Bull. Korean Math. Soc. **59** (2022), No. 6, pp. 1349–1357
https://doi.org/10.4134/BKMS.b210425
pISSN: 1015-8634 / eISSN: 2234-3016

AN ASSOCIATED SEQUENCE OF IDEALS OF AN INCREASING SEQUENCE OF RINGS

Ali Benhissi and Abdelamir Dabbabi

Abstract. Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings. We say that $\mathcal{I} = (I_n)_{n\geq 0}$ is an associated sequence of ideals of $\mathcal{A}$ if $I_0 = A_0$ and for each $n \geq 1$, I_n is an ideal of A_n contained in I_{n+1}. We define the polynomial ring and the power series ring as follows: $\mathcal{I}[X] = \{f = \sum_{i=0}^{n} a_i X^i \in \mathcal{A}[X] : n \in \mathbb{N}, a_i \in I_i\}$ and $\mathcal{I}[[X]] = \{f = \sum_{i=0}^{+\infty} a_i X^i \in \mathcal{A}[[X]] : a_i \in I_i\}$. In this paper we study the Noetherian and the SFT properties of these rings and their consequences.

Introduction

In this paper, a ring means a commutative ring with identity and every considered module are left side and unitary. Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings. According to Y. Haouat in [4], we define the polynomial and the power series subrings of $A[[X]]$, where $A = \bigcup_{n=0}^{+\infty} A_n$, by $\mathcal{A}[X] = \{f = \sum_{i=0}^{n} a_i X^i \in A[X],\ a_i \in A_i,\ 0 \leq i \leq n\}$ and $\mathcal{A}[[X]] = \{f = \sum_{i=0}^{+\infty} a_i X^i \in A[[X]],\ a_i \in A_i,\ i \geq 0\}$. Y. Haouat in [4] had proved that $\mathcal{A}[X]$ is Noetherian if and only if $\mathcal{A}[[X]]$ is Noetherian if and only if the ring A_0 is Noetherian and the A_0-module A is finitely generated, where X is one indeterminate over A. We prove that we have the same even when X has more than one variable. This construction includes the rings of the form $A + XB[X]$ and $A + XB[[X]]$, where $A \subseteq B$ is a ring extension. On the other hand, M. D'anna, C. A. Finocchiaro and M. Fontana in [2] had shown that if $A \subseteq B$ is a ring extension and I an ideal of B, then $A + XI[X]$ is Noetherian if and only if the ring A is Noetherian, I is an idempotent ideal of B and it is finitely generated as an A-module. The naturel question in this case is when is $A + XI[[X]]$ Noetherian? In this work, we generalize these forms of ring as follows. Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings. We say that $\mathcal{I} = (I_n)_{n\geq 0}$ is an associated sequence of ideals of $\mathcal{A}$ if $I_0 = A_0$ and for each $n \geq 1$, I_n is an ideal of A_n contained in I_{n+1}. We define the polynomial ring and the power series ring as follows. $\mathcal{I}[X] = \{f = \sum_{i=0}^{n} a_i X^i \in \mathcal{A}[X] : n \in \mathbb{N}, a_i \in I_i\}$ and

Received June 1, 2021; Accepted August 1, 2022.
2020 *Mathematics Subject Classification.* Primary 13B25, 13B35, 13E05.
Key words and phrases. Noetherian ring, SFT rings, power series ring, I-adic topology.

$\mathcal{I}[[X]] = \{f = \sum_{i=0}^{+\infty} a_i X^i \in \mathcal{A}[[X]] : a_i \in I_i\}$. We give a sufficient condition for the rings $\mathcal{I}[X]$ and $\mathcal{I}[[X]]$ to be Noetherian and we answer the question posed above. In fact, we show that A is Noetherian and the ideal I is idempotent and it is finitely generated as an A-module if and only if $A + XI[[X]]$ is Noetherian. We find the result of S. Hizem and A. Benhissi in [5] when $I = B$. This generalization covers another kind of ring, those of the form $A + X^n I[X]$ and $A + X^n I[[X]]$, where $n \geq 1$ an integer.

