{3}\Big[g(PY,Z)(PX)^{\top} - g(PX,Z)(PY)^{\top} \\ & + g(JPY,Z)(JPX)^{\top} - g(JPX,Z)(JPY)^{\top}\Big] \\ & + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{10}$$

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = {} & \frac{1}{12}\Big[g(X,U)\phi Y - g(Y,U)\phi X - 2g(JX,Y)U\Big] \\ & + \frac{1}{3}\Big[\mu(X)TY - \mu(Y)TX \\ & + g(TX,U)(\phi TY - \mu(Y)U) \\ & - g(TY,U)(\phi TX - \mu(X)U)\Big], \end{aligned} \tag{11}$$

where $\cdot^{\top}$ means the tangential part.

In view of (10), the Ricci tensor is given by

$$\begin{aligned} Ric(X,Y) = {} & \frac{5}{4}g(X,Y) - \frac{1}{4}\eta(X)\eta(Y) + \frac{1}{3}\Big[g(TU,U)g(TX,Y) \\ & + \mu(X)\mu(Y) + \mu(U)g(TX,\phi Y) + \mu(U)\mu(X)\eta(Y) \\ & + g(TX,U)g(TY,U)\Big] + Hg(AX,Y) - g(A^2X,Y), \end{aligned}$$

where $H = trace(A)$ is the mean curvature of M. Thus the Ricci operator S may be expressed as

$$\begin{aligned} SX = {} & \frac{5}{4}X - \frac{1}{4}\eta(X)U + \frac{1}{3}\Big[g(TU,U)TX + \mu(X)\mu^{\sharp} - \mu(U)\phi TX \\ & + \mu(U)\mu(X)U + g(TX,U)TU\Big] + HAX - A^2X \end{aligned} \tag{12}$$

for all $X \in TM$, where $\mu^{\sharp}$ denotes the dual vector field of μ with respect to g.

Proposition 2.1 ([12]). *For a real hypersurface M of NK $\mathbb{S}^3\times\mathbb{S}^3$, the following formulas hold*:

$$\nabla_X U = -G(X,\xi)+\phi AX, \tag{13}$$

$$(\nabla_X\phi)Y = G(X,Y)^\top - g(AX,Y)U+\eta(Y)AX, \tag{14}$$

$$\begin{aligned}(\nabla_X T)Y = &\frac{1}{2}[JG(X,PY)+JPG(X,Y)]^\top\\ &- g(AX,Y)\phi TU + g(AX,Y)\mu(U)U+\mu(Y)AX,\end{aligned} \tag{15}$$

$$\begin{aligned}(\nabla_X\mu)Y = &\frac{1}{2}g(G(X,PY)+PG(X,Y),U)-g(AX,TY)\\ &- g(AX,Y)g(TU,U).\end{aligned} \tag{16}$$

If M is a Hopf hypersurface in NK $\mathbb{S}^3\times\mathbb{S}^3$, i.e., $AU=\alpha U$ for a smooth function α on M, then taking inner product of the Codazzi equation (11) with U and a straightforward computation, we obtain:

Proposition 2.2 ([14, Lemma 2.1]). *If M is a Hopf hypersurface of NK $\mathbb{S}^3\times\mathbb{S}^3$, then it holds that*

$$\begin{aligned}&\frac{1}{6}g(\phi X,Y)-\frac{2}{3}\Big[g(PX,\xi)g(PY,U)-g(PX,U)g(PY,\xi)\Big]\\ &= g((\alpha I-A)G(X,\xi),Y)+g(G((\alpha I-A)X,\xi),Y)\\ &\quad -\alpha g((A\phi+\phi A)X,Y)+2g(A\phi AX,Y)\end{aligned} \tag{17}$$

for any vector fields $X,Y\in\{U\}^\perp$.

In order to choose a suitable frame of hypersurface, following [15], we define

$$\mathfrak{D}(p):=\mathrm{Span}\{\xi(p),U(p),P\xi(p),PU(p)\},\quad p\in M.$$

It is clear that $\mathfrak{D}$ defines a distribution with dimension 2 or 4 since J is anti-commuting with P, and that it is invariant under both J and P. Denote by $\mathfrak{D}^\perp$ the orthogonal complimentary distribution of $\mathfrak{D}$, which is a 2-dimensional subdistribution and also invariant under both J and P.

Case (I). If $\dim\mathfrak{D}=4$ holds in an open set, then there exist a unit tangent vector field $e_1\in\mathfrak{D}$ and functions a,b,c with $c>0$ such that

$$P\xi = a\xi+bU+ce_1,\quad a^2+b^2+c^2=1. \tag{18}$$

Put $e_2=Je_1$, $e_3=\sqrt{3}c^{-1}G(U,PU)$, $e_4=Je_3$ and $e_5=U$, then $\{e_i\}_{i=1}^5$ forms a well-defined orthonormal frame field of M with the following properties:

$$\begin{cases} P\xi = a\xi+ce_1+be_5, & Pe_1 = c\xi-ae_1-be_2,\\ Pe_2 = ce_5-be_1+ae_2, & Pe_3=e_3,\\ Pe_4=-e_4, & Pe_5 = b\xi+ce_2-ae_5.\end{cases} \tag{19}$$

Denote $e_6=\xi$, we obtain the matrix $(G_{ij})=(G(e_i,e_j))_{5\times 6}$:

$$(G_{ij})=\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & e_6 & e_5 & -e_4 & -e_3\\ 0 & 0 & e_5 & -e_6 & -e_3 & e_4\\ -e_6 & -e_5 & 0 & 0 & e_2 & e_1\\ -e_5 & e_6 & 0 & 0 & e_1 & -e_2\\ e_4 & e_3 & -e_2 & -e_1 & 0 & 0\end{pmatrix}. \tag{20}$$

From the second term of (7) and (19), we infer to

$$\begin{cases} Te_1=-ae_1-be_2, & \mu(e_1)=c,\\ Te_2=ce_5-be_1+ae_2, & \mu(e_2)=0,\\ Te_3=e_3, & \mu(e_3)=0,\\ Te_4=-e_4, & \mu(e_4)=0,\\ Te_5=ce_2-ae_5, & \mu(e_5)=b.\end{cases} \tag{21}$$

Thus from (12) we have

$$Se_1=\frac{5}{4}e_1+\frac{1}{3}\Big[(a^2-b^2)e_1+c\mu^\sharp+2bae_2+bce_5\Big]+HAe_1-A^2e_1, \tag{22}$$

$$Se_2=\frac{5}{4}e_2+\frac{1}{3}\Big[2abe_1+(b^2-a^2+c^2)e_2-2ace_5\Big]+HAe_2-A^2e_2, \tag{23}$$

$$Se_3=\frac{5}{4}e_3+\frac{1}{3}\Big[-ae_3-be_4\Big]+HAe_3-A^2e_3, \tag{24}$$

$$Se_4=\frac{5}{4}e_4+\frac{1}{3}\Big[ae_4-be_3\Big]+HAe_4-A^2e_4, \tag{25}$$

$$Se_5=e_5+\frac{1}{3}\Big[-2ace_2+bce_1+(2a^2+b^2)e_5+b\mu^\sharp\Big]+HAe_5-A^2e_5. \tag{26}$$

Moreover, Choosing $(X,Y)=(e_1,e_2),\ (e_1,e_3),\ (e_1,e_4),\ (e_2,e_3),\ (e_2,e_4),$ (e_3,e_4) in (17) respectively and making use of (19) and (20), we can obtain the following equations:

$$-\frac{1}{2}=\frac{1}{\sqrt{3}}a_{23}+\frac{1}{\sqrt{3}}a_{14}-\alpha(a_{22}+a_{11})+2g(A\phi Ae_1,e_2), \tag{27}$$

$$0=-\frac{2}{\sqrt{3}}\alpha+\frac{1}{\sqrt{3}}a_{33}+\frac{1}{\sqrt{3}}a_{11}-\alpha(a_{23}-a_{14})+2g(A\phi Ae_1,e_3), \tag{28}$$

$$0=\frac{1}{\sqrt{3}}a_{34}-\frac{1}{\sqrt{3}}a_{12}-\alpha(a_{24}+a_{13})+2g(A\phi Ae_1,e_4), \tag{29}$$

$$0=-\frac{1}{\sqrt{3}}a_{34}+\frac{1}{\sqrt{3}}a_{12}+\alpha(a_{13}+a_{24})+2g(A\phi Ae_2,e_3), \tag{30}$$

$$0=\frac{2}{\sqrt{3}}\alpha-\frac{1}{\sqrt{3}}a_{22}-\frac{1}{\sqrt{3}}a_{44}+\alpha(a_{14}-a_{23})+2g(A\phi Ae_2,e_4), \tag{31}$$

$$\frac{1}{6}=\frac{1}{\sqrt{3}}(-a_{14}-a_{23})-\alpha(a_{44}+a_{33})+2g(A\phi Ae_3,e_4). \tag{32}$$

