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THE CLASS OF WEAK w-PROJECTIVE MODULES IS A PRECOVER

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Abstract. Let R be a commutative ring with identity. Denote by $\mathrm{w}\mathcal{P}_w$ the class of weak w-projective R-modules and by $\mathrm{w}\mathcal{P}_w{}^{\perp}$ the right orthogonal complement of $\mathrm{w}\mathcal{P}_w$. It is shown that $(\mathrm{w}\mathcal{P}_w, \mathrm{w}\mathcal{P}_w{}^{\perp})$ is a hereditary and complete cotorsion theory, and so every R-module has a special weak w-projective precover. We also give some necessary and sufficient conditions for weak w-projective modules to be w-projective. Finally it is shown that when we discuss the existence of a weak w-projective cover of a module, it is enough to consider the w-envelope of the module.

1. Introduction

Throughout this paper R is always a commutative ring with identity. We first review some related concepts of w-modules. A finitely generated ideal J of R is called a *GV-ideal* if the homomorphism $R \to \mathrm{Hom}_R(J, R)$ induced by the inclusion map $J \hookrightarrow R$ is an isomorphism. Denote by $\mathrm{GV}(R)$ the set of GV-ideals of R. For any R-module N, set

$$\mathrm{tor}_{\mathrm{GV}(R)}(N) = \{x \in N \mid \text{there exists } J \in \mathrm{GV}(R) \text{ such that } Jx = 0\},$$

which is a submodule of N, called the *total GV-torsion submodule* of N. If $\mathrm{tor}_{\mathrm{GV}(R)}(N) = N$, then N is called a *GV-torsion module*; if $\mathrm{tor}_{\mathrm{GV}(R)}(N) = 0$, then N is called a *GV-torsion-free module*. A GV-torsion-free module N is called a *w-module* if $\mathrm{Ext}^1_R(R/J, N) = 0$ for any $J \in \mathrm{GV}(R)$. Denote by $\mathcal{W}$ the class of w-modules. The set of maximal w-ideals of R is denoted by w-$\mathrm{Max}(R)$. By [10, Theorem 6.2.15], an R-module T is a GV-torsion module if and only if $T_{\mathfrak{m}} = 0$ for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$.

We also need the concept of strong w-modules. An R-module N is called a *strong w-module* if $\mathrm{Ext}^k_R(T, N) = 0$ for any GV-torsion module T and any $k \geqslant 1$. For a discussion of strong w-modules, please refer to [12]. Denote by $\mathcal{W}_\infty$ the class of strong w-modules.

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Since the w-operation on an integral domain can establish the concept of w-modules, which allows the w-operation to work in the category of modules, in 1997 the concepts of w-projective modules and w-flat modules over an integral domain were first introduced [8]. In [4] the definition of w-flat modules was extended to any commutative ring as follows. A module M is called a *w-flat module* if the functor $M \otimes -$ preserves a w-exact sequence into a w-exact sequence. In [11] the concepts of the w-flat dimension (w-fd) of a module and the w-weak global dimension (w-w.gl.dim) of a ring have been successively introduced. Using the w-weak global dimension of a ring, a Prüfer v-multiplication domain (PVMD for short) can be characterized homologically as an integral domain of w-w.gl.dim$(R) \leqslant 1$.

In 2015, the concept of w-projective modules was also extended to any commutative ring [9]. Let M be an R-module. Set $L(M) := (M/\mathrm{tor}_{\mathrm{GV}(R)}(M))_w$. Then M is called a *w-projective module* if $\mathrm{Ext}_R^1(L(M), N)$ is a GV-torsion module for any w-module N. Denoted by $\mathcal{P}_w$ the class of w-projective modules. One can use the w-projective modules to introduce the w-projective ideals. One hopes that some rings that used to be described by ideals can be described by the w-projective modules. For example, in [11] it is proved that an integral domain R is a PVMD if and only if every finitely generated submodule of a projective module is w-projective. As we all know, an integral domain R is a Dedekind domain if and only if each nonzero ideal is invertible; R is a Krull domain if and only if each nonzero ideal is w-invertible. Therefore, in the above sense, Krull domains can actually be considered as w-Dedekind domains. But a Dedekind domain is exactly an integral domain with global dimension at most 1, in other words, every submodule of a projective module is projective. In [14], the authors can only prove that an integral domain R is a Krull domain if and only if every submodule of a finitely generated projective module is w-projective. That is to say, the concept of w-projective modules cannot be used to obtain a complete characterization of the Krull domains corresponding to the Dedekind domains.

In order to give a complete homological characterization of Krull domains, the concept of weak w-projective modules is introduced in [12] with the aid of w-projective modules. Denote by ${}_R\mathfrak{M}$ the category of all R-modules. Set

$$\mathcal{P}_w{}^{\dagger_\infty} = \left\{ N \in \mathfrak{M} \;\middle|\; \begin{array}{l} N \text{ is GV-torsion-free and} \\ \mathrm{Ext}_R^k(M, N) = 0 \text{ for any } M \in \mathcal{P}_w \text{ and any } k \geqslant 1 \end{array} \right\}.$$

An R-module M is called a *weak w-projective module* if $\mathrm{Ext}_R^1(M, N) = 0$ for any $N \in \mathcal{P}_w{}^{\dagger_\infty}$. Denote by $\mathrm{w}\mathcal{P}_w$ the class of weak w-projective modules. In [12] the authors pointed out: Every w-projective module must be weak w-projective. Conversely, every weak w-projective module of finite type and any weak w-projective ideal of an integral domain are all w-projective. At the same time, in [12] it is also given an example of a weak w-projective module over a UFD, which is not w-projective. In [12] it is also introduced the concept of the w-projective dimension (w-pd) of a module and the global w-projective dimension (w-gl.dim) of a ring. With the help of the concepts of weak w-projective modules and the global w-projective dimension of a ring, in [12] the authors give a homological characterization of Krull domains: An integral domain R is a Krull domain if and only if every submodule of a projective module is weak w-projective, equivalently, w-gl.dim$(R) \leq 1$.

