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SOME REMARKS ON PROBLEMS OF SUBSET SUM

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Abstract. Let $A = \{a_1 < a_2 < \cdots\}$ be a sequence of integers and let $P(A) = \{\sum \varepsilon_i a_i : a_i \in A, \varepsilon_i = 0 \text{ or } 1, \sum \varepsilon_i < \infty\}$. Burr posed the following question: Determine conditions on integers sequence B that imply either the existence or the non-existence of A for which $P(A)$ is the set of all non-negative integers not in B. In this paper, we focus on some problems of subset sum related to Burr's question.

1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a sequence of integers $A = \{a_1 < a_2 < \cdots\}$, let

$$P(A) = \left\{\sum \varepsilon_i a_i : a_i \in A, \varepsilon_i = 0 \text{ or } 1, \sum \varepsilon_i < \infty\right\}.$$

Here $0 \in P(A)$.

In 1970, Burr [1] asked the following question: Determine conditions on integers sequence B that imply either the existence or the non-existence of A for which $P(A)$ is the set of all non-negative integers not in B. He showed the following result (unpublished):

Theorem A ([1]). *Let $B = \{4 \le b_1 < b_2 < \cdots\}$ be a sequence of integers for which $b_{n+1} \ge b_n^2$ for $n = 1, 2, \ldots$. Then there exists $A = \{a_1 < a_2 < \cdots\}$ for which $P(A) = \mathbb{N} \setminus B$.*

Burr [1] ever mentioned that if B grows "sufficiently rapidly", then there exists a sequence A such that $P(A) = \mathbb{N} \setminus B$. More previous work has helped to clarify what "sufficiently rapidly" means.

In 1996, Hegyvári [6] improved Burr's result by relaxing the restriction "$b_{n+1} \ge b_n^2 (n \ge 1)$" to "$b_{n+1} \ge 5b_n (n \ge 1)$".

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Theorem B ([6], Theorem 1)**.** *Let $B = \{7 \le b_1 < b_2 < \cdots\}$ be a sequence of integers. Suppose that for every n, $b_{n+1} \ge 5b_n$. Then there exists a sequence of integers $A = \{a_1 < a_2 < \cdots\}$ for which $P(A) = \mathbb{N} \setminus B$.*

In 2012, Chen and Fang [2] precisely extended Hegyvári's result by elementary but not easy argument.

Theorem C ([2], Theorem 1)**.** *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers with $b_1 \in \{4, 7, 8\} \cup \{b : b \ge 11, b \in \mathbb{N}\}$ and $b_{n+1} \ge 3b_n + 5$ for all $n \ge 1$. Then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ for which $P(A) = \mathbb{N} \setminus B$.*

Theorem D ([2], Theorem 2)**.** *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of positive integers with $b_1 \in \{3, 5, 6, 9, 10\}$ or $b_2 = 3b_1 + 4$ or $b_1 = 1, b_2 = 9$ or $b_1 = 2, b_2 = 15$. Then there is no a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ for which $P(A) = \mathbb{N} \setminus B$.*

In 2013, Chen and Wu [3] further relaxed the restriction "$b_{n+1} \ge 3b_n + 5$" of Theorem D.

Theorem E ([3], Theorem 1)**.** *If $B = \{b_1 < b_2 < \cdots\}$ is a sequence of integers with $b_1 \in \{4, 7, 8\} \cup \{b : b \ge 11, b \in \mathbb{N}\}$, $b_2 \ge 3b_1 + 5$, $b_3 \ge 3b_2 + 3$ and $b_{n+1} > 3b_n - b_{n-2}$ for all $n \ge 3$, then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that $P(A) = \mathbb{N} \setminus B$ and*

$$P(A_s) = [0, 2b_s] \setminus \{b_1, \ldots, b_s, 2b_s - b_{s-1}, \ldots, 2b_s - b_1\},$$

where $A_s = A \cap [0, b_s - b_{s-1}]$ for all $s \ge 2$.

Theorem F ([3], Theorem 2)**.** *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers and $d_1 = 10$, $d_2 = 3b_1 + 4$, $d_3 = 3b_2 + 2$ and $d_{n+1} = 3b_n - b_{n-2}$ $(n \ge 3)$. If $b_m = d_m$ for some $m \ge 1$ and $b_n > d_n$ for all $n \ne m$, then there is no a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that*

$$P(A_s) = [0, 2b_s] \setminus \{b_1, \ldots, b_s, 2b_s - b_{s-1}, \ldots, 2b_s - b_1\}, \tag{1.1}$$

where $A_s = A \cap [0, b_s - b_{s-1}]$ for all $s \ge 2$.