Case (II). If $\dim\mathfrak{D} = 2$ holds in an open set, then $P\{U\}^\perp = \{U\}^\perp$ and we can write

$$P\xi = a\xi + bU, \quad a^2 + b^2 = 1. \tag{33}$$

Now, $\mathfrak{D}^\perp$ is a 4-dimensional distribution and invariant under both J and P. Hence we can choose a unit $e_1 \in \mathfrak{D}^\perp$ such that $Pe_1 = e_1$. Put $e_2 = Je_1$, $e_3 = -\sqrt{3}G(e_1, \xi)$, $e_4 = Je_3$ and $e_5 = U$. Then we obtain a local frame field of M with the following properties:

$$\begin{cases} P\xi = a\xi + be_5, & Pe_1 = e_1, \\ Pe_2 = -e_2, & Pe_3 = -ae_3 - be_4, \\ Pe_4 = -be_3 + ae_4, & Pe_5 = b\xi - ae_5. \end{cases} \tag{34}$$

Denote $e_6 = \xi$, then we can calculate the matrix $(G_{ij})_{5\times 6} = (G(e_i, e_j))$ with the same expression as (20). From (7) and (34), we have

$$\begin{cases} Te_1 = e_1, & \mu(e_1) = 0, \\ Te_2 = -e_2, & \mu(e_2) = 0, \\ Te_3 = -ae_3 - be_4, & \mu(e_3) = 0, \\ Te_4 = -be_3 + ae_4, & \mu(e_4) = 0, \\ Te_5 = -ae_5, & \mu(e_5) = b. \end{cases} \tag{35}$$

In this case, from (12) we obtain

$$Se_1 = \frac{5}{4}e_1 - \frac{1}{3}\Big[ae_1 + be_2\Big] + HAe_1 - A^2e_1, \tag{36}$$

$$Se_2 = \frac{5}{4}e_2 + \frac{1}{3}\Big[ae_2 - be_1\Big] + HAe_2 - A^2e_2, \tag{37}$$

$$Se_3 = \frac{5}{4}e_3 + \frac{1}{3}\Big[(a^2 - b^2)e_3 + 2bae_4\Big] + HAe_3 - A^2e_3, \tag{38}$$

$$Se_4 = \frac{5}{4}e_4 + \frac{1}{3}\Big[(b^2 - a^2)e_4 + 2bae_3\Big] + HAe_4 - A^2e_4, \tag{39}$$

$$SU = U + \frac{1}{3}\Big[2a^2U + b\mu + b^2U\Big] + (H\alpha - \alpha^2)U. \tag{40}$$

As in Case I, choosing different vector fields X, Y in (17) and using (20) and (34), we also have the following equations:

$$\frac{1}{6} = \frac{1}{\sqrt{3}}a_{23} + \frac{1}{\sqrt{3}}a_{14} - \alpha(a_{22} + a_{11}) + 2g(A\phi Ae_1, e_2), \tag{41}$$

$$0 = -\frac{2}{\sqrt{3}}\alpha + \frac{1}{\sqrt{3}}a_{33} + \frac{1}{\sqrt{3}}a_{11} - \alpha(a_{23} - a_{14}) + 2g(A\phi Ae_1, e_3), \tag{42}$$

$$0 = \frac{1}{\sqrt{3}}a_{34} - \frac{1}{\sqrt{3}}a_{12} - \alpha(a_{24} + a_{13}) + 2g(A\phi Ae_1, e_4), \tag{43}$$

$$0 = -\frac{1}{\sqrt{3}}a_{34} + \frac{1}{\sqrt{3}}a_{12} + \alpha(a_{13} + a_{24}) + 2g(A\phi Ae_2, e_3), \tag{44}$$

$$0 = \frac{2}{\sqrt{3}}\alpha - \frac{1}{\sqrt{3}}(a_{44} + a_{22}) + \alpha(a_{14} - a_{23}) + 2g(A\phi Ae_2, e_4), \tag{45}$$

$$\frac{1}{6} = \frac{1}{\sqrt{3}}(-a_{14} - a_{23}) - \alpha(a_{44} + a_{33}) + 2g(A\phi Ae_3, e_4). \tag{46}$$

3. Hopf hypersurfaces in NK $\mathbb{S}^3 \times \mathbb{S}^3$ with commuting Ricci tensor

In this section we consider the Hopf hypersurface M with commuting Ricci tensor, i.e., its Ricci operator S satisfies (1). In order to prove Theorem 1.2, we need some lemmas.

Lemma 3.1. *Let M be a real Hopf hypersurface in the NK $\mathbb{S}^3 \times \mathbb{S}^3$ with commuting Ricci tensor. Then for any vector field $X \in \mathfrak{X}(M)$, the following equation holds*:

$$\begin{aligned} &SG(X,U)^\top - \eta(AX)\Big\{\frac{1}{3}\Big[2g(TU,U)TU + \mu(U)\mu^\sharp - \mu(U)\phi TU \\ &\quad + \mu(U)\mu(U)U\Big] + (H\alpha - \alpha^2 + 1)U\Big\} + \Big\{\frac{5}{4}AX - \frac{1}{4}\eta(AX)U \\ &\quad + \frac{1}{3}\Big[g(TU,U)TAX + \mu(AX)\mu^\sharp - \mu(U)\phi TAX \\ &\quad + \mu(U)\mu(AX)U + g(TAX,U)TU\Big] + HA^2X - A^3X\Big\} \\ &= G(X,SU)^\top - g(AX,SU)U + \eta(SU)AX \\ &\quad - \frac{1}{4}\phi\nabla_X U + \frac{1}{3}\Big[2g((\nabla_X T)U,U)\phi TU \\ &\quad + 3g(T\nabla_X U,U)\phi TU + 2g(TU,U)\phi(\nabla_X T)U \\ &\quad + (\nabla_X\mu)(U)\phi\mu + \mu(U)\phi\nabla_X\mu^\sharp - (\nabla_X\mu)(U)\phi^2TU - \mu(\nabla_X U)\phi^2TU \\ &\quad - \mu(U)\phi(\nabla_X\phi)TU - \mu(U)\phi^2(\nabla_X T)U + \mu(U)\mu(U)\phi\nabla_X U \\ &\quad + g(TU,U)\phi T\nabla_X U\Big] + H\phi(\nabla_X A)U - \phi(\nabla_X A^2)U. \end{aligned} \tag{47}$$

Proof. The Ricci tensor of a real hypersurface M is commuting, i.e., the Ricci operator S satisfies

$$S\phi Y = \phi SY \tag{48}$$

for every vector field Y on M. Let us take the covariant derivative of equation (48) along vector field X, namely

$$(\nabla_X S)\phi Y + S(\nabla_X\phi)Y = (\nabla_X\phi)SY + \phi(\nabla_X S)Y. \tag{49}$$

By (12) and (14), we compute

$$\begin{aligned} &S(\nabla_X\phi)Y \\ &= SG(X,Y)^\top - g(AX,Y)SU + \eta(Y)SAX \\ &= SG(X,Y)^\top - g(AX,Y)\Big\{\frac{1}{3}\Big[2g(TU,U)TU + \mu(U)\mu^\sharp - \mu(U)\phi TU \\ &\quad + \mu(U)\mu(U)U\Big] + (H\alpha - \alpha^2 + 1)U\Big\} + \eta(Y)\Big\{\frac{5}{4}AX - \frac{1}{4}\eta(AX)U \end{aligned} \tag{50}$$

$$
\begin{aligned}
&+\frac{1}{3}\Big[g(TU,U)TAX+\mu(AX)\mu^\sharp-\mu(U)\phi TAX\\
&+\mu(U)\mu(AX)U+g(TAX,U)TU\Big]+HA^2X-A^3X\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&(\nabla_X S)Y\\
=\;&-\frac{1}{4}(\nabla_X\eta)(Y)U-\frac{1}{4}\eta(Y)\nabla_X U+\frac{1}{3}\Big[g((\nabla_X T)U,U)TY\\
&+g(T\nabla_X U,U)TY+g(TU,\nabla_X U)TY+g(TU,U)(\nabla_X T)Y\\
&+(\nabla_X\mu)(Y)\mu^\sharp+\mu(Y)\nabla_X\mu^\sharp-(\nabla_X\mu)(U)\phi TY-\mu(\nabla_X U)\phi TY\\
&-\mu(U)(\nabla_X\phi)TY-\mu(U)\phi(\nabla_X T)Y+(\nabla_X\mu)(U)\mu(Y)U\\
&+\mu(\nabla_X U)\mu(Y)U+\mu(U)(\nabla_X\mu)(Y)U+\mu(U)\mu(Y)\nabla_X U\\
&+g((\nabla_X T)Y,U)TU+g(TY,\nabla_X U)TU+g(TY,U)\nabla_X TU\Big]\\
&+X(H)AY+H(\nabla_X A)Y-(\nabla_X A^2)Y.
\end{aligned}
$$