Arnold in [1] has introduced the SFT property. He called a ring A to be SFT, if for each ideal I of A there exist a finitely generated ideal $F \subseteq I$ and an integer $n \geq 1$ such that $x^n \in F$ for every $x \in I$. This property have played an important role in the Krull dimension theory of power series ring. B. G. Kang and M. H. Park in [6] had showed that if A is an SFT Prüfer domain, then the mixed ring $A[X_1]][X_2]] \cdots [X_k]]$ is SFT, where $k \geq 1$ is an integer and $[X]] = [X]$ or $[X]] = [[X]]$. By using their result and some results of the I-adic topology, we give sufficient conditions for the rings of the form $\mathcal{I}[X]$ and $\mathcal{I}[[X]]$ to be SFT, where $\mathcal{I} = (I_n)_{n\geq 0}$ is an associated sequence of ideals of a given increasing sequence of rings $\mathcal{A} = (A_n)_{n\geq 0}$. This helps us to give sufficient conditions for the rings of the form $A + XI[X]$ and $A + XI[[X]]$ to be SFT as an easy consequence, where $A \subseteq B$ is a ring extension and I is an ideal of B. At the end of this paper, we prove that for a ring extension $A \subseteq B$, if A is an SFT Prüfer domain and B is a finitely generated A-module, then B is also SFT.

1. The Noetherian property

We start this section by the following definition.

Definition. Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings. We say that $\mathcal{I} = (I_n)_{n\geq 0}$ is an associated sequence of ideals of $\mathcal{A}$ if $I_0 = A_0$ and for each $n \geq 1$, I_n is an ideal of A_n contained in I_{n+1}.

Example 1.1. Let A be a ring, $A_0 = A = I_0$ and for each $n \geq 1$, $A_n = A[X_1, \ldots, X_n]$ and $I_n = \langle X_1, \ldots, X_n \rangle A_n$. Then $\mathcal{I} = (I_n)_{n\geq 0}$ is an associated sequence of ideals of $\mathcal{A} = (A_n)_{n\geq 0}$.

Notation 1. Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n\geq 0}$ be an associated sequence of ideals of $\mathcal{A}$.

(1) We denote by $\mathcal{I}[X] = \{f = \sum_{i=0}^{n} a_i X^i \in \mathcal{A}[X] : n \in \mathbb{N}, a_i \in I_i\}$.
(2) We denote by $\mathcal{I}[[X]] = \{f = \sum_{i=0}^{+\infty} a_i X^i \in \mathcal{A}[[X]] : a_i \in I_i\}$.

Remark 1.2. Under the same notations, the set $\mathcal{I}[X]$ (resp. $\mathcal{I}[[X]]$) is a subring of $\mathcal{A}[X]$ (resp. $\mathcal{A}[[X]]$).

Proposition 1.3. *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n\geq 0}$ be an associated sequence of ideals of $\mathcal{A}$. If $\mathcal{I}[X]$ is Noetherian, then*

(1) *The ring A_0 is Noetherian.*

(2) *For every $n \geq 1$, the A_0-module I_n is finitely generated.*
(3) *There exists $N \in \mathbb{N}$ such that, for each $n \geq N$, $I_n = I_N = I_N^2$.*

Proof. (1) Let J be an ideal of A_0. Then $J\mathcal{I}[X]$ is an ideal of $\mathcal{I}[X]$, hence finitely generated. It yields that there exist $a_1, \dots, a_n \in J$ such that, $J\mathcal{I}[X] = \langle a_1, \dots, a_n \rangle \mathcal{I}[X]$. Thus $J = \langle a_1, \dots, a_n \rangle A_0$. Therefore, A_0 is a Noetherian ring.

(2) Let $n \geq 1$. The ideal J of $\mathcal{I}[X]$ generated by $\{aX^n,\ a \in I_n\}$ is finitely generated. Then there exist $a_1, \dots, a_k \in I_n$ such that

$$J = \langle a_1 X^n, \dots, a_k X^n \rangle \mathcal{I}[X].$$

Let $a \in I_n$. There exist $f_1, \dots, f_k \in \mathcal{I}[X]$ such that, $aX^n = \sum_{i=1}^k f_i a_i X^n$. Since X^n is a regular element, $a = \sum_{i=1}^k f_i a_i$. Thus $a = \sum_{i=1}^k f_i(0) a_i$ with $f_i(0) \in I_0 = A_0$. Hence the A_0-module I_n is finitely generated.