Moreover, Chen and Wu [3] posed the following problem:

Problem 1 ([3], Problem 1)**.** *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of positive integers. Let $d_1 = 10$, $d_2 = 3b_1 + 4$, $d_3 = 3b_2 + 2$ and $d_{n+1} = 3b_n - b_{n-2}$ $(n \ge 3)$. If $b_m = d_m$ for some $m \ge 3$ and $b_n > d_n$ for all $n \ne m$. Is it true that there is no a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ with $P(A) = \mathbb{N} \setminus B$?*

With the further research of Burr's question, many related problems arise. For the related problems, see [4, 5, 7–9].

In this paper, we give a further contribution to this problem:

Theorem 1.1. *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers with $b_1 \in \{4, 7, 8\} \cup \{b : b \ge 11, b \in \mathbb{N}\}$, if $3b_1 + 5 \le b_2 \le 4b_1 - 2$, $b_3 = 3b_2 + 2$ and*

$b_{n+1} = 3b_n + 4b_{n-1}$ *for all* $n \geq 3$, *then there exists a sequence of positive integers* $A = \{a_1 < a_2 < \cdots\}$ *such that, for all* $k \geq 4$,

$$P(A_k) = [0, b_k + b_{k-1}] \setminus \{b_1, \ldots, b_k, b_k + b_{k-1} - b_i : i = 1, \ldots, k-2\},$$

where $A_k = A \cap [0, b_{k-1} + 2b_{k-2} - b_{k-3}]$.

Corollary 1.2. *Let* B *be as defined above. Then there exists a sequence of positive integers* $A = \{a_1 < a_2 < \cdots\}$ *such that* $P(A) = \mathbb{N} \setminus B$.

Remark 1.3. By Theorem F, choose $m = 3$, we know that if $B = \{11 \leq b_1 < b_2 < \cdots\}$ is a sequence of integers with

$$b_2 \geq 3b_1 + 5, b_3 = 3b_2 + 2, b_{n+1} \geq 3b_n - b_{n-2} \ (n \geq 3), \tag{1.2}$$

then there is no a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that (1.1). Our results show that given positive integers sequences B satisfying (1.2), although there is no a sequence of positive integers A satisfies "local" property (1.1), the sequence A satisfies other new "local" property, so that the sequence A still satisfies "global" property: $P(A) = \mathbb{N} \setminus B$. This result also shows that the answer to Problem 1 is negative for $m = 3$.

Moreover, we obtain a supplement result to Theorem D.

Theorem 1.4. *Let* $B = \{3 \leq b_1 < b_2 < \cdots\}$ *be a sequence of integers. If* $b_2 \in [b_1 + 2, 2b_1] \cup \{3b_1 + 2, 3b_1 + 3\}$, *then there is no a sequence of positive integers* $A = \{a_1 < a_2 < \cdots\}$ *such that* $P(A) = \mathbb{N} \setminus B$.

2. Lemmas

Lemma 2.1 ([8], Lemma 2.2)**.** *Let* $b_1 \in \{4, 7, 8\} \cup [11, \infty)$ *be an integer. Then there exists a sequence of positive integers* A_1 *with* $A_1 \subset [0, b_1 - 1]$ *such that* $P(A_1) = [0, b_1 - 1]$.

Lemma 2.2 ([8], Lemma 2.3)**.** *Let* $A = \{a_1 < a_2 < \cdots\}$ *and* $B = \{b_1 < b_2 < \cdots\}$ *be two sequences of positive integers. For any integer* $t \geq 3$, *let*

$$\begin{aligned} &P(\{a_1, \ldots, a_{k+t-1}\}) \\ &= [0, a_{k+2} + \cdots + a_{k+t-1} + 2b_1] \setminus \{b_1, a_{k+2} + \cdots + a_{k+t-1} + b_1\}. \end{aligned}$$

(i) *If* $a_{k+2} + \cdots + a_{k+t-1} + b_1 \geq a_{k+t}$ *and* $a_{k+2} + \cdots + a_{k+t-1} \neq a_{k+t}$, *then*

$$P(\{a_1, \ldots, a_{k+t}\}) = [0, a_{k+2} + \cdots + a_{k+t} + 2b_1] \setminus \{b_1, a_{k+2} + \cdots + a_{k+t} + b_1\}.$$

(ii) *If* $a_{k+2} + \cdots + a_{k+t-1} + b_1 < a_{k+t}$, *then* $b_3 > b_2 + b_1$.