Since $\phi U=0$, we further obtain

$$
\begin{aligned}
(51)\quad &\phi(\nabla_X S)Y\\
=\;&-\frac{1}{4}\eta(Y)\phi\nabla_X U+\frac{1}{3}\Big[g((\nabla_X T)U,U)\phi TY\\
&+g(T\nabla_X U,U)\phi TY+g(TU,\nabla_X U)\phi TY+g(TU,U)\phi(\nabla_X T)Y\\
&+(\nabla_X\mu)(Y)\phi\mu^\sharp+\mu(Y)\phi\nabla_X\mu^\sharp-(\nabla_X\mu)(U)\phi^2TY-\mu(\nabla_X U)\phi^2TY\\
&-\mu(U)\phi(\nabla_X\phi)TY-\mu(U)\phi^2(\nabla_X T)Y+\mu(U)\mu(Y)\phi\nabla_X U\\
&+g((\nabla_X T)Y,U)\phi TU+g(TY,\nabla_X U)\phi TU+g(TY,U)\phi\nabla_X TU\Big]\\
&+X(H)\phi AY+H\phi(\nabla_X A)Y-\phi(\nabla_X A^2)Y.
\end{aligned}
$$

Putting $Y=U$ in (49) gives

$$
S(\nabla_X\phi)U=(\nabla_X\phi)SU+\phi(\nabla_X S)U. \tag{52}
$$

Letting $Y=U$ in (50) and (51) and inserting the resulting equations into (52) yields (47). □

Lemma 3.2. *Let M be a real Hopf hypersurface in the NK $\mathbb{S}^3\times\mathbb{S}^3$ with commuting Ricci tensor. If* $\dim\mathfrak{D}=2$, *for any vector fields $X,Y\in\mathfrak{X}(M)$, the following equation holds*:

$$
\begin{aligned}
(53)\qquad &-\frac{1}{4}(\nabla_X\eta)(\phi Y)+\frac{1}{3}\Big[2b^2g(\nabla_X U,\phi Y)-bg((\nabla_X\phi)T\phi Y,U)\\
&-2ag((\nabla_X T)\phi Y,U)-ag(T\phi Y,\nabla_X U)\Big]\\
&+Hg((\nabla_X A)\phi Y,U)-g((\nabla_X A^2)\phi Y,U)
\end{aligned}
$$

$$
\begin{aligned}
&+(\frac{5}{3}+H\alpha-\alpha^2)[g(G(X,Y)^\top,U)-g(AX,Y)]\\
&=g(G(X,SY)^\top,U)-g(AX,SY).
\end{aligned}
$$

Proof. Taking the inner product of (49) with U, we have

$$
g((\nabla_X S)\phi Y,U)+g(S(\nabla_X\phi)Y,U)=g((\nabla_X\phi)SY,U). \tag{54}
$$

Noticing that $TU=-aU$ and $\mu^\sharp=bU$ due to (35), using (14) and (12) we thus have

$$
\begin{aligned}
g((\nabla_X S)\phi Y,U)= &-\frac{1}{4}(\nabla_X\eta)(\phi Y)+\frac{1}{3}\Big[2b^2g(\nabla_X U,\phi Y)-bg((\nabla_X\phi)T\phi Y,U)\\
&-2ag((\nabla_X T)\phi Y,U)-ag(T\phi Y,\nabla_X U)\Big]\\
&+Hg((\nabla_X A)\phi Y,U)-g((\nabla_X A^2)\phi Y,U),
\end{aligned}
$$

$$
\begin{aligned}
g(S(\nabla_X\phi)Y,U)&=(\frac{5}{3}+H\alpha-\alpha^2)g((\nabla_X\phi)Y,U)\\
&=(\frac{5}{3}+H\alpha-\alpha^2)[g(G(X,Y)^\top,U)-g(AX,Y)+\eta(Y)\eta(AX)],
\end{aligned}
$$

$$
g((\nabla_X\phi)SY,U)=g(G(X,SY)^\top,U)-g(AX,SY)+\eta(SY)\eta(AX).
$$

Substituting the previous equations into (54) gives (53). $\square$

Next, we separate the proof of Theorem 1.2 into the proofs of two lemmas, depending on the dimension of $\mathfrak{D}$.

Lemma 3.3. *For the case* $\dim\mathfrak{D}=4$, *the mean curvature* $H=2\alpha$.

Proof. By $\phi SU=S\phi U=0$, we see $SU=\eta(SU)U$. Thus $0=g(SU,e_2)=-\frac{2}{3}ac$ and $0=g(SU,e_1)=\frac{2}{3}bc$. That means $b=a=0$ and $c=1$. Thus $Te_5=e_2$ and $\mu^\sharp=e_1$ from (21). Let us write $Ae_i=\sum_{j=1}^5 a_{ij}e_j$, where $a_{ij}=a_{ji}$ for $1\le i,j\le 5$. By $AU=\alpha U$, it is clear that $a_{15}=a_{25}=a_{35}=a_{45}=0$ and $a_{55}=\alpha$.

Let us choose $Y=e_1$ and $Y=e_3$ in (48), respectively. Then it follows from Eqs. (22)-(25) that

$$
H\phi Ae_1-\phi A^2e_1=HAe_2-A^2e_2, \tag{55}
$$

$$
H\phi Ae_3-\phi A^2e_3=HAe_4-A^2e_4. \tag{56}
$$

Since $TU=e_2$ and $\mu(U)=0$ (see (21)), Equation (47) may be simplified as

$$
\begin{aligned}
&SG(X,U)^\top+\frac{1}{3}\Big[\mu(AX)e_1+g(AX,e_2)e_2\Big]+HA^2X-A^3X\\
&=G(X,SU)^\top+\frac{1}{4}\eta(AX)U+(H\alpha-\alpha^2-\frac{1}{4})AX\\
&-\frac{1}{4}\phi\nabla_X U+\frac{1}{3}\Big[-2g((\nabla_X T)U,U)e_1-3g(\nabla_X U,e_2)e_1
\end{aligned}
$$

$$+ 2(\nabla_X\mu)(U)e_2 + \mu(\nabla_X U)e_2\Big] + H\phi(\nabla_X A)U - \phi(\nabla_X A^2)U.$$

Moreover, using (13), (15) and (16), we conclude that

$$\begin{aligned}
&SG(X,U)^\top + \frac{1}{3}\Big[\mu(AX)e_1 + g(AX,e_2)e_2\Big] + HA^2X - A^3X \\
=\;& G(X,SU)^\top + (H\alpha-\alpha^2)AX + \frac{1}{4}\phi G(X,\xi) \\
&+ \frac{1}{3}\Big[-g([JG(X,e_2)+JPG(X,U)]^\top, U)e_1 \\
&- 3g(-G(X,\xi)+\phi AX, e_2)e_1 \\
&+ [g(G(X,e_2)+PG(X,U),U) - 2g(AX,e_2)]e_2 \\
&+ \mu(-G(X,\xi)+\phi AX)e_2\Big] \\
&+ H\phi(\alpha\nabla_X U - A\nabla_X U) - \phi(\alpha^2\nabla_X U - A^2\nabla_X U).
\end{aligned} \tag{57}$$

By (20) and (25), we know

$$SG(e_1,U)^\top = -\frac{1}{\sqrt{3}}Se_4 = -\frac{1}{\sqrt{3}}(\frac{5}{4}e_4 + HAe_4 - A^2e_4).$$

Therefore, substituting this into (57) with $X=e_1$ and taking the inner product with e_1, we obtain from (20) that

$$\begin{aligned}
&-\frac{1}{\sqrt{3}}g(HAe_4 - A^2e_4, e_1) + \frac{1}{3}a_{11} + g(HA^2e_1 - A^3e_1, e_1) \\
=\;& (H\alpha-\alpha^2)a_{11} - a_{11} - Hg(\alpha\nabla_{e_1}U - A\nabla_{e_1}U, e_2) \\
&+ g(\alpha^2\nabla_{e_1}U - A^2\nabla_{e_1}U, e_2) \\
=\;& (H\alpha-\alpha^2-1)a_{11} - Hg(\alpha\phi Ae_1 - A(\frac{1}{\sqrt{3}}e_3+\phi Ae_1), e_2) \\
&+ g(\alpha^2\phi Ae_1 - A^2(\frac{1}{\sqrt{3}}e_3+\phi Ae_1), e_2) \\
=\;& -a_{11} + \frac{1}{\sqrt{3}}g(HAe_3 - A^2e_3, e_2) + g(HA\phi Ae_1 - A^2\phi Ae_1, e_2).
\end{aligned} \tag{58}$$

On the other hand, applying $A\phi$ in (55), we get

$$-HA^2e_1 + A^3e_1 = HA\phi Ae_2 - A\phi A^2e_2.$$

Inserting the previous relation and (56) into (58) implies

$$\frac{1}{3}a_{11} = -a_{11}.$$

That means $a_{11}=0$.