Let $\mathcal{A}$ be a class of modules, M be an R-module, $A \in \mathcal{A}$, and $\varphi : A \to M$ be a homomorphism. Then (A, φ) is called an $\mathcal{A}$-*precover* of M if for any $A' \in \mathcal{A}$ and any homomorphism $f : A' \to M$, the following diagram

$$\begin{array}{ccc} & & A' \\ & \overset{h}{\swarrow} & \downarrow f \\ A & \xrightarrow{\varphi} & M \end{array}$$

is commutative, equivalently, for any $A' \in \mathcal{A}$, $\mathrm{Hom}_R(A', A) \overset{\varphi_*}{\to} \mathrm{Hom}_R(A', M) \to 0$ is an exact sequence. Let (A, φ) be an $\mathcal{A}$-precover of a module M. When $A' = A$, $f = \varphi$, and the above diagram is commutative, it is said that (A, φ) is an $\mathcal{A}$-*cover* of M if h is an isomorphism. If any R-module M has $\mathcal{A}$-precover (resp., cover), then we say that $\mathcal{A}$ is a precover (resp, cover) class.

Let $\mathcal{S}$ be a class of modules. Set

$$^{\perp}\mathcal{S} := \{A \in \mathfrak{M} \mid \mathrm{Ext}^1_R(A, C) = 0 \text{ for any } C \in \mathcal{S}\}$$

and

$$\mathcal{S}^{\perp} := \{B \in \mathfrak{M} \mid \mathrm{Ext}^1_R(C, B) = 0 \text{ for any } C \in \mathcal{S}\},$$

are called the *left orthogonal complement* and the *right orthogonal complement* of $\mathcal{S}$, respectively [3]. Then obviously one has $\mathrm{w}\mathcal{P}_w = {}^{\perp}(\mathcal{P}_w^{\dagger\infty})$. In [1], the authors introduced and studied the right orthogonal complement of the class of w-flat modules. Also set

$$^{\perp_\infty}\mathcal{S} := \{A \in \mathfrak{M} \mid \mathrm{Ext}^k_R(A, C) = 0 \text{ for any } C \in \mathcal{S} \text{ and any } k \geqslant 1\},$$

and

$$\mathcal{S}^{\perp_\infty} := \{B \in \mathfrak{M} \mid \mathrm{Ext}^k_R(C, B) = 0 \text{ for any } C \in \mathcal{S} \text{ and any } k \geqslant 1\}.$$

In recent years, the cotorsion theory has received great attention from researchers. Let $\mathcal{A}, \mathcal{B}$ be two classes of modules. Then $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion theory* if $\mathcal{B} = \mathcal{A}^{\perp}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$. In addition, $(\mathcal{A}, \mathcal{B})$ is called a *hereditary cotorsion theory* if whenever $0 \to A_1 \to A \to A_2 \to 0$ is exact with $A, A_2 \in \mathcal{A}$, one has $A_1 \in \mathcal{A}$. And $(\mathcal{A}, \mathcal{B})$ is called a *complete cotorsion theory* if for any R-module M, there is an exact sequence $0 \to K \to A \to M \to 0$ with $A \in \mathcal{A}$ and $K \in \mathcal{B}$. When a formulated pair $(\mathcal{A}, \mathcal{B})$ of modules becomes a cotorsion pair, the classical homology method can be used very smoothly to characterize rings and modules. For the projective modules, a well-known theorem of Kaplansky states that a projective module over an arbitrary ring is a direct sum of countably generated projective modules. In 2020, Wang and Qiao established the w-version of Kaplansky's theorem [13]: If M is a w-projective w-module, then M has a w-projective w-$\aleph_0$-continuous ascending chain (see the definition later). Using this result, this article obtains the main result: $(\mathrm{w}\mathcal{P}_w, \mathrm{w}\mathcal{P}_w{}^{\perp})$ is a hereditary and complete cotorsion theory, and so every module has a special weak w-projective precover.

2. Basic results

Denoted by $\mathcal{FT}$ the class of GV-torsion-free modules. Let $\mathcal{S}$ be a class of modules. Define:

$$\begin{aligned}\mathcal{S}^{\dagger} &:= \mathcal{S}^{\perp} \cap \mathcal{FT}\\ &= \{N \in \mathfrak{M} \mid N \text{ is GV-torsion-free and } \mathrm{Ext}_R^1(M,N) = 0 \text{ for any } M \in \mathcal{S}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}^{\dagger_\infty} &:= \mathcal{S}^{\perp_\infty} \cap \mathcal{FT}\\ &= \left\{ N \in \mathfrak{M} \;\middle|\; \begin{array}{l} N \text{ is GV-torsion-free and} \\ \mathrm{Ext}_R^k(M,N) = 0 \text{ for any } M \in \mathcal{S} \text{ and any } k \geqslant 1 \end{array} \right\}.\end{aligned}$$

Set

$$\mathrm{GV}(R)^* := \{R/J \mid J \in \mathrm{GV}(R)\}.$$

Obviously $\mathrm{GV}(R)^*$ is a set of modules.

Proposition 2.1. *Let $\mathcal{S}, \mathcal{S}_1$ be classes of modules. Then:*

(1) $\mathcal{S} \subseteq {}^{\perp}(\mathcal{S}^{\dagger_\infty}) \subseteq {}^{\perp}(\mathcal{S}^{\dagger})$.

(2) *If $\mathcal{S} \subseteq \mathcal{S}_1$, then $\mathcal{S}_1^{\dagger} \subseteq \mathcal{S}^{\dagger}$ and $\mathcal{S}_1^{\dagger_\infty} \subseteq \mathcal{S}^{\dagger_\infty}$.*

(3) $(\mathcal{S} \cup \mathcal{S}_1)^{\dagger} = \mathcal{S}^{\dagger} \cap \mathcal{S}_1^{\dagger}$.