(3) The ideal J of $\mathcal{I}[X]$ generated by $\{a_n X^n,\ n \geq 1,\ a_n \in I_n\}$ is finitely generated. Then there exist $k \geq 1$ and $a_{i,j} \in I_i, 1 \leq i \leq k,\ 1 \leq j \leq n_i$ such that, $J = \langle a_{i,j} X^i,\ 1 \leq i \leq k,\ 1 \leq j \leq n_i \rangle \mathcal{I}[X]$. Let $n \geq k+1$, $a \in I_n$. There exist $f_{i,j} \in \mathcal{I}[X],\ 1 \leq i \leq k,\ 1 \leq j \leq n_i$ such that, $aX^n = \sum_{i=1}^k \sum_{j=1}^{n_i} f_{i,j} a_{i,j} X^i$. For $1 \leq i \leq k,\ 1 \leq j \leq n_i$, denote $f_{i,j} = \sum_{l=0}^{N_{i,j}} \alpha_{i,j,l} X^l$ with $\alpha_{i,j,l} \in I_l, 0 \leq l \leq N_{i,j}$. Thus $a = \sum_{i=1}^k \sum_{j=1}^{n_i} a_{i,j} \alpha_{i,j,n-i} \in I_k(I_{n-k} + \cdots + I_{n-1}) = I_k I_{n-1}$. Hence $I_n \subset I_k I_{n-1} \subseteq I_k$. Therefore, we have the equality.

Let $a \in I_k$. The ideal L of $\mathcal{I}[X]$ generated by $\{aX^i,\ i \geq k\}$ is finitely generated. Thus there exists $n \geq 1$ such that, $L = \langle aX^k, \dots, aX^{k+n} \rangle \mathcal{I}[X]$. Since $aX^{2k+n} \in L$, there exist $f_0, \dots, f_n \in \mathcal{I}[X]$ such that $aX^{2k+n} = \sum_{i=0}^n f_i a X^{k+i}$. For $1 \leq i \leq n$, denote $f_i = \sum_{j=0}^{n_i} a_{i,j} X^j$. By identification of the coefficient of X^{2k+n}, we obtain that $a = \sum_{i=0}^n a a_{i,k+n-i} \in I_k^2$. Hence $I_k = I_k^2$. □

Example 1.4. Let A be a ring, $B = A[Y]$, $n \geq 1$ and $J_n = Y^n B$. Then for each $m \geq 1$, the ring $B + X^m J_n[X] = A[Y] + X^m Y^n A[X, Y]$ is not Noetherian. Indeed, let $I_0 = A_k = B$ for every $k \geq 0$, $I_1 = \cdots = I_{m-1} = \{0\}$ for each $k \geq m$, $I_k = J_n$ and $\mathcal{I} = (I_k)_{k \geq 0}$. It is clear that $\mathcal{I}[X] = B + X^m J_n[X]$. If $B + X^m J_n[X]$ is Noetherian, by Proposition 1.3, J_n is an idempotent ideal of B which is not the case.

Example 1.5. Let $n, m \geq 2$ be two integers, $A_0 = A_1 = A_2 = \cdots = \mathbb{Z}$, $I_0 = \mathbb{Z}$, $I_k = n^{m-k}\mathbb{Z}$ for $1 \leq k \leq m-1$ and $I_k = n\mathbb{Z}$ for each $k \geq m$. Then $\mathcal{I}[X]$ is not Noetherian with $\mathcal{I} = (I_k)_{k \geq 0}$, because I_m is not idempotent.

Recall that for a ring A, every finitely generated idempotent ideal is principal. In fact, this kind of ideal is generated by an idempotent element (by [3, Lemma 1]). By using this result we have the following proposition.