By (55) and the symmetry of A, we have

$$\begin{aligned}
g(HA^2e_1 - A^3e_1, e_2) &= g(HA^2e_2 - A^3e_2, e_1) \\
&= g(HA\phi Ae_1 - A\phi A^2e_1, e_1) = g(A\phi Ae_1, Ae_1).
\end{aligned}$$

Also, we can get

$$\begin{aligned} g(HA^2e_1 - A^3e_1, e_2) &= g(-HA\phi Ae_2 + A\phi A^2e_2, e_2) \\ &= -g(A\phi Ae_2, Ae_2). \end{aligned}$$

Comparing the above two formulas yields

$$g(A\phi Ae_1, Ae_1) + g(A\phi Ae_2, Ae_2) = 0. \tag{59}$$

Letting $X = e_1$ in (57) and taking the inner product with e_2 implies $a_{12} = 0$ by (59).

Similarly, letting $X = e_2$ in (57) and taking an inner product of the resulting relation with e_2, we have

$$\begin{aligned} &-\frac{1}{\sqrt{3}}g(Se_3, e_2) + \frac{1}{3}a_{22} + g(HA^2e_2 - A^3e_2, e_2) \\ &= (H\alpha - \alpha^2)a_{22} + \frac{1}{3}\Big[-2a_{22} + \mu(\phi Ae_2)\Big] \\ &\quad + Hg(\phi(\nabla_{e_2}A)U, e_2) - g(\phi(\nabla_{e_2}A^2)U, e_2). \end{aligned}$$

Applying (24), (55) and (56) in the above relation, we obtain $a_{22} = 0$. Hence the product of (55) with e_2 implies

$$a_{13}^2 + a_{14}^2 = a_{23}^2 + a_{24}^2. \tag{60}$$

Now it follows from (17) and (59) that

$$2\alpha(a_{24} - a_{13}) + \sum_i a_{3i}a_{i1} - \sum_i a_{4i}a_{i2} = 0. \tag{61}$$

On the other hand, taking the inner product of (56) with e_2, we find

$$Ha_{13} - \sum_i a_{3i}a_{i1} = Ha_{24} - \sum_i a_{4i}a_{i2}.$$

By combining with (61), it implies

$$(H - 2\alpha)(a_{24} - a_{13}) = 0. \tag{62}$$

Next we set $a_{24} = a_{13}$ then from (60) we have

$$(a_{14} + a_{23})(a_{14} - a_{23}) = 0. \tag{63}$$

In the following we divide into two cases to discuss.

Case I: $a_{24} = a_{13} \neq 0$. Because $a_{12} = 0$, the inner product of (55) with e_1 yields $\sum_i a_{1i}a_{i2} = 0$, i.e.,

$$a_{13}a_{32} + a_{14}a_{42} = 0.$$

This shows $a_{32} + a_{14} = 0$. By (27), we have $-\frac{1}{4} = a_{13}^2 + a_{14}^2$, which is impossible.

Case II: $a_{24} = a_{13} = 0$. If $a_{32} + a_{14} = 0$ we have $-\frac{1}{4} = a_{14}^2$ from (27), which is impossible, thus $a_{23} = a_{14}$ from (63). If $a_{34} \neq 0$, then from (29) and (30) we get $a_{14} = \frac{1}{2\sqrt{3}}$. On the other hand, by virtue of (27), we obtain

$$a_{14}^2 - \frac{1}{\sqrt{3}} a_{14} - \frac{1}{4} = 0, \tag{64}$$

but $a_{14} = \frac{1}{2\sqrt{3}}$ does not satisfy the above equation. That means that $a_{34} = 0$. Therefore, from (28) and (31) we have

$$(a_{33} - a_{44})(a_{14} - \frac{1}{2\sqrt{3}}) = 0.$$

Since Equation (64) has two solutions: $a_{14} = \frac{\sqrt{3}}{2}$ or $-\frac{1}{2\sqrt{3}}$, the above relation yields

$$a_{33} = a_{44}.$$

Putting $X = e_4$ in (57) and taking the inner product with e_1, we obtain

$$(H + \alpha) a_{44} a_{14} + \frac{2}{3\sqrt{3}} + \frac{4}{3} a_{14} = 0. \tag{65}$$

Recalling (31), we have

$$\frac{2}{\sqrt{3}} \alpha - \frac{1}{\sqrt{3}} a_{44} + 2 a_{14} a_{44} = 0.$$

Since $a_{14} = \frac{\sqrt{3}}{2}$ or $a_{14} = -\frac{1}{2\sqrt{3}}$, the above relation correspondingly yields $a_{44} = -\alpha$ or $a_{44} = \alpha$.

Now substituting $a_{14} = \frac{\sqrt{3}}{2}$ and $a_{44} = -\alpha$ into (65) gives a contradiction:

$$\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = 0.$$

Here we have used $H = a_{33} + a_{44} + \alpha = 2a_{44} + \alpha$. When $a_{14} = -\frac{1}{2\sqrt{3}}$ and $a_{44} = \alpha$, (65) implies $\alpha = 0$. That means that $a_{33} = a_{44} = \alpha = 0$.

In conclusion, we proved $a_{23} = a_{14}$ and the other $a_{ij} = 0$ for $1 \leq i, j \leq 5$. This implies M satisfies $\phi A + A\phi = 0$. According to [14], there does not admit a hypersurface that satisfies the condition. Thus by (62), we have $H = 2\alpha$. $\square$

Lemma 3.4. *For* $\dim \mathfrak{D} = 2$, *the mean curvature* $H = 2\alpha$ *is constant.*

Proof. First, from (40) we get

$$g(SU, U) = \frac{5}{3} + H\alpha - \alpha^2. \tag{66}$$

From the commuting condition $\phi S = S\phi$, we derive from (36)-(39) that

$$\phi S e_1 = S e_2 \Rightarrow -\frac{2}{3}\Big[a e_2 - b e_1\Big] + H\phi A e_1 - \phi A^2 e_1 = H A e_2 - A^2 e_2, \tag{67}$$

$$\phi S e_3 = S e_4 \Rightarrow \frac{2}{3}\Big[(a^2 - b^2) e_4 - 2ba e_3\Big] + H\phi A e_3 - \phi A^2 e_3 = H A e_4 - A^2 e_4. \tag{68}$$

Since $SU=\eta(SU)U$, we conclude from (66) that

$$(1+\frac{1}{3}(2a^2+b^2)+H\alpha-\alpha^2)U+\frac{1}{3}b\mu=(\frac{5}{3}+H\alpha-\alpha^2)U,$$

that is,

$$b\mu^\sharp=b^2U.$$

In the following we divide into two cases to discuss.