(iii) *If* $a_{k+2} + \cdots + a_{k+t-1} = a_{k+t}$ *and* $a_{k+t} + b_1 < a_{k+t+1}$, *then* $b_3 > b_2 + b_1$.

(iv) *If* $a_{k+2} + \cdots + a_{k+t-1} = a_{k+t}$ *and* $a_{k+t} + b_1 \geq a_{k+t+1}$, *then*

$$P(\{a_1, \ldots, a_{k+t+1}\}) = [0, a_{k+2} + \cdots + a_{k+t+1} + 2b_1] \setminus \{b_1, a_{k+2} + \cdots + a_{k+t+1} + b_1\}.$$

The following lemma is contained in the proof of [8, Theorem 1.3]. For the sake of readability, we give a self-contained proof.

Lemma 2.3. *Let b_1, b_2 be two positive integers satisfying $b_1 \in \{4,7,8\} \cup \{b : b \geq 11, b \in \mathbb{N}\}$. If $b_2 \geq 3b_1 + 5$, then there exists a finite sequence of positive integers $A = \{a_1 < \cdots < a_k < a_{k+1} < \cdots < a_{k+s} < b_1 + b_2\}$ such that*

$$P(\{a_1, \ldots, a_{k+s}\}) = [0, b_1 + b_2] \backslash \{b_1, b_2\},$$

where k, s are the indexes such that $a_k < b_1 < a_{k+1}$ and

$$sb_1 + \frac{s(s+1)}{2} \leq b_2 + 1 \leq (s+1)b_1 + \frac{s(s+3)}{2}.$$

Proof. By Lemma 2.1, there exists $A_1 = \{a_1 < a_2 < \cdots < a_k\} \subset [0, b_1 - 1]$ such that

$$P(A_1) = [0, b_1 - 1], \tag{2.1}$$

where k is the indexes such that $a_k < b_1 < a_{k+1}$. For $i = 3, 4, \ldots$, let

$$T_i = \left[ib_1 + \frac{i(i+1)}{2}, (i+1)b_1 + \frac{i(i+3)}{2} \right].$$

For all $i \geq 3$, we have $\min T_{i+1} = \max T_i + 1$. Thus $T_i \cap T_j = \emptyset$ for all $i \neq j$. Hence

$$[3b_1 + 6, +\infty] = \bigcup_{i=3}^{\infty} T_i.$$

Since $b_2 \geq 3b_1 + 5$, we know that there exists an $s \geq 3$ such that $b_2 + 1 \in T_s$. Thus

$$sb_1 + \frac{s(s+1)}{2} \leq b_2 + 1 \leq (s+1)b_1 + \frac{s(s+3)}{2}.$$

Let

$$r = b_2 + 1 - (b_1 + 1) - (b_1 + 2) - \cdots - (b_1 + s).$$

Then $0 \leq r \leq b_1 + s$. Hence,

$$b_2 + 1 = (b_1 + 1) + (b_1 + 2) + \cdots + (b_1 + s) + r, \quad 0 \leq r \leq b_1 + s. \tag{2.2}$$

By the proof of [8, Theorem 1.3], we know that there exist $r_2, \ldots, r_s$ and $\varepsilon(r)$ such that

$$r = r_2 + \cdots + r_s + \varepsilon(r), \quad 0 \leq r_2 \leq r_3 \leq \cdots \leq r_s \leq b_1 - 1,$$

where $r_j - r_{j-1} \leq b_1 - 2$ for any $3 \leq j \leq s$; $\varepsilon(0) = 0$, $\varepsilon(r) = 1$ $(r \geq 1)$.