Proposition 1.6. *Let $\mathcal{A} = (A_n)_{n \geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n \geq 0}$ an associated sequence of ideals of $\mathcal{A}$. Assume that*

(1) *The ring A_0 is Noetherian.*

(2) *For each $n \geq 1$, the A_0-module I_n is finitely generated.*
(3) *There exists $N \in \mathbb{N}$ such that, for every $n \geq N$, $I_n = I_N = I_N^2$ and I_1 contains the idempotent generator of I_N.*

Then the ring $\mathcal{I}[X]$ is Noetherian.

Proof. For $1 \leq k \leq N$, let $\{a_{k,i},\ 1 \leq i \leq n_k\}$ be a generator family of A_0-module I_k and e_N an idempotent generator of I_N (i.e., $I_N = e_N A_N$ with $e_N^2 = e_N$). Add a family of indeterminates $Y = \{Y_{k,j},\ 1 \leq k \leq N,\ 1 \leq j \leq n_k\}$ over A_0. Let $A = \bigcup_{n=0}^{+\infty} A_n$. Let $\phi : A_0[X,Y] \longrightarrow A[X]$ be the A_0-homomorphism of rings such that $\phi(X) = e_N X$, $\phi(Y_{k,j}) = a_{k,j}X^k$, $1 \leq k \leq N$, $1 \leq j \leq n_k$. It is easy to check that $\phi(A_0[X,Y]) \subseteq \mathcal{I}[X]$. Conversely, let $f = \sum_{i=0}^{N} \alpha_i X^i + \sum_{N+1}^{l} \alpha_i X^i \in \mathcal{I}[X]$. For $1 \leq i \leq N$, we put $\alpha_i = \sum_{j=1}^{n_i} \beta_{i,j} a_{i,j}$ with $\beta_{i,j} \in A_0$. For $N+1 \leq i \leq l$, $\alpha_i \in I_i = I_N$, then we put $\alpha_i = \sum_{j=1}^{n_N} \gamma_{i,j} a_{N,j}$ with $\gamma_{i,j} \in A_0$. It yields that

$$\begin{aligned} f &= \alpha_0 + \sum_{i=1}^{N}\sum_{j=1}^{n_i} \beta_{i,j} a_{i,j} X^i + \sum_{i=N+1}^{l}\sum_{j=1}^{n_N} \gamma_{i,j} a_{N,j} X^i \\ &= \alpha_0 + \sum_{i=1}^{N}\sum_{j=1}^{n_i} \beta_{i,j} a_{i,j} X^i + \sum_{j=1}^{n_N}\Big(\sum_{i=N+1}^{l} \gamma_{i,j} X^{i-N}\Big) a_{N,j} X^N \\ &= \alpha_0 + \sum_{i=1}^{N}\sum_{j=1}^{n_i} \beta_{i,j} a_{i,j} X^i + \sum_{j=1}^{n_N}\Big(\sum_{i=N+1}^{l} \gamma_{i,j} X^{i-N}\Big) e_N a_{n,j} X^N \\ &= \alpha_0 + \sum_{i=1}^{N}\sum_{j=1}^{n_i} \beta_{i,j} a_{i,j} X^i + \sum_{j=1}^{n_N}\Big(\sum_{i=N+1}^{l} \gamma_{i,j} (e_N X)^{i-N}\Big) a_{N,j} X^N = \phi(g), \end{aligned}$$

where $g = \alpha_0 + \sum_{i=1}^{N} \sum_{j=1}^{n_i} \beta_{i,j} Y_{i,j} + \sum_{j=1}^{n_N} (\sum_{i=N+1}^{l} \gamma_{i,j} X^{i-N}) Y_{N,j} \in A_0[X,Y]$. Thus $\mathcal{I}[X] \subseteq \phi(A_0[X,Y])$. Therefore, $\mathcal{I}[X] = \phi(A_0[X,Y])$. Hence $\mathcal{I}[X] \simeq A_0[X,Y]/ker(\phi)$ is Noetherian because A_0 is Noetherian. □

Corollary 1.7 ([4, Chap. V, Proposition 1.2])**.** *Let $\mathcal{A} = (A_n)_{n \geq 0}$ be an increasing sequence of rings. The following statements are equivalent*:

(1) *The ring $\mathcal{A}[X]$ is Noetherian.*
(2) *The ring A_0 is Noetherian, the sequence $\mathcal{A}$ is stationary and for each $n \geq 1$ the A_0-module A_n is finitely generated.*

Proof. (1) ⇒ (2) It follows from Proposition 1.3.