Case (a): If $b\neq 0$ in an open set of M, then $\mu^\sharp=bU$. Differentiating this along any vector field X gives

$$\nabla_X\mu=X(b)U+b\nabla_XU.$$

Making use of (13) and (16), we have

$$\begin{aligned}\frac{1}{2}g(G(X,PY)&+PG(X,Y),U)-g(AX,TY)+ag(AX,Y)\\ &=X(b)\eta(Y)+bg(-G(X,\xi)+\phi AX,Y).\end{aligned}\tag{69}$$

If we take $(X,Y)=(e_1,e_2),(e_1,e_3)$ in (69), respectively, then

$$(1+a)a_{12}=ba_{11},\tag{70}$$

$$aa_{13}+ba_{14}=\frac{1}{2\sqrt{3}}b.\tag{71}$$

Similarly, choosing $(X,Y)=(e_2,e_2)$, (e_2,e_3) in (69), respectively, we obtain

$$(1+a)a_{22}=ba_{21},\tag{72}$$

$$aa_{23}+ba_{24}=\frac{1}{2\sqrt{3}}a.\tag{73}$$

Since $TU=-aU$ and $\phi U=0$ (see (35)), Equation (47) becomes

$$\begin{aligned}SG(X,U)^\top&-\alpha\eta(X)(H\alpha-\alpha^2+\frac{5}{3})U+\frac{5}{4}AX-\frac{1}{4}\eta(AX)U\\ &+\frac{1}{3}\Big[-aTAX+(2b^2+a^2)\eta(AX)U-b\phi TAX\Big]+HA^2X-A^3X\\ =G(X,SU)^\top&-g(AX,SU)U+\eta(SU)AX-\frac{1}{4}\phi\nabla_XU\\ &+\frac{1}{3}\Big[-2a\phi(\nabla_XT)U-b\phi(\nabla_X\phi)TU-b\phi^2(\nabla_XT)U\\ &\qquad+2b^2\phi\nabla_XU-a\phi T\nabla_XU\Big]\\ &+H\phi\Big(\alpha(-G(X,\xi)+\phi AX)-A(-G(X,\xi)+\phi AX)\Big)\\ &-\phi\Big(\alpha^2(-G(X,\xi)+\phi AX)-A^2(-G(X,\xi)+\phi AX)\Big).\end{aligned}\tag{74}$$

From (13), (14) and (15), we compute

$$\left\{\begin{array}{l}\nabla_{e_3}U=-\frac{1}{\sqrt{3}}e_1+\phi Ae_3,\\ G(e_3,U)=\frac{1}{\sqrt{3}}e_2,\\ (\nabla_{e_3}T)U=\frac{1}{2\sqrt{3}}[be_2+ae_1+e_1]+bAe_3,\\ (\nabla_{e_3}\phi)TU=-\frac{1}{\sqrt{3}}ae_2-aAe_3.\end{array}\right. \tag{75}$$

Substituting the above relations into (74) with $X=e_3$ yields

$$\begin{aligned}&\frac{1}{\sqrt{3}}\Big[-\frac{(1+a)^2}{2}e_2+\frac{1-a}{2}be_1\Big]-\Big[aTAe_3+b\phi TAe_3\Big]\\ &+3HA^2e_3-3A^3e_3\\ &=(2-b^2)Ae_3-\Big[ba\phi Ae_3+a\phi T\phi Ae_3\Big]-3H\phi A\phi Ae_3+3\phi A^2\phi Ae_3.\end{aligned} \tag{76}$$

By taking the inner product with e_3, it further implies that

$$a^2a_{33}=-aba_{34}. \tag{77}$$

Similarly, choosing $X=e_4$ in (74) and applying the same method, we get

$$\begin{aligned}&\frac{1}{\sqrt{3}}\Big[-\frac{(1-a)^2}{2}e_1+\frac{1+a}{2}be_2\Big]-\Big[aTAe_4+b\phi TAe_4\Big]+3HA^2e_4-3A^3e_4\\ &=(2-b^2)Ae_4-[ab\phi Ae_4+a\phi T\phi Ae_4]-3H\phi A\phi Ae_4+3\phi A^2\phi Ae_4.\end{aligned}$$

Moreover, the inner product of the above equation with e_4 gives

$$a^2a_{44}=aba_{34}. \tag{78}$$

Letting $X=e_1$ in (74) yields

$$\begin{aligned}&\frac{1}{\sqrt{3}}\Big[b^2e_4+2bae_3\Big]-\Big[aTAe_1+b\phi TAe_1\Big]+3HA^2e_1-3A^3e_1\\ &=(2-b^2)Ae_1-\Big[ab\phi Ae_1+a\phi T\phi Ae_1\Big]-3H\phi A\phi Ae_1+3\phi A^2\phi Ae_1.\end{aligned} \tag{79}$$

Thus the inner product of (68) with e_2 yields

$$Ha_{13}-\sum_i a_{3i}a_{i1}=Ha_{24}-\sum_i a_{4i}a_{i2}. \tag{80}$$

On the other hand, as the proof of Lemma 3.3, making use of $g(HA^2e_1-A^3e_1,e_2)=g(HA^2e_2-A^3e_2,e_1)$, we derive

$$g(A\phi Ae_1,Ae_1)+g(A\phi Ae_2,Ae_2)=0.$$

Hence, it follows from (17) with $(X,Y)=(e_1,Ae_1)$ and (e_2,Ae_2) that

$$2\alpha(a_{24}-a_{13})+\sum_i a_{3i}a_{i1}-\sum_i a_{4i}a_{i2}=0. \tag{81}$$

Combining (80) with (81) implies

$$(H-2\alpha)(a_{24}-a_{13})=0. \tag{82}$$

Choosing $(X,Y)=(e_3,Ae_3)$ and $(X,Y)=(e_4,Ae_4)$ in (17), respectively, gives

$$\frac{1}{6}a_{34}=\frac{1}{\sqrt{3}}(2\alpha a_{13}-\sum_i a_{1i}a_{i3})-\alpha\sum_i a_{4i}a_{i3}+2g(A\phi Ae_3,Ae_3)$$

and

$$-\frac{1}{6}a_{34}=\frac{1}{\sqrt{3}}(-2\alpha a_{24}+\sum_i a_{2i}a_{i4})+\alpha\sum_i a_{3i}a_{i4}+2g(A\phi Ae_4,Ae_4).$$

Thus

$$\begin{aligned}0=&-\frac{1}{\sqrt{3}}\{2\alpha(a_{24}-a_{13})+\sum_i a_{3i}a_{i1}-\sum_i a_{4i}a_{i2}\}\\&+2g(A\phi Ae_3,Ae_3)+2g(A\phi Ae_4,Ae_4),\end{aligned}$$

that is, $g(A\phi Ae_3,Ae_3)+g(A\phi Ae_4,Ae_4)=0$ by (81). Therefore, taking the inner product of (76) with e_4 implies

$$2aba_{33}-4aba_{44}-[3(a^2-b^2)+1]a_{34}=0. \tag{83}$$

This leads to $a_{34}=0$ from (77) and (78).

In the following we assume $a_{24}=a_{13}$ and separate two cases to discuss.

Case (a)-(i). If $a\neq 0$, then from (77) and (78), we have

$$a_{44}=\frac{b}{a}a_{34}=0,\quad a_{33}=-\frac{b}{a}a_{34}=0.$$

Then $\sum_i a_{3i}a_{i1}=\sum_i a_{4i}a_{i2}$ by (81), i.e.,

$$a_{31}a_{11}+a_{23}a_{12}=a_{41}a_{12}+a_{24}a_{22}.$$

So, by virtue of (70)-(73), the previous equation yields

$$(4a^2-1)a_{31}a_{12}=0. \tag{84}$$

On the other hand, by (46) we obtain

$$\begin{aligned}\frac{1}{6}&=\frac{1}{\sqrt{3}}(-a_{14}-a_{23})+2(a_{13}a_{24}-a_{23}a_{14})\\&=-\frac{1}{3}+\frac{1}{\sqrt{3}}(\frac{b}{a}+\frac{a}{b})a_{13}+2\Big[a_{13}^2-(\frac{1}{2\sqrt{3}}-\frac{b}{a}a_{24})(\frac{1}{2\sqrt{3}}-\frac{a}{b}a_{13})\Big]\\&=\frac{1}{\sqrt{3}}\frac{b^2+a^2}{ab}a_{13}+2\Big[-\frac{1}{12}+\frac{1}{2\sqrt{3}}\frac{1}{ab}a_{13}\Big]-\frac{1}{3}\\&=\frac{2}{\sqrt{3}}\frac{1}{ab}a_{13}-\frac{1}{2},\end{aligned}$$

that is,

$$a_{13}=\frac{\sqrt{3}}{3}ab. \tag{85}$$

Substituting this into (84) gives

$$(1-4a^2)a_{12}=0. \tag{86}$$

If $a_{12}=0$, then $a_{11}=a_{22}=0$ by (70) and (72). Moreover, from (41) and (46), we get $a_{23}=-a_{14}$. Hence we follows from (43) that $-\alpha(a_{24}+a_{13})=0$, this means $\alpha=0$ or $a_{24}+a_{13}=0$. Because the latter will lead to $a_{13}=0$, which is impossible due to (85), thus in this case we prove $\alpha=0$ and $H=0$.