Proof. These are obvious. □

For $k \geqslant 1$, set

$$\mathcal{W}_k := \{N \in \mathcal{FT} \mid \mathrm{Ext}_R^i(R/J, N) = 0 \text{ for any } J \in \mathrm{GV}(R) \text{ and any } 1 \leqslant i \leqslant k\}.$$

By convention, we set $\mathcal{W}_0 := \mathcal{FT}$. A module N is called a *w_k-module* if $N \in \mathcal{W}_k$. It is known that a GV-torsion-free module N is a w-module if and only if $\mathrm{Ext}_R^1(C,N) = 0$ for any GV-torsion-module C ([10, Theorem 6.2.7]).

Lemma 2.2. (1) *If $1 \leqslant i \leqslant k$, then $\mathcal{W}_k \subseteq \mathcal{W}_i$.*

(2) *$\mathcal{W}_k$ is closed under extensions.*

(3) *Let $N \in \mathcal{W}_k$. Then $N \in \mathcal{W}_{k+1}$ if and only if $\mathrm{Ext}_R^{k+1}(M,N) = 0$ for any GV-torsion module M.*

Proof. (1) and (2) are trivial. We will prove only (3). It is enough to show the necessity. Assume that $N \in \mathcal{W}_{k+1}$. If $k = 0$, then $N \in \mathcal{W}_1 = \mathcal{W}$. Thus by [10, Theorem 6.2.7], $\mathrm{Ext}_R^1(M,N) = 0$ for any GV-torsion module M. Consider the case $k = 1$. Let M be a GV-torsion module. Then for any $x \in M$, there exists $I_x \in \mathrm{GV}(R)$ such that $I_x x = 0$. Set $F := \bigoplus_{x \in M} R/I_x$. Then F is a GV-torsion module. Let e_x denote the element in F that takes the value $1+I_x$ at the component x, and the other components take the value 0. Define $h : F \to M$

by $h(e_x) = x$. Then h is an epimorphism. Set $A := \mathrm{Ker}(h)$. Then it follows from the exact sequence $0 = \mathrm{Ext}^1_R(A, N) \to \mathrm{Ext}^2_R(M, N) \to \mathrm{Ext}^2_R(F, N) = 0$ that $\mathrm{Ext}^2_R(M, N) = 0$. Now the assertion follows by induction. □

Proposition 2.3. *The following are equivalent for a GV-torsion-free module N.*

(1) $N \in \mathcal{W}_\infty$.
(2) $\mathrm{Ext}^i_R(R/J, N) = 0$ *for any* $J \in \mathrm{GV}(R)$ *and any* $i \geqslant 1$.

Proof. (1) ⇒ (2) This is trivial.

(2) ⇒ (1) Let $k \geqslant 1$ and set

$$\mathcal{W}_k^{'} := \left\{ N \in \mathcal{FT} \;\middle|\; \begin{array}{l} \mathrm{Ext}^i_R(M, N) = 0 \text{ for any GV-torsion module } M \\ \text{and any } 1 \leqslant i \leqslant k \end{array} \right\}.$$

By Lemma 2.2, $\mathcal{W}_k^{'} = \mathcal{W}_k$. Thus $N \in \bigcap_{k=1}^{\infty} \mathcal{W}_k^{'} = \mathcal{W}_\infty$. □

Let M and N be R-modules. A homomorphism $f : M \to N$ is called a *w-monomorphism* (resp., a *w-epimorphism*, a *w-isomorphism*) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism) for any maximal w-ideal $\mathfrak{m}$ of R. And M is said to be *w-isomorphic* to N provided that there exist an R-module L and two w-isomorphisms $f : L \to M$ and $g : L \to N$.

Theorem 2.4. *Let $\mathcal{S}$ be a class of modules such that $\mathcal{S} \subseteq \mathcal{FT}$. Set $\mathcal{A} := {}^{\perp}\mathcal{S}$. Then the following are equivalent.*

(1) *$\mathcal{A}$ is closed under w-isomorphisms.*
(2) $\mathrm{GV}(R) \cup \mathrm{GV}(R)^* \subseteq \mathcal{A}$.
(3) $\mathcal{S} \subseteq \mathcal{W}_2$.

Proof. (1) ⇒ (2) Let $J \in \mathrm{GV}(R)$. Since $R \in \mathcal{A}$ and J and R are w-isomorphic, it follows that $J \in \mathcal{A}$. Also since R/J and 0 are w-isomorphic, it follows that $R/J \in \mathcal{A}$.

(2) ⇒ (3) Let $N \in \mathcal{S}$. Then $\mathrm{Ext}^1_R(R/J, N) = 0$ and $\mathrm{Ext}^1_R(J, N) = 0$ for any $J \in \mathrm{GV}(R)$. Thus N is a w_2-module. Therefore $\mathcal{S} \subseteq \mathcal{W}_2$.

(3) ⇒ (1) Let $f : M \to M^{'}$ be a w-isomorphism. By [10, Proposition 6.3.4], there exist a module B and exact sequences $0 \to A \to M \to B \to 0$ and $0 \to B \to M^{'} \to C \to 0$, where A and C are GV-torsion modules. If $M \in \mathcal{A}$, then for any $N \in \mathcal{S}$ it follows from the exact sequence $0 = \mathrm{Hom}_R(A, N) \to \mathrm{Ext}^1_R(B, N) \to \mathrm{Ext}^1_R(M, N) = 0$ that $\mathrm{Ext}^1_R(B, N) = 0$. Again by the exact sequence $0 = \mathrm{Ext}^1_R(C, N) \to \mathrm{Ext}^1_R(M^{'}, N) \to \mathrm{Ext}^1_R(B, N) = 0$ it follows that $\mathrm{Ext}^1_R(M^{'}, N) = 0$, that is, $M^{'} \in \mathcal{A}$.