(2) ⇒ (1) The idempotent generator of A_n is $1 \in A_1$ for each $n \geq 1$. By Proposition 1.6, the ring $\mathcal{A}[X]$ is Noetherian. □

Proposition 1.8. *Let $\mathcal{A} = (A_n)_{n \geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n \geq 0}$ an associated sequence of ideals of $\mathcal{A}$. If $\mathcal{I}[[X]]$ is Noetherian, then*

(1) *The ring A_0 is Noetherian.*

(2) *For each $n \geq 1$, the A_0-module I_n is finitely generated.*
(3) *There exists $N \in \mathbb{N}$ such that, for each $n \geq N$, $I_n = I_N = I_N^2$.*

Proof. The same proof as in the case of polynomial ring in Proposition 1.3. □

Lemma 1.9. *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n\geq 0}$ an associated sequence of ideals of $\mathcal{A}$. Assume that there exists $N \in \mathbb{N}$ such that, for each $n \geq N$, $I_n = I_N = I_N^2$, the A_0-module I_N is finitely generated and I_1 contains the idempotent generator of I_N. Then $\mathcal{I}[[X]]$ is the completion of $\mathcal{I}[X]$ for the $X^N I_N \mathcal{I}[X]$-adic topology.*

Proof. Denote $J = X^N I_N \mathcal{I}[[X]]$. Since $\bigcap_{n=1}^{+\infty} J^n = \{0\}$, $\mathcal{I}[[X]]$ is Hausdorff for its J-adic topology. We have $J \bigcap \mathcal{I}[X] = X^N I_N \mathcal{I}[X]$. Indeed, let $f = \sum_{i=N}^{k} a_i X^i \in J \bigcap \mathcal{I}[X]$. $f = \sum_{i=N}^{k} a_i X^N (e_N X)^{i-N}$ with e_N is the idempotent generator of I_N. Since $N \leq i \leq k$, $a_i \in I_i = I_N$ and $(e_N X)^{i-N} \in \mathcal{I}[X]$, $f \in X^N I_N \mathcal{I}[X]$. Therefore, the J-adic topology of $\mathcal{I}[[X]]$ induces the $X^N I_N \mathcal{I}[X]$-adic topology over $\mathcal{I}[X]$. Let $f = \sum_{i=0}^{+\infty} a_i X^i \in \mathcal{I}[[X]]$ and for each $k \geq 1$, $g_k = \sum_{i=0}^{k} a_i X^i$. For every $l \geq 1$ and $k \geq lN$, $f - g_k = \sum_{i=k+1}^{+\infty} a_i X^i \in X^{k+1} I_N \mathcal{I}[X]$. Indeed, $f - g_k = \sum_{i=k+1}^{k+1+N} a_i X^{k+1} (e_N X)^{i-(k+1)} + e_N X^{k+1} \sum_{i=k+2+N}^{+\infty} a_i X^{i-(k+1)}$ with $e_N \in I_N$, $\sum_{i=k+2+N}^{+\infty} a_i X^{i-(k+1)} \in \mathcal{I}[[X]]$, for each $k+1 \leq i \leq k+1+N$, $a_i \in I_i = I_N$ (because $i \geq N$) and $e_N X \in \mathcal{I}[[X]]$. By the same process we show that $X^{k+1} I_N \mathcal{I}[[X]] \subseteq X^l(N) I_N \mathcal{I}[[X]]$. It yields that $f \in X^{k+1} I_N \mathcal{I}[[X]] \subseteq X^{lN} I_N \mathcal{I}[[X]] = (X^N I_N)^l \mathcal{I}[[X]] = J^l$ (because I_N is idempotent). Then f is the limit of $(g_k)_{k\geq 1}$ in $\mathcal{I}[X]$ for its J-adic topology.