If $a_{12}\neq 0$, then $a^2=\frac{1}{4}$ and $b^2=\frac{3}{4}$ by (86) and (33). From (71) and (73), we further have

$$a_{14}=\frac{1}{4\sqrt{3}},\ a_{23}=-\frac{1}{4\sqrt{3}}.$$

Due to $a_{24}=a_{13}=\frac{\sqrt{3}}{3}ab$ and $a_{12}\neq 0$, using (70) and (72), we follows from (41) that $\alpha=0$. But from (44) and (72) we obtain $a_{12}=0$, which is a contradiction.

In summary, in the Case (a)-(i), we have proved $H=\alpha=0$.

Case (a)-(ii). If $a=0$, then $b^2=1$. Eqs. (70)-(73) become

$$\begin{cases} a_{12}=ba_{11}, a_{22}=ba_{21},\\ a_{24}=0, a_{14}=\frac{1}{2\sqrt{3}}.\end{cases}$$

As $a_{13}=a_{24}$, from (43) we get $a_{12}=0$, thus $a_{11}=a_{22}=0$. Moreover, Eqs. (42) and (45) are simplified as

$$0=-\frac{\sqrt{3}}{2}\alpha-\alpha a_{23}, \tag{87}$$

$$0=\frac{5}{2\sqrt{3}}\alpha-\frac{1}{\sqrt{3}}a_{44}-\alpha a_{23}+2a_{23}a_{44}. \tag{88}$$

Now putting $X=Y=e_1$ in (53), we conclude

$$\frac{2}{3}b\Big[ba_{11}-aa_{12}\Big]-\frac{2}{3}a_{11}=g(G(e_1,HAe_1-A^2e_1)^\top,U)-\frac{2}{3}aa_{11},$$

then

$$Ha_{14}=\sum_i a_{1i}a_{i4}. \tag{89}$$

Since $a_{34}=a_{11}=a_{22}=0$, we know $a_{33}=-\alpha$ from (89). Moreover, the inner product of (68) with e_1 and (89) imply

$$Ha_{23}=\sum_i a_{2i}a_{i3},$$

i.e., $(a_{44}+\alpha)a_{23}=0$. If $\alpha\neq 0$, then $a_{23}=-\frac{\sqrt{3}}{2}$ by (87) and $a_{44}=-\alpha$. Inserting this into (88) gives

$$0=\frac{5}{2\sqrt{3}}\alpha+\frac{11}{2\sqrt{3}}\alpha.$$

It is impossible, thus we see $\alpha=0$. That means $a_{33}=a_{44}=0$ by (88), i.e., $H=0$.

Summarizing Case (a)-(i) and Case (a)-(ii), we have proved $H=\alpha=0$ when $a_{13}=a_{24}$. If $a_{13}\neq a_{24}$ in an open subset, we get from (82) that

$$H=2\alpha.$$

According to [13, Lemma 2.2], the mean curvature H is constant.

Case (b). If $b=0$, then $a^2=1$ and $\mu^\sharp=0$ by (35). In terms of the proof in Case (a), we know from (71) and (73) with $b=0$ that $a_{13}=0$ and $a_{23}=\frac{1}{2\sqrt{3}}$. Furthermore, from (77) and (78) we also know $a_{33}=a_{44}=0$. In terms of (46) we obtain $a_{14}=-\frac{1}{2\sqrt{3}}=-a_{23}$. Here we have used $a_{34}=0$ due to (83).

In the same way, we assume $a_{24}=a_{13}$. Then we obtain from (44), (45) and (42) that $a_{12}=\alpha=a_{11}=0$. This means $H=a_{22}$. Finally, if $H=a_{22}=0$, we proved $a_{23}=-a_{14}$ and the other $a_{ij}=0$ for $1\leq i,j\leq 5$. This implies that M satisfies $\phi A=A\phi$. According to [15, Claim 4.2], we can obtain $\nabla_U U=0$. Thus, following from [12, Lemma 5.2] and [12, Eq. (5.10)], we have

$$0=\frac{1}{2}\left\|\phi A-A\phi\right\|^2+\frac{2}{3}\Theta-\frac{1}{3}-\|A\|^2+Hg(AU,U)=\frac{1}{6},$$

which is impossible. Here we have used $\Theta=1$ for $\dim\mathfrak{D}=2$.

Therefore it completes the proof Lemma 3.4 from Case (a) and Case (b). □

Combining Lemma 3.3 with Lemma 3.4, we complete the proof of Theorem 1.2.

4. Hopf hypersurfaces in NK $\mathbb{S}^3\times\mathbb{S}^3$ with pseudo-anti commuting Ricci tensor

In this section we study the Hopf hypersurface M with pseudo-anti commuting Ricci tensor. Namely, the equation (2) is satisfied.

Lemma 4.1. *Let M be a real Hopf hypersurface in the NK $\mathbb{S}^3\times\mathbb{S}^3$ with pseudo-anti commuting Ricci tensor. Then for any vector field $X\in\mathfrak{X}(M)$, the following equation holds*:

$$\begin{aligned}
&SG(X,U)^\top-\eta(AX)\Big\{\frac{1}{3}\Big[2g(TU,U)TU+\mu(U)\mu^\sharp-\mu(U)\phi TU\\
&+\mu(U)\mu(U)U\Big]+(H\alpha-\alpha^2+1-2\kappa)U\Big\}+\Big\{\frac{5}{4}AX-\frac{1}{4}\eta(AX)U\\
&+\frac{1}{3}\Big[g(TU,U)TAX+\mu(AX)\mu^\sharp-\mu(U)\phi TAX\\
&+\mu(U)\mu(AX)U+g(TAX,U)TU\Big]+HA^2X-A^3X\Big\}\\
(90)\quad =\;&-G(X,SU)^\top+2\kappa G(X,U)^\top+g(AX,SU)U+\Big(2\kappa-\eta(SU)\Big)AX\\
&+\frac{1}{4}\phi\nabla_XU-\frac{1}{3}\Big[2g((\nabla_XT)U,U)\phi TU\\
&+3g(T\nabla_XU,U)\phi TU+2g(TU,U)\phi(\nabla_XT)U\\
&+(\nabla_X\mu)(U)\phi\mu+\mu(U)\phi\nabla_X\mu^\sharp-(\nabla_X\mu)(U)\phi^2TU-\mu(\nabla_XU)\phi^2TU
\end{aligned}$$

$$
\begin{aligned}
&- \mu(U)\phi(\nabla_X\phi)TU - \mu(U)\phi^2(\nabla_X T)U + \mu(U)\mu(U)\phi\nabla_X U\\
&+ g(TU,U)\phi T\nabla_X U\Big] - H\phi(\nabla_X A)U + \phi(\nabla_X A^2)U.
\end{aligned}
$$

Proof. Applying the same method as Lemma 3.1 and utilizing the following condition:

$$
S\phi Y + \phi SY = 2\kappa\phi Y, \quad \forall Y \in TM, \tag{91}
$$

we have

$$
(\nabla_X S)\phi Y + S(\nabla_X\phi)Y + (\nabla_X\phi)SY + \phi(\nabla_X S)Y = 2\kappa(\nabla_X\phi)Y. \tag{92}
$$

Putting $Y = U$ in (92) and using (14), we have

$$
S(\nabla_X\phi)U = -(\nabla_X\phi)SU - \phi(\nabla_X S)U + 2\kappa\big(G(X,U)^\top - \eta(AX)U + AX\big). \tag{93}
$$

Letting $Y = U$ in (50) and (51) and inserting the resulting equations into (93) yields (90). □

Lemma 4.2. *Let M be a real Hopf hypersurface of the NK $\mathbb{S}^3\times\mathbb{S}^3$ with pseudo-anti commuting Ricci tensor. If* $\dim\mathfrak{D} = 2$, *for any vector fields $X, Y \in \mathfrak{X}(M)$, the following equation holds*:

$$
\begin{aligned}
&- \frac{1}{4}(\nabla_X\eta)(\phi Y) + \frac{1}{3}\Big[2b^2 g(\nabla_X U, \phi Y) - bg((\nabla_X\phi)T\phi Y, U)\\
&- 2ag((\nabla_X T)\phi Y, U) - ag(T\phi Y, \nabla_X U)\Big]\\
&+ Hg((\nabla_X A)\phi Y, U) - g((\nabla_X A^2)\phi Y, U)\\
&+ (\frac{5}{3} + H\alpha - \alpha^2)[g(G(X,Y)^\top, U) - g(AX,Y) + 2\eta(Y)\eta(AX)]\\
=\ & - g(G(X,SY)^\top, U) + g(AX, SY)\\
&+ 2\kappa\Big[g(G(X,Y)^\top, U) - g(AX,Y) + \eta(Y)\eta(AX)\Big].
\end{aligned} \tag{94}
$$

Proof. As the proof of Lemma 3.2, taking the inner product of (92) with U, we have

$$
\begin{aligned}
&g((\nabla_X S)\phi Y, U) + g(S(\nabla_X\phi)Y, U)\\
=\ & - g((\nabla_X\phi)SY, U) + 2\kappa\Big[g(G(X,Y)^\top, U) - g(AX,Y) + \eta(Y)\eta(AX)\Big].
\end{aligned}
$$

Through a series of calculations that are the same as the proof of Lemma 3.2, we can get the required equation. □

Lemma 4.3. *The case* $\dim\mathfrak{D} = 4$ *does not occur if* $\kappa > \frac{19}{12}$.