Let $a_{k+1} = b_1 + 1$ and

$$a_{k+s} = b_1 + s + r_s + \varepsilon(r), \ a_{k+t} = b_1 + t + r_t, \ 2 \leq t \leq s - 1. \tag{2.3}$$

By (2.2)-(2.3), we have

$$a_{k+2} + \cdots + a_{k+s} + b_1 = b_2. \tag{2.4}$$

$$a_{k+t-1} < a_{k+t} \leq a_{k+t-1} + b_1, \quad 2 \leq t \leq s. \tag{2.5}$$

Since $a_{k+1} = b_1 + 1$, by (2.1) we have

$$P(\{a_1, \ldots, a_{k+1}\}) = [0, 2b_1] \backslash \{b_1\},$$

$$a_{k+2} + P(\{a_1, \ldots, a_{k+1}\}) = [a_{k+2}, a_{k+2} + 2b_1] \backslash \{a_{k+2} + b_1\}.$$

Noting that $a_{k+1} < a_{k+2} \leq a_{k+1} + b_1$, we have

$$P(\{a_1, \ldots, a_{k+2}\}) = [0, a_{k+2} + 2b_1] \backslash \{b_1, a_{k+2} + b_1\}.$$

By (2.5) we know that for all integers $3 \leq t \leq s$ we have

$$\begin{aligned} &a_{k+2} + \cdots + a_{k+t-1} + b_1 \geq a_{k+t-1} + b_1 \geq a_{k+t}, \\ &a_{k+2} + \cdots + a_{k+t-1} \geq a_{k+t-1} + a_{k+2} > a_{k+t-1} + b_1 \geq a_{k+t}, \end{aligned} \tag{2.6}$$

thus

$$a_{k+2} + \cdots + a_{k+t-1} \neq a_{k+t}. \tag{2.7}$$

By (2.6) and (2.7), repeat Lemma 2.2(i) $s - 2$ times, we have

$$P(\{a_1, \ldots, a_{k+s}\}) = [0, a_{k+2} + \cdots + a_{k+s} + 2b_1] \backslash \{b_1, a_{k+2} + \cdots + a_{k+s} + b_1\}.$$

Hence, by (2.4) we have $P(\{a_1, \ldots, a_{k+s}\}) = [0, b_1 + b_2] \backslash \{b_1, b_2\}$.

This completes the proof of Lemma 2.3. □

3. Proof of Theorem 1.1

We shall construct a set sequence $\{A_k\}_{k=3}^{\infty}$ such that, for $k \geq 4$

(i) $A_k = A_{k-1} \cup \{b_{k-1} + 2b_{k-3}, b_{k-1} + b_{k-2} - b_{k-3}, b_{k-1} + 2b_{k-2} - b_{k-3}\}$;

(ii) $P(A_k) = [0, b_k + b_{k-1}] \setminus \{b_1, \ldots, b_k, b_k + b_{k-1} - b_i : i = 1, \ldots, k-2\}$.

By Lemma 2.3, there exists $A_1 = \{a_1 < \cdots < a_{k+s} < b_1 + b_2\}$ such that

$$P(\{a_1, \ldots, a_{k+s}\}) = [0, b_1 + b_2] \backslash \{b_1, b_2\}, \tag{3.1}$$

where k, s are the indexes such that $a_k < b_1 < a_{k+1}$ and

$$sb_1 + \frac{s(s+1)}{2} \leq b_2 + 1 \leq (s+1)b_1 + \frac{s(s+3)}{2}.$$

Let $a_{k+s+1} = b_1 + b_2$, $a_{k+s+2} = 2b_2 - 2b_1 + 2$. Noting that

$$\max A_1 = a_{k+s} < b_1 + b_2 < 2b_2 - 2b_1 + 2,$$

we have

$$b_1 + b_2 + P(\{a_1, \ldots, a_{k+s}\}) = [b_1 + b_2, 2b_1 + 2b_2] \backslash \{2b_1 + b_2, b_1 + 2b_2\}. \tag{3.2}$$

By (3.1), (3.2) and $b_3 = 3b_2 + 2$, we have

$$P(\{a_1, \ldots, a_{k+s+1}\}) = [0, 2b_1 + 2b_2] \backslash \{b_1, b_2, 2b_1 + b_2, b_1 + 2b_2\},$$

$$a_{k+s+2} + P(\{a_1, \ldots, a_{k+s+1}\}) = [2b_2 - 2b_1 + 2, b_3 + b_2] \setminus \mathcal{B}_0,$$

where $\mathcal{B}_0 = \{2b_2 - b_1 + 2, 3b_2 - 2b_1 + 2, b_3, b_3 + b_2 - b_1\}$.