Conversely, let $(g_k)_{k\geq 0}$ be a Cauchy's sequence of $\mathcal{I}[X]$ for its $X^N I_N \mathcal{I}[X]$-adic topology and $g = g_0 + \sum_{i=1}^{+\infty} (g_i - g_{i-1})$. Since $(g_k)_{k\geq 0}$ is a Cauchy's sequence, $g \in \mathcal{I}[[X]]$. Let $l \geq 1$. There exists $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $g_{k+1} - g_k \in (X^N I_N \mathcal{I}[X])^l \subseteq J^l$. Thus $g - g_k = \sum_{i=k+1}^{+\infty} (g_i - g_{i-1}) = \sum_{i=lN}^{+\infty} a_j X^j = \sum_{j=lN}^{2lN} a_j X^N (e_N X)^{j-lN} + e_N X^{lN} \sum_{j=2lN+1}^{+\infty} a_j X^{j-lN} \in J^l$ for every $k \geq k_0$. Hence, g is the limit of $(g_k)_{k\geq 0}$ in $\mathcal{I}[[X]]$ for its J-adic topology. □

Theorem 1.10. *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n\geq 0}$ an associated sequence of ideals of $\mathcal{A}$. Assume that*

(1) *The ring A_0 is Noetherian.*
(2) *For each $n \geq 1$, the A_0-module I_n is finitely generated.*
(3) *There exists $N \in \mathbb{N}$ such that, for each $n \geq N$, $I_n = I_N = I_N^2$ and I_1 contains the idempotent generator of I_N.*

Then the ring $\mathcal{I}[[X]]$ is Noetherian.

Proof. By Proposition 1.6, the ring $\mathcal{I}[X]$ is Noetherian. By Lemma 1.9, $\mathcal{I}[[X]]$ is the completion of $\mathcal{I}[X]$ for its $X^N I_N \mathcal{I}[X]$-adic topology. Hence $\mathcal{I}[[X]]$ is Noetherian. □

In the next corollary the equivalence (1) $\Rightarrow$ (2) is shown in [2, Example 5.11]. It is an easy consequence of Propositions 1.3 and 1.6.

Corollary 1.11. *Let $A \subseteq B$ be a ring extension and I an ideal of B. The following statements are equivalent*:

(1) *The ring A is Noetherian, the ideal I is idempotent and it is a finitely generated A-module.*
(2) *The ring $A + XI[X]$ is Noetherian.*
(3) *The ring $A + XI[[X]]$ is Noetherian.*

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 1.6.
(2) $\Rightarrow$ (1) It follows from Proposition 1.3.
(1) $\Rightarrow$ (3) It follows from Theorem 1.10.
(3) $\Rightarrow$ (1) It follows from Proposition 1.8. $\square$

Notation 2. Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $X = \{X_1, \dots, X_k\}$ a finite set of indeterminates over $A = \bigcup_{n=0}^{+\infty} A_n$.

(1) For $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, denote $X^\alpha = X_1^{\alpha_1} \cdots X_k^{\alpha_k}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_k$.
(2) Denote $\mathcal{A}[X] = \{f = \sum_{\alpha\in\mathbb{N}^k} a_\alpha X^\alpha \in A[X],\ a_\alpha \in A_{|\alpha|}\}$.
(3) Denote $\mathcal{A}[[X]] = \{f = \sum_{\alpha\in\mathbb{N}^k} a_\alpha X^\alpha \in A[[X]],\ a_\alpha \in A_{|\alpha|}\}$.

Proposition 1.12. *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $X = \{X_1, \dots, X_k\}$ a finite set of indeterminates over $A = \bigcup_{n=0}^{+\infty} A_n$. The following statements are equivalent*:

(1) *The ring A_0 is Noetherian, the sequence $\mathcal{A}$ is stationary and the A_0-module A_n is finitely generated for every $n \geq 1$.*
(2) *The polynomial ring $\mathcal{A}[X]$ is Noetherian.*
(3) *The power series ring $\mathcal{A}[[X]]$ is Noetherian.*