On the other hand, assume that $M^{'} \in \mathcal{A}$. By Lemma 2.2, $\mathrm{Ext}^2_R(C, N) = 0$. By the exact sequence $0 = \mathrm{Ext}^1_R(M^{'}, N) \to \mathrm{Ext}^1_R(B, N) \to \mathrm{Ext}^2_R(C, N) = 0$ it follows that $\mathrm{Ext}^1_R(B, N) = 0$. Also by the exact sequence $0 = \mathrm{Ext}^1_R(B, N) \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^1_R(A, N) = 0$, it follows that $\mathrm{Ext}^1_R(M, N) = 0$, i.e., $M \in \mathcal{A}$. Therefore $\mathcal{A}$ is closed under w-isomorphisms. □

Corollary 2.5. *Let $\mathcal{S}$ be a class of modules. Set $\mathcal{A} := {}^{\perp}\mathcal{S}$. If $\mathcal{S} \subseteq \mathcal{W}_{\infty}$, then $\mathcal{A}$ is closed under w-isomorphisms.*

Proof. This follows directly from Theorem 2.4 and the fact that $\mathcal{W}_{\infty} \subseteq \mathcal{W}_2$. □

Example 2.6. (1) It is easy to see that $(\mathrm{GV}(R)^*)^{\dagger} = \mathcal{W}$.

(2) By Proposition 2.3, $(\mathrm{GV}(R)^*)^{\dagger_{\infty}} = \mathcal{W}_{\infty}$.

(3) By Theorem 2.4, $(\mathrm{GV}(R)^* \cup \mathrm{GV}(R))^{\dagger} = \mathcal{W}_2$.

Proposition 2.7. *Let $\mathcal{S}$ be a class of modules satisfying $\mathrm{GV}(R)^* \subseteq \mathcal{S}$. Then:*

(1) *$\mathcal{S}^{\dagger} \subseteq \mathcal{W}$ and $\mathcal{S}^{\dagger_{\infty}} \subseteq \mathcal{W}_{\infty}$.*

(2) *If $\mathrm{GV}(R) \subseteq \mathcal{S}$, then $\mathcal{S}^{\dagger} \subseteq \mathcal{W}_2$.*

Proof. This follows immediately from Example 2.6. □

3. The class of weak w-projective modules is a precover

Let $\mathcal{A}$ be a class of modules and M be an R-module. If there is a continuous ascending chain of submodules of M:

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{\alpha} \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_{\lambda} = M \tag{3.1}$$

such that $M_{\alpha+1}/M_{\alpha} \in \mathcal{A}$ for any $\alpha < \lambda$, then M is called an *$\mathcal{A}$-filtered module.* A continuous ascending chain (3.1) is called an *$\mathcal{A}$-filtration* of M.

In order to determine when $(\mathcal{S}, \mathcal{S}^{\perp})$ is a complete cotorsion theory, the following lemma is very effective and will be used later.

Lemma 3.1 (Eklof–Trlifaj)**.** *Let $\mathcal{S}$ be a set of modules. Then:*

(1) *Let N be an R-module. Then there exists a short exact sequence $0 \to N \to Q \to A \to 0$, where $Q \in \mathcal{S}^{\perp}$ and A is an $\mathcal{S}$-filtered module, and thus $A \in {}^{\perp}(\mathcal{S}^{\perp})$.*

(2) *$({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is a complete cotorsion theory.*

Proof. See [2] or [7, Theorem 2.2]. □

In order to make Lemma 3.1 apply to the context of a class of related modules, we make corresponding modifications to it, but note that the idea belongs to Eklof–Trlifaj essentially.

Lemma 3.2. *Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1 \subseteq \mathcal{FT}$.*

(1) *Let N be a GV-torsion-free module. Then there exists an exact sequence*

$$0 \to N \to Q \to A \to 0, \tag{3.2}$$

where $Q \in \mathcal{S}^{\dagger}$ and A is an $\mathcal{S}$-filtered module such that $A \in {}^{\perp}(\mathcal{S}^{\dagger})$.

(2) *Let M be an R-module. Then there exists an exact sequence*

$$0 \to B \to P \to M \to 0, \tag{3.3}$$

where $P \in {}^{\perp}(\mathcal{S}^{\dagger})$ and $B \in \mathcal{S}^{\dagger}$.

Proof. (1) Set $X := \bigoplus_{S\in\mathcal{S}_1} S$ and $Y := \bigoplus_{J\in \mathrm{GV}(R)} R/J$. Then X is a GV-torsion-free module and Y is a GV-torsion module. Set $S = X\oplus Y$. Then $\mathcal{S}^{\perp} = \{S\}^{\perp}$. Thus we may assume that $\mathcal{S}$ is the class of modules composed of the fixed module S and its direct sums. Let $0 \to K_1 \xrightarrow{\mu_1} F_1 \to X \to 0$ and $0 \to K_2 \xrightarrow{\mu_2} F_2 \to Y \to 0$ be exact sequences, where F_1 and F_2 are free modules. Set $F := F_1 \oplus F_2$ and $K := K_1 \oplus K_2$. Then $0 \to K \xrightarrow{\mu} F \to S \to 0$ is an exact sequence, where $\mu := \mu_1 \oplus \mu_2$. Since X is GV-torsion-free, K_1 is a w-module. Since Y is GV-torsion, we have $(K_2)_w = F_2$

Take a regular cardinal λ so that K has a generating system Z with $|Z| < \lambda$.

Set $Q_0 := N$. Then Q_0 is GV-torsion-free. For $\alpha < \lambda$, if Q_α has been constructed, select a free module $F_\alpha^{'}$ and an epimorphism $\delta_\alpha : F_\alpha^{'} \to Q_\alpha$. Set $I_\alpha := \mathrm{Hom}_R(K, Q_\alpha)$ to be a new index set and define $\mu_\alpha : K^{(I_\alpha)} \to F^{(I_\alpha)}$ as the homomorphism of direct sums, which is induced by μ. Then μ_α is a monomorphism and $\mathrm{Coker}(\mu_\alpha) = S^{(I_\alpha)}$.