Write

$$A_3 = A_1 \cup \{b_1 + b_2, 2b_2 - 2b_1 + 2\}.$$

Since $b_2 \leq 4b_1 - 2$, we have

$$2b_2 - 2b_1 + 2 \leq 2b_1 + b_2 < 2b_2 - b_1 + 2 < b_1 + 2b_2 < 3b_2 - 2b_1 + 2 \leq 2b_1 + 2b_2,$$

we have

$$P(A_3) = [0, b_3 + b_2] \setminus \{b_1, b_2, b_3, b_3 + b_2 - b_1\}.$$

To obtain the set A_4 satisfying (i) and (ii), we shall add three integers $b_3+2b_1, b_3+b_2-b_1, b_3+2b_2-b_1$ to set A_3.

First, we have the following observation

$$\max A_3 = 2b_2 - 2b_1 + 2 < b_3 + 2b_1 < b_3 + b_2 - b_1 < b_3 + 2b_2 - b_1.$$

Second, noting that

$$b_3 + 2b_1 + P(A_3) = [b_3 + 2b_1, 2b_3 + b_2 + 2b_1] \setminus \mathcal{B}_{3,1},$$

where

$$\mathcal{B}_{3,1} = \Big\{b_3 + 3b_1, b_3 + b_2 + 2b_1, 2b_3 + 2b_1, 2b_3 + b_2 + b_1\Big\}.$$

Then by $b_3 + 2b_1 < b_3 + b_2 - b_1 < b_3 + 3b_1 < b_2 + b_3$, we have

$$P(A_3 \cup \{b_3 + 2b_1\}) = [0, 2b_3 + b_2 + 2b_1] \setminus \mathcal{B}_{3,2},$$

where

$$\mathcal{B}_{3,2} = \Big\{b_1, b_2, b_3, b_3 + b_2 + 2b_1, 2b_3 + 2b_1, 2b_3 + b_2 + b_1\Big\}.$$

Noting that

$$b_3 + b_2 - b_1 + P(A_3 \cup \{b_3 + 2b_1\}) = [b_3 + b_2 - b_1, 3b_3 + 2b_2 + b_1] \setminus \mathcal{B}_{3,3},$$

where

$$\mathcal{B}_{3,3} = \Big\{b_3 + b_2, b_3 + 2b_2 - b_1, 2b_3 + b_2 - b_1, 2b_3 + 2b_2 + b_1, 3b_3 + b_2 + b_1, 3b_3 + 2b_2\Big\}.$$

Since

$$b_3 + b_2 < b_3 + b_2 + 2b_1 < b_3 + 2b_2 - b_1 < 2b_3 + 2b_1 < 2b_3 + b_2 - b_1 < 2b_3 + b_2 + b_1,$$

we have

$$P(A_3 \cup \{b_3 + 2b_1, b_3 + b_2 - b_1\}) = [0, 3b_3 + 2b_2 + b_1] \setminus \mathcal{B}_{3,4},$$

where

$$\mathcal{B}_{3,4} = \Big\{b_1, b_2, b_3, 2b_3 + 2b_2 + b_1, 3b_3 + b_2 + b_1, 3b_3 + 2b_2\Big\}.$$

Noting that

$$b_3 + 2b_2 - b_1 + P(A_3 \cup \{b_3 + 2b_1, b_3 + b_2 - b_1\}) = [b_3 + 2b_2 - b_1, 4b_3 + 4b_2] \setminus \mathcal{B}_{3,5},$$

where

$$\mathcal{B}_{3,5} = \Big\{b_3 + 2b_2, b_3 + 3b_2 - b_1, 2b_3 + 2b_2 - b_1, 3b_3 + 4b_2, 4b_3 + 3b_2, 4b_3 + 4b_2 - b_1\Big\}.$$

Since

$$b_3 + 2b_2 < b_3 + 3b_2 - b_1 < 2b_3 + 2b_2 - b_1 < 2b_3 + 2b_2 + b_1 < 3b_3 + b_2 + b_1 < 3b_3 + 2b_2,$$

we have

$$P(A_3 \cup \{b_3 + 2b_1, b_3 + b_2 - b_1, b_3 + 2b_2 - b_1\}) = [0, 4b_3 + 4b_2] \setminus \mathcal{B}_{3,6},$$

where

$$\mathcal{B}_{3,6} = \Big\{b_1, b_2, b_3, 3b_3 + 4b_2, 4b_3 + 3b_2, 4b_3 + 4b_2 - b_1\Big\}.$$