Proof. (1) $\Rightarrow$ (2) and (3). Since $\mathcal{A}$ is stationary, there exists $N \in \mathbb{N}$ such that $A = A_N$ (because $A_n \subseteq A_N$ for each $n \geq 0$). But the A_0-module A_N is finitely generated, hence the A_0-module A is finitely generated. Thus the $A_0[X]$-module (resp. $A_0[[X]]$-module) $A[X]$ (resp. $A[[X]]$) is finitely generated. Since A_0 is Noetherian, the ring $A_0[X]$ (resp. $A_0[[X]]$) is Noetherian. It yields that the $A_0[X]$-module (resp. $A_0[[X]]$-module) $A[X]$ (resp. $A[[X]]$) is Noetherian. Therefore, the $A_0[X]$-submodule (resp. $A_0[[X]]$-submodule) $\mathcal{A}[X]$ (resp. $\mathcal{A}[[X]]$) of $A[X]$ (resp. of $A[[X]]$) is Noetherian. Thus the rings $\mathcal{A}[X]$ and $\mathcal{A}[[X]]$ are Noetherian.

(2) $\Rightarrow$ (1) It is clear that if $\mathcal{A}[X]$ is Noetherian, so is $\mathcal{A}[X_1]$. By Proposition 1.3, we have the result.

(3) $\Rightarrow$ (1) Same proof as (2) $\Rightarrow$ (1) by using Proposition 1.8 instead of Proposition 1.3. $\square$

2. The SFT property

Proposition 2.1. *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n\geq 0}$ an associated sequence of ideals of $\mathcal{A}$. Assume that*

(1) *A_0 is an SFT Prüfer domain.*
(2) *For each $n \geq 1$, the A_0-module I_n is finitely generated.*
(3) *There exists $N \in \mathbb{N}$ such that, for each $n \geq N$, $I_n = I_N = I_N^2$ and I_1 contains the idempotent generator of I_N.*

Then the ring $\mathcal{I}[X]$ is SFT.

Proof. As in the proof of Proposition 1.6, we show that $\mathcal{I}[X] \simeq A_0[X, Y]/J$, where Y is a finite set of indeterminates over A_0 and J an ideal of $A_0[X, Y]$. By [6, Proposition 10], $A_0[X, Y]$ is an SFT ring, and so is $\mathcal{I}[X]$. □

Example 2.2. Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings. Assume that

(1) A_0 is an SFT Prüfer domain.
(2) The A_0-module A_n is finitely generated for each $n \geq 1$.
(3) The sequence $\mathcal{A}$ is stationary.

Then the ring $\mathcal{A}[X]$ is SFT.

Corollary 2.3. *Let $A \subseteq B$ be a ring extension and I an idempotent ideal of B. Assume that A is an SFT Prüfer domain and the A-module I is finitely generated. Then $A + XI[X]$ is an SFT ring.*

Theorem 2.4. *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $\mathcal{I} = (I_n)_{n\geq 0}$ an associated sequence of ideals of $\mathcal{A}$. Assume that*

(1) *A_0 is an SFT Prüfer domain.*
(2) *For each $n \geq 1$, the A_0-module I_n is finitely generated.*
(3) *There exists $N \in \mathbb{N}$ such that, for each $n \geq N$, $I_n = I_N = I_N^2$ and I_1 contains the idempotent generator of I_N.*

Then the ring $\mathcal{I}[[X]]$ is SFT.

Proof. As in the proof of Proposition 1.6, we show that there exist $Y = \{Y_1, \ldots, Y_k\}$ a finite family of indeterminates over A_0 and an ideal J of $A_0[X, Y]$ such that, $\mathcal{I}[X] \simeq A_0[X, Y]/J$. Let $\phi : \mathcal{I}[X] \longrightarrow A_0[X, Y]/J$ be such an isomorphism and $F = \phi(X^N I_N \mathcal{I}[X])$. By Lemma 1.9, $\mathcal{I}[[X]]$ is the completion of $\mathcal{I}[X]$ for its $X^N I_N \mathcal{I}[X]$-adic topology, it yields that the ring $\mathcal{I}[[X]]$ is isomorphic to the completion of $A_0[X, Y]/J$ for its F-adic topology. We have $I_N = \langle a_1, \ldots, a_m \rangle A_0$ with $a_1, \ldots, a_m \in I_N$. Therefore,