Define $\varphi_\alpha : K^{(I_\alpha)} \oplus F_\alpha^{'} = (\bigoplus_{f\in I_\alpha} K_f) \oplus F_\alpha^{'} \to Q_\alpha$, where $K_f = K$, by $\varphi_\alpha([u_f], z) = \sum_{f\in I_\alpha} f(u_f) + \delta_\alpha(z)$, where $u_f \in K_f, z \in F_\alpha^{'}$. Since δ_α is an epimorphism, so is φ_α. In addition, for any $f \in I_\alpha$, let $i_f : K \to K^{(I_\alpha)}$ and $j_f : F \to F^{(I_\alpha)}$ be the natural imbeddings. Then one has

$$f = \varphi_\alpha i_f \qquad \text{and} \qquad j_f\mu = \mu_\alpha i_f. \tag{3.4}$$

Now assume that if $\beta \leqslant \alpha$, then Q_β has been constructed (if α is a limit ordinal, set $Q_\alpha = \bigcup_{\beta<\alpha} Q_\beta$), in particular, Q_α has been constructed. Construct the following pushout diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^{(I_\alpha)} \oplus F_\alpha^{'} & \xrightarrow{\mu_\alpha\oplus 1} & F^{(I_\alpha)} \oplus F_\alpha^{'} & \longrightarrow & S^{(I_\alpha)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_\alpha} & & \downarrow{\scriptstyle \psi_\alpha} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & Q_\alpha & \xrightarrow{h_\alpha} & Q_{\alpha+1} & \longrightarrow & Q_{\alpha+1}/Q_\alpha & \longrightarrow & 0
\end{array}$$

One gets $Q_{\alpha+1}$. At this time ψ_α is an epimorphism. As you can see from the above diagram, if Q_α is a GV-torsion-free module, then $\mathrm{Ker}(\psi_\alpha) = \mathrm{Ker}(\varphi_\alpha)$ is a w-module, and thus $Q_{\alpha+1}$ is also a GV-torsion-free module. Hence by a transfinite induction, we see that each Q_α is a GV-torsion-free module.

Set $Q := \bigcup_{\alpha<\lambda} Q_\alpha = \varinjlim_{\alpha<\lambda} Q_\alpha$. Then Q is a GV-torsion-free module. Set $A := Q/N$ and $A_\alpha := Q_\alpha/N$. Then $A_{\alpha+1}/A_\alpha \cong Q_{\alpha+1}/Q_\alpha \cong S^{(I_\alpha)}$. Since $Q = \bigcup_{\alpha<\lambda} Q_\alpha$, one gets that $A = \bigcup_{\alpha<\lambda} A_\alpha$. Thus A is an $\mathcal{S}$-filtered module, and thus one has $A \in {}^{\perp}(\mathcal{S}^{\perp})$. Since $\mathcal{S}^{\dagger} \subseteq \mathcal{S}^{\perp}$, one has $A \in {}^{\perp}(\mathcal{S}^{\dagger})$.

Let us prove that $Q \in \mathcal{S}^{\perp}$. For this, it is sufficient to prove that $\mu^* : \mathrm{Hom}_R(F, Q) \to \mathrm{Hom}_R(K, Q)$ is an epimorphism. Let $g : K \to Q$ be a homomorphism. Since the generating system Z of K satisfies $|Z| < \lambda$ and $Q =$

$\bigcup\limits_{\alpha<\lambda} Q_\alpha$, there exists an ordinal $\alpha < \lambda$ such that $\mathrm{Im}(g) \subseteq Q_\alpha$. Thus there exists a homomorphism $f : K \to Q_\alpha$ such that $g(x) = f(x)$ for any $x \in K$. By the pushout diagram above and (3.4), one has $\psi_\alpha j_f \mu = \psi_\alpha \mu_\alpha i_f = h_\alpha \varphi_\alpha i_f = h_\alpha f$. Define $\sigma : F \to Q$ by $\sigma(z) = \psi_\alpha j_f(z) \in Q_{\alpha+1} \subseteq Q$. Then one can verify directly that $g = \sigma\mu = \mu^*(\sigma)$. Thus μ^* is an epimorphism. Therefore $Q \in \mathcal{S}^\perp \cap \mathcal{FT} = \mathcal{S}^\dagger$.

(2) Take an exact sequence $0 \to N \to F \to M \to 0$, where F is a projective module. Then N is a GV-torsion-free module. By (1), there is an exact sequence $0 \to N \to Q \to A \to 0$, where $Q \in \mathcal{S}^\dagger$ and $A \in {}^\perp(\mathcal{S}^\dagger)$. Consider the following commutative diagram with two exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the square diagrams in the upper left and lower corners are pushout diagrams. Since $F, A \in {}^\perp(\mathcal{S}^\dagger)$, one has $P \in {}^\perp(\mathcal{S}^\dagger)$. Therefore one gets the exact sequence (3.3) by taking $B := Q$. □

Let $\mathcal{A}$ be a class of modules. Then an $\mathcal{A}$-precover $f : C \to M$ of M is said to be *special* if f is surjective and $\mathrm{Ker}(f) \in \mathcal{A}^\perp$. In other words, there is an exact sequence $0 \to K \to C \to M \to 0$ with $C \in \mathcal{A}$ and $K \in \mathcal{A}^\perp$.

Theorem 3.3. *Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1 \subseteq \mathcal{FT}$. Set $\mathcal{A} := {}^\perp(\mathcal{S}^\dagger)$. If $\mathcal{A}$ is closed under w-isomorphisms, then $(\mathcal{A}, \mathcal{A}^\perp)$ is a complete cotorsion theory.*

Proof. Note that $(\mathcal{A}, \mathcal{A}^\perp)$ is the cotorsion theory generated by $\mathcal{S}^\dagger$. Let us prove that any module M has a special $\mathcal{A}$-precover.