Let

$$A_4 = A_3 \cup \{b_3 + 2b_1, b_3 + b_2 - b_1, b_3 + 2b_2 - b_1\}. \tag{3.3}$$

Since $b_4 = 3b_3 + 4b_2$, we have

$$P(A_4) = [0, b_4 + b_3] \setminus \{b_1, b_2, b_3, b_4, b_4 + b_3 - b_2, b_4 + b_3 - b_1\}. \tag{3.4}$$

By (3.3) and (3.4), we know that the result is true for $k = 4$.

Suppose that the result is true for $k(\geq 4)$. That is,

$$A_k = A_{k-1} \cup \{b_{k-1} + 2b_{k-3}, b_{k-1} + b_{k-2} - b_{k-3}, b_{k-1} + 2b_{k-2} - b_{k-3}\},$$
$$P(A_k) = [0, b_k + b_{k-1}] \setminus \{b_1, \dots, b_k, b_k + b_{k-1} - b_i : i = 1, \dots, k-2\}.$$

Now we consider the case $k+1$. we shall add three integers $b_k + 2b_{k-2}, b_k + b_{k-1} - b_{k-2}, b_k + 2b_{k-1} - b_{k-2}$ to set A_k.

First, we have the following observation

$$\max A_k = b_{k-1}+2b_{k-2}-b_{k-3} < b_k+2b_{k-2} < b_k+b_{k-1}-b_{k-2} < b_k+2b_{k-1}-b_{k-2}.$$

Second, noting that

$$b_k + 2b_{k-2} + P(A_k) = [b_k + 2b_{k-2}, 2b_k + b_{k-1} + 2b_{k-2}] \setminus \mathcal{B}_{k,1},$$

where

$$\mathcal{B}_{k,1} = \Big\{b_k + 2b_{k-2} + b_i, 2b_k + b_{k-1} + 2b_{k-2} - b_i : i = 1, \dots, k-1\Big\}.$$

Since

$$\begin{aligned} &\mathbf{b_k + 2b_{k-2}} < b_k + 2b_{k-2} + b_1 < \cdots < b_k + 2b_{k-2} + b_{k-3} \\ &< \mathbf{b_k + 3b_{k-2}} \neq b_k + b_{k-1} - b_{k-2} \\ &< \mathbf{b_k + b_{k-1} - b_{k-3}} < \cdots < \mathbf{b_k + b_{k-1} - b_1}, \end{aligned}$$

we have

$$P(A_k \cup \{b_k + 2b_{k-2}\}) = [0, 2b_k + b_{k-1} + 2b_{k-2}] \setminus \mathcal{B}_{k,2},$$

where

$$\mathcal{B}_{k,2} = \Big\{b_1, \dots, b_k, 2b_k + b_{k-1} + 2b_{k-2} - b_i : i = 1, \dots, k\Big\}.$$

Noting that

$$\begin{aligned} &b_k + b_{k-1} - b_{k-2} + P(A_k \cup \{b_k + 2b_{k-2}\}) \\ &= [b_k + b_{k-1} - b_{k-2}, 3b_k + 2b_{k-1} + b_{k-2}] \setminus \mathcal{B}_{k,3}, \end{aligned}$$

where

$$\mathcal{B}_{k,3} = \Big\{b_k + b_{k-1} - b_{k-2} + b_i, 3b_k + 2b_{k-1} + b_{k-2} - b_i : i = 1, \dots, k\Big\}.$$

Since

$$\begin{aligned} &\mathbf{b_k+b_{k-1}-b_{k-2}} < b_k+b_{k-1}-b_{k-2}+b_1 < \cdots < b_k+b_{k-1} \\ &< \mathbf{b_k+b_{k-1}+2b_{k-2}} < b_k+2b_{k-1}-b_{k-2} < 2b_k+2b_{k-2} < 2b_k+b_{k-1}-b_{k-2} \\ &< \mathbf{2b_k+b_{k-1}+b_{k-2}} < \cdots < 2b_k+b_{k-1}+2b_{k-2}-b_1, \end{aligned}$$

we have

$$\begin{aligned} &P(A_k \cup \{b_k + 2b_{k-2}, b_k + b_{k-1} - b_{k-2}\}) \\ &= [0, 3b_k + 2b_{k-1} + b_{k-2}] \setminus \mathcal{B}_{k,4}, \end{aligned}$$

where

$$\mathcal{B}_{k,4} = \Big\{b_1, \ldots, b_k, 3b_k + 2b_{k-1} + b_{k-2} - b_i : i = 1, \ldots, k\Big\}.$$