$$X^N I_N = \langle a_1 X^N, \ldots, a_m X^N \rangle A_0.$$

Thus $X^N I_N \mathcal{I}[X] = \langle a_1 X^N, \ldots, a_m X^N \rangle \mathcal{I}[X]$ is a finitely generated ideal of $\mathcal{I}[X]$. Hence F is a finitely generated ideal of $A_0[X, Y]/J$. Then

$$\mathcal{I}[[X]] \simeq (A_0[X, Y]/J)[[Z]]/Q \simeq (A_0[X, Y][[Z]]/J[[Z]])/Q,$$

where $Z = \{Z_1, \ldots, Z_m\}$ is a finite family of indeterminates over A_0 and Q is an ideal of $(A_0[X,Y]/J)[[Z]]$. By [6, Proposition 10], $A_0[X,Y][[Z]]$ is an SFT ring, so is $(A_0[X,Y][[Z]]/J[[Z]])/Q$. Hence $\mathcal{I}[[X]]$ is an SFT ring. □

The next two corollaries are a simple application of the previous theorem.

Corollary 2.5. *Let $A \subseteq B$ be a ring extension and I an idempotent ideal of B. Assume that A is an SFT Prüfer domain and the A-module I is finitely generated. Then $A + XI[[X]]$ is an SFT ring.*

Corollary 2.6. *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings. Assume that*

(1) *A_0 is an SFT Prüfer domain.*
(2) *The A_0-module A_n is finitely generated for every $n \geq 1$.*
(3) *The sequence $\mathcal{A}$ is stationary.*

Then the ring $\mathcal{A}[[X]]$ is SFT.

In case of Noetherian rings, we know that if $A \subseteq B$ is a ring extension such that, the ring A is Noetherian and B is a finitely generated A-module, then B is also a Noetherian ring. It is natural to ask if the result holds in the case of SFT rings. The following proposition shows a partial answer to this question.

Proposition 2.7. *Let $A \subseteq B$ be a ring extension. Assume that A is an SFT Prüfer domain and B is a finitely generated A-module. Then B is an SFT-ring.*

Proof. Let I be an ideal of B. By Corollary 2.3, $XI[X]$ is an SFT-ideal of $A + XB[X]$, then there exist $k \geq 1$ and $f_1, \ldots, f_n \in XI[X]$ such that, for each $g \in XI[X]$, $g^k \in \langle f_1, \ldots, f_n \rangle$. Let $a \in I$. Since $a^k X^k = (aX)^k \in \langle f_1, \ldots, f_n \rangle$, $a^k \in F = c(f_1) + \cdots + c(f_n)$, where $c(f_i)$ is the ideal of B generated by the coefficients of f_i, $1 \leq i \leq n$. Then F is a finitely generated subideal of I. Thus B is an SFT-ring. □

References

[1] J. T. Arnold, *Krull dimension in power series rings*, Trans. Amer. Math. Soc. **177** (1973), 299–304. https://doi.org/10.2307/1996598
[2] M. D'Anna, C. A. Finocchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, in Commutative algebra and its applications, 155–172, Walter de Gruyter, Berlin, 2009.
[3] R. Gilmer, *An existence theorem for non-Noetherian rings*, Amer. Math. Monthly **77** (1970), 621–623. https://doi.org/10.2307/2316741
[4] Y. Haouat, *Thèse de doctorat*, Faculté des Sciences de Tunis, 1988.
[5] S. Hizem and A. Benhissi, *When is $A + XB[[X]]$ Noetherian?*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 1, 5–7. https://doi.org/10.1016/j.crma.2004.11.017
[6] B. G. Kang and M. H. Park, *Krull dimension of mixed extensions*, J. Pure Appl. Algebra **213** (2009), no. 10, 1911–1915. https://doi.org/10.1016/j.jpaa.2009.02.010

Ali Benhissi
Mathematics of Department
Faculty of Sciences of Monastir
Monastir University
Tunisia
Email address: ali_benhissi@yahoo.fr

Abdelamir Dabbabi
Mathematics of Department
Faculty of Sciences of Monastir
Monastir University
Tunisia
Email address: amir.dababi.25@gmail.com