By Lemma 3.2, there is an exact sequence (3.3), where $P \in \mathcal{A}$ and $B \in \mathcal{S}^\dagger \subseteq \left({}^\perp(\mathcal{S}^\dagger)\right)^\perp = \mathcal{A}^\perp$. Therefore M has a special $\mathcal{A}$-precover. □

Proposition 3.4. *Let $\mathcal{S}$ be a class of modules such that $\mathrm{GV}(R)^* \subseteq \mathcal{S}$. Set $\mathcal{B} := {}^\perp(\mathcal{S}^{\dagger_\infty})$. Then:*

(1) *$\mathcal{S}^{\dagger_\infty}$ is closed under direct products, direct summands, and cokernels of monomorphisms.*
(2) *$\mathcal{B}$ is closed under direct sums, direct summands, kernels of epimorphisms, and w-isomorphisms.*
(3) *$\mathcal{B}^\dagger = \mathcal{B}^{\dagger_\infty} = \mathcal{S}^{\dagger_\infty}$.*

Proof. (1) Obviously $\mathcal{S}^{\dagger_\infty}$ is closed under direct products and direct summands. Obviously $\mathcal{S}^{\perp_\infty}$ is closed under cokernels of monomorphisms. By [12, Proposition 2.2(2)], $\mathcal{W}_\infty$ is also closed under cokernels of monomorphisms. Since $\mathcal{S}^{\dagger_\infty} = \mathcal{S}^{\perp_\infty} \cap \mathcal{W}_\infty$, $\mathcal{S}^{\dagger_\infty}$ is closed under cokernels of monomorphisms.

(2) Obviously $\mathcal{B}$ is closed under direct sums and direct summands. By (1), $\mathcal{B}$ is closed under kernels of epimorphisms. By Corollary 2.5, $\mathcal{B}$ is closed under w-isomorphisms.

(3) Obviously we have that $\mathcal{S}^{\dagger_\infty} \subseteq \left({}^{\perp}(\mathcal{S}^{\dagger_\infty})\right)^{\perp} \cap \mathcal{FT} = \mathcal{B}^{\dagger}$. Since $\mathcal{B}$ is closed under kernels of epimorphisms, we have $\mathcal{B}^{\perp_\infty} = \mathcal{B}^{\perp}$. Thus we have $\mathcal{B}^{\dagger} = \mathcal{B}^{\perp_\infty} \cap \mathcal{FT} = \mathcal{B}^{\dagger_\infty}$. Since $\mathcal{S} \subseteq \mathcal{B}$, it follows that $\mathcal{B}^{\dagger} = \mathcal{B}^{\dagger_\infty} \subseteq \mathcal{S}^{\dagger_\infty}$. Therefore $\mathcal{B}^{\dagger} = \mathcal{S}^{\dagger_\infty}$. □

Let M be an R-module. Then M is said to be *w-$\aleph_0$-generated* if there exist a countably generated free module F and a w-epimorphism $\phi : F \to M$.

Let M be a w-projective w-module. If there is a continuous ascending chain of w-projective w-submodules of M:

$$0 = M_0 \subseteq M_1' \subseteq M_2' \subseteq \cdots \subseteq M_\alpha' \subseteq \cdots \subseteq M_\lambda' = M$$

such that each factor $M_{\alpha+1}'/M_\alpha'$ is a w-$\aleph_0$-generated w-projective module, then it is said that M has a *w-projective w-$\aleph_0$-continuous ascending chain*. It follows from [13, Theorem 3.5] that if M is a w-projective w-module, then M has a w-projective w-$\aleph_0$-continuous ascending chain.

Proposition 3.5. (1) $\mathrm{w}\mathcal{P}_w{}^{\dagger} = \mathcal{P}_w^{\dagger_\infty}$.

(2) *Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1$ is the class of w-projective w-$\aleph_0$-generated w-modules. Then $\mathcal{S}^{\dagger_\infty} = \mathcal{P}_w^{\dagger_\infty}$.*

(3) *Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1 = \{R\}$. Then $\mathcal{S}^{\dagger_\infty} = \mathcal{W}_\infty$.*

Proof. (1) This follows immediately from Proposition 3.4 by setting $\mathcal{S} := \mathcal{P}_w$.

(2) Since $\mathcal{S} \subseteq \mathcal{P}_w$, we have $\mathcal{P}_w^{\dagger_\infty} \subseteq \mathcal{S}^{\dagger_\infty}$. Let $N \in \mathcal{S}^{\dagger_\infty}$. For any w-projective w-module P, by [13, Theorem 3.5] P is an $\mathcal{S}_1$-filtered module. Thus $\mathrm{Ext}_R^i(P, N) = 0$ for any $i \geqslant 1$. By Proposition 2.7, N is a strong w-module. Let P be a w-projective module. Then one has the following two exact sequences:

$$0 \to \mathrm{tor}_{\mathrm{GV}(R)}(P) \to P \to P/\mathrm{tor}_{\mathrm{GV}(R)}(P) \to 0$$

and

$$0 \to Q \to Q_w \to Q_w/Q \to 0,$$

where $Q := P/\mathrm{tor}_{\mathrm{GV}(R)}(P)$ is GV-torsion-free. Considering two long exact sequences induced by the above two exact sequences, it follows that $\mathrm{Ext}_R^i(P, N) = 0$ for any w-projective module P and any $i \geqslant 1$ since $\mathcal{P}_w$ is closed under w-isomorphisms. Thus $N \in \mathcal{P}_w^{\perp_\infty} \cap \mathcal{FT} = \mathcal{P}_w^{\dagger_\infty}$. Therefore $\mathcal{S}^{\dagger_\infty} = \mathcal{P}_w^{\dagger_\infty}$.