Noting that

$$\begin{aligned} &b_k + 2b_{k-1} - b_{k-2} + P(A_k \cup \{b_k + 2b_{k-2}, b_k + b_{k-1} - b_{k-2}\}) \\ &= [b_k + 2b_{k-1} - b_{k-2}, 4b_k + 4b_{k-1}] \setminus \mathcal{B}_{k,5}, \end{aligned}$$

where

$$\mathcal{B}_{k,5} = \Big\{b_k + 2b_{k-1} - b_{k-2} + b_i, 4b_k + 4b_{k-1} - b_i : i = 1, \ldots, k\Big\}.$$

Since

$$\begin{aligned} &\mathbf{b_k + 2b_{k-1} - b_{k-2}} < b_k + 2b_{k-1} - b_{k-2} + b_1 < \cdots < 2b_k + 2b_{k-1} - b_{k-2} \\ &< \mathbf{2b_k + 2b_{k-1} + b_{k-2}} < \cdots < \mathbf{3b_k + 2b_{k-1} + b_{k-2} - b_1}, \end{aligned}$$

we have

$$P(A_k \cup \{b_k + 2b_{k-2}, b_k + b_{k-1} - b_{k-2}, b_k + 2b_{k-1} - b_{k-2}\}) = [0, 4b_k + 4b_{k-1}] \setminus \mathcal{B}_{k,6},$$

where

$$\mathcal{B}_{k,6} = \Big\{b_1, \ldots, b_k, 4b_k + 4b_{k-1} - b_i : i = 1, \ldots, k\Big\}.$$

Write

$$A_{k+1} = A_k \cup \{b_k + 2b_{k-2}, b_k + b_{k-1} - b_{k-2}, b_k + 2b_{k-1} - b_{k-2}\}.$$

Since $b_{k+1} = 3b_k + 4b_{k-1}$, we have

$$P(A_{k+1}) = [0, b_{k+1} + b_k] \setminus \{b_1, \ldots, b_{k+1}, b_{k+1} + b_k - b_i : i = 1, \ldots, k-1\}.$$

This completes the proof of Theorem 1.1.

4. Proof of Corollary 1.2

Let $A_k (k = 3, 4, \ldots)$ be as in Lemma 2.3. Write

$$A = \bigcup_{k=4}^{\infty} A_k.$$

For any $n \in P(A)$, we may assume that $n \le b_{k-1} + 2b_{k-2} - b_{k-3}$ for some $k \ge 4$. For all $i \ge k$, we have

$$A \setminus A_i \subseteq [b_{k-1} + 2b_{k-2} - b_{k-3} + 1, +\infty).$$

Thus, we have $n \in P(A_k)$. By Theorem 1.1 we have

$$n \notin \{b_1, \ldots, b_k, b_k + b_{k-1} - b_i : i = 1, \ldots, k-2\}. \tag{4.1}$$

Noting that $n \le b_{k-1} + 2b_{k-2} - b_{k-3} < b_k$, we know that if $n \in B$, then $n \in \{b_1, \dots, b_k\}$, which contradicts with (4.1). Hence, we have $n \notin B$. That is, $n \in \mathbb{N} \setminus B$.

Conversely, if $n' \in \mathbb{N} \setminus B$, then $n' \notin B$, let $n' < b_{k'}$, we have

$$n' \notin \{b_1, \dots, b_{k'}, b_{k'} + b_{k'-1} - b_i : i = 1, \dots, k' - 2\}.$$

By Theorem 1.1 we have $n' \in P(A_{k'})$. So $n' \in P(A)$.

Hence $P(A) = \mathbb{N} \setminus B$.

This completes the proof of Corollary 1.2.

5. Proof of Theorem 1.4

By Theorem D, we know that if $b_1 \in \{3, 5, 6, 9, 10\}$, then there is no a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ for which $P(A) = \mathbb{N} \setminus B$. Now, it is sufficient to consider a positive integers sequence $B = \{b_1 < b_2 < \cdots\}$ with $b_1 \in \{4, 7, 8\} \cup \{b : b \ge 11, b \in \mathbb{N}\}$.