Proof. By $\phi SU + S\phi U = 2\kappa\phi U = 0$, we derive $\phi SU = -S\phi U = 0$. We see $SU = \eta(SU)U$. Thus $0 = g(SU, e_2) = -\frac{2}{3}ac$ and $0 = g(SU, e_1) = \frac{2}{3}bc$. That means that $b = a = 0$ and $c = 1$.

Let us choose $Y=e_1$ and $Y=e_3$ in (91), respectively. Then it follows from Eqs. (22)-(25) that

$$(\frac{19}{6}-2\kappa)e_2+HAe_2-A^2e_2=\phi A^2e_1-H\phi Ae_1, \tag{95}$$

$$(\frac{5}{2}-2\kappa)e_4+HAe_4-A^2e_4=\phi A^2e_3-H\phi Ae_3. \tag{96}$$

Since $TU=e_2$ and $\mu(U)=0$, Equation (90) may be simplified as

$$\begin{aligned}
&SG(X,U)^\top+\frac{1}{3}\Big[\mu(AX)e_1+g(AX,e_2)e_2\Big]+HA^2X-A^3X\\
=&-G(X,SU)^\top+2\kappa G(X,U)^\top+(2H\alpha-2\alpha^2+\frac{9}{4}-2\kappa)\eta(AX)U\\
&-(H\alpha-\alpha^2+\frac{9}{4}-2\kappa)AX+\frac{1}{4}\phi\nabla_XU-\frac{1}{3}\Big[-2g((\nabla_XT)U,U)e_1\\
&-3g(\nabla_XU,e_2)e_1+2(\nabla_X\mu)(U)e_2+\mu(\nabla_XU)e_2\Big]\\
&-H\phi(\nabla_XA)U+\phi(\nabla_XA^2)U.
\end{aligned}$$

Moreover, using (13), (15) and (16), we conclude

$$\begin{aligned}
&SG(X,U)^\top+\frac{1}{3}\Big[\mu(AX)e_1+g(AX,e_2)e_2\Big]+HA^2X-A^3X\\
=&-G(X,SU)^\top+2\kappa G(X,U)^\top+(2H\alpha-2\alpha^2+\frac{5}{2}-2\kappa)\eta(AX)U\\
&-(H\alpha-\alpha^2+\frac{5}{2}-2\kappa)AX-\frac{1}{4}\phi G(X,\xi)\\
&-\frac{1}{3}\Big[-g([JG(X,e_2)+JPG(X,U)]^\top,U)e_1\\
&\qquad-3g(-G(X,\xi)+\phi AX,e_2)e_1+[g(G(X,e_2)+PG(X,U),U)\\
&\qquad-2g(AX,e_2)]e_2+\mu(-G(X,\xi)+\phi AX)e_2\Big]\\
&-H\phi(\alpha\nabla_XU-A\nabla_XU)+\phi(\alpha^2\nabla_XU-A^2\nabla_XU).
\end{aligned} \tag{97}$$

On the other hand, applying ϕA in (95), we have

$$HA^2e_1-A^3e_1=(2\kappa-\frac{19}{6})Ae_1+HA\phi Ae_2-A\phi A^2e_2. \tag{98}$$

Letting $X=e_1$ in (97) and taking the inner product with e_1, using (98), (96), we obtain from (20) that

$$\frac{1}{3}a_{11}+(2\kappa-\frac{19}{6})a_{11}=(2\kappa-\frac{3}{2})a_{11}.$$

That means $a_{11}=0$.

By (95) and the symmetry of A, we can get

$$g(A\phi Ae_1,Ae_1)+g(A\phi Ae_2,Ae_2)=0. \tag{99}$$

Letting $X = e_1$ in (97) and taking the inner product with e_2 implies $a_{12} = 0$ by (99).

Similarly, letting $X = e_2$ in (97) and taking an inner product of the resulting relation with e_2, we get $a_{22} = 0$ by applying (24), (95) and (96). From these and the product of (95) with e_2 we follow

$$a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 = \frac{19}{6} - 2\kappa.$$

Thus, if $\kappa > \frac{19}{12}$ the above relation is impossible. $\square$

Lemma 4.4. *For the case* $\dim\mathfrak{D} = 2$*, the mean curvature* $H = \alpha$ *is constant.*

Proof. First, from the pseudo-anti commuting condition $\phi S + S\phi = 2\kappa\phi$, using (36)-(39), we derive

$$\phi Se_1 + Se_2 = 2\kappa e_2 \Rightarrow (\frac{5}{2} - 2\kappa)e_2 + HAe_2 - A^2e_2 = \phi A^2e_1 - H\phi Ae_1, \tag{100}$$

$$\phi Se_3 + Se_4 = 2\kappa e_4 \Rightarrow (\frac{5}{2} - 2\kappa)e_4 + HAe_4 - A^2e_4 = \phi A^2e_3 - H\phi Ae_3. \tag{101}$$

Case (a): If $b \neq 0$ in an open set of M, then $\mu^\sharp = bU$.

Since $TU = -aU$ and $\phi U = 0$ (see (35)), Equation (90) becomes

$$\begin{aligned}
&SG(X,U)^\top - \alpha\eta(X)(H\alpha - \alpha^2 + \frac{5}{3})U + \frac{5}{4}AX - \frac{1}{4}\eta(AX)U\\
&+ \frac{1}{3}\Big[- aTAX + (2b^2 + a^2)\eta(AX)U - b\phi TAX\Big] + HA^2X - A^3X\\
= &- G(X,SU)^\top + 2\kappa G(X,U)^\top + g(AX,SU)U\\
&+ \big(2\kappa - \eta(SU)\big)AX - 2\kappa\eta(AX)U + \frac{1}{4}\phi\nabla_X U\\
&- \frac{1}{3}\Big[- 2a\phi(\nabla_X T)U - b\phi(\nabla_X\phi)TU - b\phi^2(\nabla_X T)U\\
&\qquad + 2b^2\phi\nabla_X U - a\phi T\nabla_X U\Big]\\
&- H\phi\Big(\alpha(-G(X,\xi) + \phi AX) - A(-G(X,\xi) + \phi AX)\Big)\\
&+ \phi\Big(\alpha^2(-G(X,\xi) + \phi AX) - A^2(-G(X,\xi) + \phi AX)\Big).
\end{aligned} \tag{102}$$

Substituting (75) into (102) with $X = e_3$ yields

$$\begin{aligned}
&\frac{1}{\sqrt{3}}\Big[(a + \frac{1}{4} + \frac{a^2}{2})e_2 + \frac{a-1}{2}be_1\Big] - \Big[aTAe_3 + b\phi TAe_3\Big]\\
&+ 3HA^2e_3 - 3A^3e_3\\
= &(-\frac{19}{2} + b^2 + 6\kappa)Ae_3 + \Big[ba\phi Ae_3 + a\phi T\phi Ae_3\Big]\\
&+ 3H\phi A\phi Ae_3 - 3\phi A^2\phi Ae_3,
\end{aligned} \tag{103}$$

which implies

$$a^2 a_{33} = -ab a_{34} \tag{104}$$

by taking the inner product of (103) with e_3.