(3) This is trivial. □

Theorem 3.6. *Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1 \subseteq \mathcal{FT}$. Set $\mathcal{B} := {}^{\perp}(\mathcal{S}^{\dagger_\infty})$. Then $(\mathcal{B}, \mathcal{B}^{\perp})$ is a hereditary and complete cotorsion theory.*

Proof. For any $M \in \mathcal{S}$, fix a projective resolution of M. Let $\mathcal{L}_M$ be the set of all syzygies of this projective resolution of M (including M itself as -1 syzygy). Set $\mathcal{L} := \bigcup_{M\in\mathcal{S}} \mathcal{L}_M$. Then $\mathcal{L}$ is again a set. Note that $\mathcal{L}$ can be split into $\mathcal{L} = \mathrm{GV}(R)^* \cup \mathcal{L}_1$, where $\mathcal{L}_1$ is the set of all syzygies of $M \in \mathcal{S}_1$ and all non-negative syzygies of $R/J \in \mathrm{GV}(R)^*$. Then $\mathcal{L}_1 \subseteq \mathcal{FT}$.

Let $N \in \mathcal{S}^{\perp_\infty}$. For any $X \in \mathcal{L}$, there exists an exact sequence

$$0 \to X \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0, \tag{3.5}$$

where each P_i is a projective module and $M \in \mathcal{S}$. Thus one has $\mathrm{Ext}_R^1(X, N) \cong \mathrm{Ext}_R^{k+2}(M, N) = 0$. Therefore $N \in \mathcal{L}^{\perp}$.

On the other hand, let $N \in \mathcal{L}^{\perp}$. For any $Y \in \mathcal{S}$ and any $k \geqslant -1$, by considering the exact sequence (3.5), one has $\mathrm{Ext}_R^{k+2}(Y, N) \cong \mathrm{Ext}_R^1(X, N) = 0$. Thus $N \in \mathcal{S}^{\perp_\infty}$. Therefore $\mathcal{L}^{\perp} = \mathcal{S}^{\perp_\infty}$. By Theorem 3.3, $(\mathcal{B}, \mathcal{B}^{\perp})$ is a complete cotorsion theory. It follows by Proposition 3.4 that $(\mathcal{B}, \mathcal{B}^{\perp})$ is a hereditary cotorsion theory. □

Now we are ready to state the main theorem.

Theorem 3.7. $(\mathrm{w}\mathcal{P}_w, \mathrm{w}\mathcal{P}_w{}^{\perp})$ *is a hereditary and complete cotorsion theory, and so every module has a special weak w-projective precover.*

Proof. Let $\mathcal{S}_1$ be the collection of all w-$\aleph_0$-generated w-projective w-modules and set $\mathcal{S} := \mathrm{GV}(R)^* \cup \mathcal{S}_1$. Since the collection of all $\aleph_0$-generated modules is a set, $\mathcal{S}$ is also a set. By Proposition 3.5(2), $\mathcal{S}^{\dagger_\infty} = \mathcal{W}_\infty$. By Theorem 3.6, $(\mathrm{w}\mathcal{P}_w, \mathrm{w}\mathcal{P}_w{}^{\perp})$ is a hereditary and complete cotorsion theory. □

According to [5, 6], we say that a module M is a w_∞-projective module if $\mathrm{Ext}_R^1(M, N) = 0$ for any strong w-module N. Denote by $\mathcal{P}_{w_\infty}$ the class of w_∞-projective modules. Then $\mathcal{P}_{w_\infty} = {}^{\perp}\mathcal{W}_\infty$.

Theorem 3.8. $(\mathcal{P}_{w_\infty}, \mathcal{P}_{w_\infty}^{\perp})$ *is a hereditary and complete cotorsion theory.*

Proof. Set $\mathcal{S}_1 := \{R\}$ and $\mathcal{S} := \mathrm{GV}(R)^* \cup \mathcal{S}_1$. Then S is a set of modules. By Proposition 3.5, $\mathcal{S}^{\dagger_\infty} = \mathcal{W}_\infty$. Thus $\mathcal{P}_{w_\infty} = {}^{\perp}(\mathcal{S}^{\dagger_\infty})$. Now the assertion follows by Theorem 3.6. □

Proposition 3.9. *Let M be a w-module. Then there is a special weak w-projective precover of M, $\varphi : P \to M$ such that P is a w-module and* $\mathrm{Ker}(\varphi) \in \mathcal{P}_w^{\dagger_\infty}$.

Proof. We use the notation $\mathcal{L}$ as in the proof of Theorem 3.6 and the notation $\mathcal{S}$ as in Proposition 3.5(2). Then $\mathcal{L}^{\dagger} = \mathcal{S}^{\dagger_\infty} = \mathcal{P}_w^{\dagger_\infty}$. Now the assertion follows by Theorem 3.6. □

Recall that a class of modules is said to be hereditary if it is closed under isomorphic copies and submodules.

Lemma 3.10. *If $\mathcal{P}_w$ is a hereditary class of modules, then* $\mathrm{w}\mathcal{P}_w{}^{\dagger} = \mathcal{P}_w{}^{\dagger}$.

Proof. If $\mathcal{P}_w$ is a hereditary class of modules, then $\mathcal{P}_w^{\perp} = \mathcal{P}_w^{\perp_\infty}$, and thus $\mathcal{P}_w^{\dagger} = \mathcal{P}_w^{\dagger_\infty}$. Now the assertion immediately follows by applying Proposition 3.5(1). □

In the following result, we give some necessary and sufficient conditions for weak w-projective modules to be w-projective.

Theorem 3.11. *The following conditions are equivalent for a ring R:*

1. *Every weak w-projective module is w-projective.*
2. *Every weak w-projective w-module is w-projective.*
3. *$(\mathcal{P}_w, \mathcal{P}_w{}^{\perp})$ is a hereditary cotorsion theory and every w-module has a special $\mathcal{P}_w$-precover of a w-module.*

Proof. (1)⇒(3) This follows by Theorem 3.7 and Proposition 3.9.

(3)⇒(2) Let M be a weak w-projective w-module. By assumption, there is an exact sequence $0 \to A \to P \to M \to 0$ such that P is a w-projective w-module and $A \in \mathcal{P}_w{}^{\perp}$. Since any GV-torsion module is w-projective, A is a w-module. By Lemma 3.10, $A \in \mathcal{P}_w{}^{\dagger} = \mathcal{P}_w{}^{\dagger_\infty}$. Thus $\mathrm{Ext}^1_R(M, A) = 0$, and so the above exact sequence is split. Therefore M is a w-projective module.