By Lemma 2.1, there exists $A_1 = \{a_1 < a_2 < \cdots < a_k\} \subseteq [1, b_1 - 1]$ such that $P(A_1) = [0, b_1 - 1]$. Then

$$a_{k+1} + P(\{a_1, \dots, a_k\}) = [a_{k+1}, a_{k+1} + b_1 - 1].$$

Assume that there exists a sequence $A = \{a_1 < a_2 < \cdots\}$ of positive integers such that $P(A) = \mathbb{N} \setminus B$. Noting that $b_1 \notin P(A)$ and $b_2 \in [b_1 + 2, 2b_1] \cup [2b_1 + 2, \infty)$, we have $a_{k+1} = b_1 + 1$. Hence

$$P(\{a_1, \dots, a_{k+1}\}) = [0, 2b_1] \setminus \{b_1\},$$

$$a_{k+2} + P(\{a_1, \dots, a_{k+1}\}) = [a_{k+2}, a_{k+2} + 2b_1] \setminus \{a_{k+2} + b_1\}.$$

If $a_{k+2} \ge 2b_1 + 2$, then $2b_1 + 1 \notin P(A)$ and $b_2 = 2b_1 + 1$, a contradiction. So

$$a_{k+2} \le 2b_1 + 1, \tag{5.1}$$

$$P(\{a_1, \dots, a_{k+2}\}) = [0, a_{k+2} + 2b_1] \setminus \{b_1, a_{k+2} + b_1\}. \tag{5.2}$$

If $b_1 + 2 \le b_2 \le 2b_1$, then by $a_{k+2} > a_{k+1} = b_1 + 1$ and (5.2), we have

$$b_2 \ge a_{k+2} + b_1 \ge 2b_1 + 2,$$

a contradiction.

Now we consider the following two cases:

Case 1. $b_2 = 3b_1 + 3$. If $a_{k+2} \ge b_1 + 3$, then $b_2 \in [0, a_{k+2} + 2b_1]$. Since $b_2 \notin P(\{a_1, \dots, a_{k+2}\})$, we have $b_2 = a_{k+2} + b_1$. Thus

$$a_{k+2} = b_2 - b_1 = 2b_1 + 3 > 2b_1 + 1,$$

which contradicts with (5.1). Thus $a_{k+2} = b_1 + 2$ and by (5.2) we have

$$P(\{a_1, \dots, a_{k+2}\}) = [0, 3b_1 + 2] \setminus \{b_1, 2b_1 + 2\}.$$

Hence

$$a_{k+3} + P(\{a_1, \dots, a_{k+2}\}) = [a_{k+3}, a_{k+3} + 3b_1 + 2] \setminus \{a_{k+3} + b_1, a_{k+3} + 2b_1 + 2\}.$$

If $a_{k+3} \geq 2b_1 + 3$, then $2b_1 + 2 \notin P(A)$, thus $b_2 = 2b_1 + 2$, a contradiction. Hence $a_{k+3} \leq 2b_1 + 2$.

Since $a_{k+3} > a_{k+2}$, we have $a_{k+3} \geq b_1 + 3$, thus $b_1 + a_{k+3} \neq 2b_1 + 2$ and

$$P(\{a_1, \ldots, a_{k+3}\}) = [0, a_{k+3} + 3b_1 + 2] \setminus \{b_1, a_{k+3} + 2b_1 + 2\}.$$

Since $b_2 = 3b_1 + 3 \in [0, a_{k+3} + 3b_1 + 2]$ and $b_2 \notin P(\{a_1, \ldots, a_{k+3}\})$, we have

$$b_2 = 3b_1 + 3 = a_{k+3} + 2b_1 + 2 \geq 3b_1 + 5,$$

a contradiction.

Case 2. $b_2 = 3b_1 + 2$. Since $a_{k+2} \geq b_1 + 2$, then $b_2 \in [0, a_{k+2} + 2b_1]$. Since $b_2 \notin P(\{a_1, \ldots, a_{k+2}\})$, we have $b_2 = a_{k+2} + b_1$. Thus

$$a_{k+2} = b_2 - b_1 = 2b_1 + 2 > 2b_1 + 1,$$

which contradicts with (5.1).

This completes the proof of Theorem 1.4.

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