Similarly, choosing $X = e_4$ in (102) and applying the same method, we get

$$\begin{aligned} &\frac{1}{\sqrt{3}}\Big[\frac{1-a^2}{2}e_1 + \frac{1-3a}{2}be_2\Big] - \Big[aTAe_4 + b\phi TAe_4\Big] \\ &+ 3HA^2e_4 - 3A^3e_4 \\ = &(-\frac{19}{2} + b^2 + 6\kappa)Ae_4 + [ab\phi Ae_4 + a\phi T\phi Ae_4] \\ &+ 3H\phi A\phi Ae_4 - 3\phi A^2\phi Ae_4. \end{aligned} \tag{105}$$

Moreover, the inner product of the above equation with e_4 gives

$$b^2 a_{44} = -ab a_{34}. \tag{106}$$

Letting $X = e_2$ in (102) and taking the inner product with e_2 and e_1, respectively, yields

$$(1+a)^2 a_{22} = b(a-1)a_{12}, \tag{107}$$

$$-(1-a)^2 a_{12} = b(a-1)a_{22}. \tag{108}$$

Letting $X = e_1$ in (102) and taking the inner product with e_1 yields

$$(1-a)^2 a_{11} = b(1-a)a_{12}. \tag{109}$$

Since $b \neq 0$, according to (72), we can get

$$a_{12} = \frac{1+a}{b}a_{22}.$$

Hence substituting this into (107), we derive $a_{22} = a_{12} = 0$. By (109), we further obtain $a_{11} = 0$. Taking the inner product of (103) with e_4, we get

$$-2ba_{34} = (b+a)a_{33}.$$

Substituting this into (104) we derive

$$a(a-b)a_{33} = 0,$$

thus, if $a \neq b$, it is clear that $a_{33} = 0$. Further, by (104) and (106), we have $a_{34} = 0$ and $a_{44} = 0$ $(a \neq b)$. Hence we get $H = \alpha$.

If $a = b$, we have $a_{33} = -a_{34} = a_{44}$ from (104) and (106). Taking the inner product of (100) with e_2, we get

$$\frac{5}{2} - 2\kappa = \sum_i a_{i1}^2 + \sum_i a_{i2}^2.$$

Taking the inner product of (101) with e_4, we get

$$\frac{5}{2} - 2\kappa = \sum_i a_{3i}^2 + \sum_i a_{4i}^2 - H(a_{33} + a_{44}).$$

Comparing the above two formulas and substituting $a_{11} = a_{12} = a_{22} = 0$ into the resulting relation, we thus have

$$Ha_{33} = 2a_{33}^2.$$

Let us suppose $a_{33} \neq 0$. Thus $H = 2a_{33}$ and $\alpha = 0$ since $a_{33} = a_{44}$.

Taking the inner product of (101) with e_2, we find

$$Ha_{13} + Ha_{24} = \sum_i a_{3i}a_{i1} + \sum_i a_{4i}a_{i2}.$$

Letting (81) add and subtract from the above formula respectively, we have $a_{13} = -a_{23}$ and $a_{14} = -a_{24}$.

Again, taking the inner product of (100) with e_3, we derive $a_{23} = a_{14}$, then $a_{13} = a_{24}$. Taking advantage of the above relations and considering (42), we immediately get a contradiction.

In conclusion, if $b \neq 0$ in an open set of M, the mean curvature $H = \alpha$ is constant.

Case (b): If $b \equiv 0$, then $\mu^\sharp = 0$, $a = \pm 1$. So Eqs. (70)-(73) become

$$\begin{cases} (1+a)a_{12} = 0, a_{13} = 0, \\ (1+a)a_{22} = 0, a_{23} = \frac{1}{2\sqrt{3}}. \end{cases}$$

Similar to the proof of Case (a), we can get $a_{33} = 0$ from (104). Taking the inner product of (105) with e_3, we get $aba_{44} = -a^2a_{34}$, further we derive $a_{34} = 0$. Next we divide it into two cases.

Case (b)-(i). If $a = 1$, then $a_{12} = a_{22} = 0$. So Eq. (81) becomes

$$a_{24}(2\alpha - a_{44}) = 0 \tag{110}$$

and Eq. (41) becomes

$$-\alpha a_{11} = 0. \tag{111}$$

Simultaneously, from (42) and (45) we can derive $\alpha(-\frac{5}{2\sqrt{3}} + a_{14}) + \frac{2}{\sqrt{3}}a_{11} = 0$ and $(a_{14} + \frac{\sqrt{3}}{2})\alpha = 0$. If $\alpha \neq 0$, then from (42) and (45) we obtain $a_{14} = \frac{5}{2\sqrt{3}}$ or $-\frac{\sqrt{3}}{2}$, respectively, which is a contradiction. Thus $\alpha = 0$ and $a_{11} = 0$ from (42). Thus we derive $H = a_{44}$.

If $a_{24} \neq 0$, by virtue of (110), we know $a_{44} = 2\alpha = 0$. That means $H = 0$. If $a_{24} = 0$, similar to Case (b) of Lemma 3.4, we can also get a contradiction.

Case (b)-(ii). If $a = -1$, then we can derive $a_{11} = 0$ by (109) and $a_{12} = 0$ by (108). In this case Eq. (81) becomes

$$a_{24}(2\alpha - a_{22} - a_{44}) = 0 \tag{112}$$

and Eq. (41) becomes

$$-\alpha a_{22} = 0. \tag{113}$$

Therefore, if $\alpha \neq 0$, we obtain $a_{14} = \frac{5}{2\sqrt{3}}$ or $-\frac{\sqrt{3}}{2}$ from (42) and (45), which is a contradiction. That means that $\alpha = 0$ and $H = a_{22} + a_{44}$. In addition, we can get $a_{14} = -\frac{1}{2\sqrt{3}}$ from (46).

If $a_{24} \neq 0$ in an open subset, $H = a_{22} + a_{44} = 2\alpha = 0$ by (112). If $a_{24} = 0$, taking advantage of Lemma 4.2, putting $X = Y = e_1$ in (94), we conclude

$$\frac{2}{3}b\Big[ba_{11} - aa_{12}\Big] - \frac{2}{3}a_{11} = -g(G(e_1, HAe_1 - A^2e_1)^\top, U) - \frac{2}{3}aa_{11},$$

then

$$Ha_{14} = \sum_i a_{1i}a_{i4} = a_{14}a_{44}. \tag{114}$$

Since $a_{14} \neq 0$, we derive $H = a_{44}$ and $a_{22} = 0$. Moreover, by a direct calculation, taking the inner product of (101) with e_1 and utilizing (114), we obtain $H = a_{44} = 0$.

In conclusion, if $b \equiv 0$, the mean curvature $H = 0$. It completes the proof of Lemma 4.4 □

Combining Lemma 4.3 and Lemma 4.4, we complete the proof of Theorem 1.3.

5. Ricci soliton on Hopf hypersurface in NK $\mathbb{S}^3 \times \mathbb{S}^3$

Recall that a Ricci soliton on Riemannian manifold M is the metric g satisfies the following equation

$$\frac{1}{2}\mathcal{L}_V g(X, Y) + Ric(X, Y) = \rho g(X, Y), \quad \forall X, Y \in TM, \tag{115}$$

where ρ is constant and V is called the potential vector field. If V is Killing, i.e., $\mathcal{L}_V g = 0$, the Ricci soliton becomes an Einstein metric. We notice that for a Ricci soliton, Cho proved the following result.

Lemma 5.1 ([8, Lemma 3.1])**.** *For a Ricci soliton (g, V) on a Riemannian manifold, the following equation holds*:

$$\frac{1}{2}\|\mathcal{L}_V g\|^2 = dr(V) + 2\mathrm{div}(\rho V - SV),$$

where r denotes the scalar curvature of g.

In this section we consider a Ricci soliton on a Hopf hypersurface of NK $\mathbb{S}^3 \times \mathbb{S}^3$ with potential Reeb vector field U. Then by virtue of (13), the Ricci soliton formula (115) becomes

$$\frac{1}{2}g((\phi A - A\phi)X, Y) + Ric(X, Y) = \rho g(X, Y).$$

From this, we have

$$\frac{1}{2}(\phi A - A\phi)X + SX = \rho X, \quad \forall X \in TM. \tag{116}$$

Let e_1 be an arbitrary local unit tangent vector field of M with $g(e_1, U) = 0$. Put

$$e_2 = Je_1,\ e_3 = \sqrt{3}G(e_1, \xi),\ e_4 = Je_3,\ e_5 = U.$$

We can check easily that $\{e_i\}_{i=1}^5$ is a local orthonormal frame of M from (3)-(6) and get

$$trace(\phi A - A\phi) = \sum_i g((\phi A - A\phi)e_i, e_i) = 0.$$

Thus it follow from (116) that

$$r = trace(S) = 5\rho = constant.$$

Moreover, since M is Hopf, i.e., $AU = \alpha U$, taking $X = U$ in (116) we obtain $SU = \rho U$. Therefore, in view of Lemma 5.1 with the potential vector field V being U, we get $\mathcal{L}_U g = 0$. That means that M is an Einstein Hopf hypersurface. But Theorem 1.1 shows that the NK $\mathbb{S}^3 \times \mathbb{S}^3$ admits no Einstein Hopf hypersurface. We thus complete the proof of Theorem 1.4.

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