(2)⇒(1) Let M be a weak w-projective module. It follows from [12, Corollary 2.7] that $L(M)$ is a weak w-projective module. By assumption, $L(M)$ is a w-projective module. So M is a w-projective module. □

Proposition 3.12. *Let $\mathcal{A}$ be a class of modules which is closed under w-isomorphisms. Let M be a GV-torsion-free module and $\varphi : P \to M$ be an $\mathcal{A}$-cover. Then*:

1. *P is a GV-torsion-free module.*
2. *If φ is a special $\mathcal{A}$-cover and M is a w-module, then P is a w-module.*

Proof. Set $T := \mathrm{tor}_{\mathrm{GV}}(P)$ and $B := P/T$. Then B is a GV-torsion-free module. Let $\pi : P \to B$ be a natural homomorphism. Since M is a GV-torsion-free module, φ induces a homomorphism $\psi : B \to M$ such that $\psi(\overline{x}) = \varphi(x)$ for any $x \in F$, that is $\psi\pi = \varphi$. Since $\mathcal{A}$ is closed under w-isomorphisms, it follows that $B \in \mathcal{A}$. Thus there is a homomorphism $h : B \to P$ such that $\varphi h = \psi$. So $\varphi h\pi = \psi\pi = \varphi$. Hence $h\pi$ is an isomorphism, and thus π is an isomorphism. Therefore P is a GV-torsion-free module.

(2) By (1), $A := \mathrm{Ker}(\varphi)$ is also a GV-torsion-free module. Since $\mathcal{A}$ is closed under w-isomorphisms, $\mathcal{A}$ contains all GV-torsion modules. So A is a w-module. It follows from the exact sequence $0 \to A \to P \to M \to 0$ that P is a w-module. □

Theorem 3.13. *Let $\mathcal{A}$ be a class of modules closed under w-isomorphisms. Let M be a GV-torsion-free module. Then M has a special $\mathcal{A}$-cover if and only* *if M_w has a special $\mathcal{A}$-cover. In addition, if M is GV-torsion-free and B is a special $\mathcal{A}$-cover of M, then B_w is a special $\mathcal{A}$-cover of M_w.*

Proof. Let $\varphi : P \to M_w$ be an $\mathcal{A}$-cover of M_w. Set $T := M_w/M$. Then T is a GV-torsion module. Let $\pi : M_w \to T$ be a natural homomorphism. Set $g := \pi\varphi$, $A := \mathrm{Ker}(\varphi)$, and $B := \mathrm{Ker}(g)$. Then one has the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & P & \xrightarrow{g} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & M_w & \xrightarrow{\pi} & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\varphi_0 = \varphi|_B$. It follows that $\varphi_0 : B \to M$ is a special $\mathcal{A}$-precover of M.

Let $h : B \to B$ be a homomorphism such that $\varphi_0 h = \varphi_0$. By [10, Theorem 6.3.2], h can be extended only to a homomorphism $h' : P \to P$. So $\varphi h'$ is an extension of $\varphi_0 h$. Again by [10, Theorem 6.3.2], $\varphi h' = \varphi$. So h' is an isomorphism. Thus h is a monomorphism.

Let $x \in B$. Then there is $y \in P$ such that $h'(y) = x$. So $gh'(y) = \pi\varphi h'(y) = \pi\varphi(y) = g(y)$. Therefore $b := y - h'(y) = y - x \in \mathrm{Ker}(g) = B$. So $y = b + x \in B$, which results in $x = h(y)$. Thus h is an epimorphism. So h is an isomorphism, and thus $\varphi_0 : B \to M$ is an $\mathcal{A}$-cover of M.

Conversely, let $\alpha : B \to M$ be an $\mathcal{A}$-cover of M and $P := B_w$. It follows from Proposition 3.12(1) that B is a GV-torsion-free module. By [10, Theorem 6.3.2], α induces a unique homomorphism $\varphi : P \to M_w$. Set $T := P/B$ and $T_2 := M_w/M$. Then T and T_2 are GV-torsion modules. Thus one has the following commutative diagram with two exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & P & \xrightarrow{\pi} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & M & \longrightarrow & M_w & \xrightarrow{\pi_1} & T_2 & \longrightarrow & 0
\end{array}$$

Set $A := \mathrm{Ker}(\alpha)$, $D := \mathrm{Ker}(\varphi)$, and $T_1 := \mathrm{Ker}(\beta)$. It follows from the snake lemma that one has the following exact sequence: $0 \to A \to D \to T_1 \to 0$. Because $A \in \mathcal{A}^{\perp}$, one has $\mathrm{Ext}^1_R(T_1, A) = 0$. Thus $D \cong A \oplus T_1$. Since D is GV-torsion-free, it follows that $T_1 = 0$, and so $D = A$. Since α is an epimorphism, φ is also an epimorphism, and thus β is an isomorphism. Hence φ is a special $\mathcal{A}$-precover of M_w.

Now let $h : P \to P$ be a homomorphism such that $\varphi h = \varphi$. Consider the following diagram with exact two rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & P & \xrightarrow{\pi} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & P & \xrightarrow{\pi} & T & \longrightarrow & 0 \end{array}$$

Then $\pi h = \beta^{-1}\pi_1\varphi h = \beta^{-1}\pi_1\varphi = \pi$, and so the square diagram on the right is a commutative diagram. Thus $h_0 : B \to B$ makes the left square a commutative diagram. Since α is the restriction of φ on B, one has $\alpha h_0 = \alpha$. So h_0 is an isomorphism, and thus h is an isomorphism. Therefore φ is an $\mathcal{A}$-cover of M_w. □

Proposition 3.14. *Let $\mathcal{A}$ be a class of modules which is closed under w-isomorphisms. Let M be an R-module and set $T := \mathrm{tor}_{\mathrm{GV}}(M)$. If $\varphi : P \to M/T$ is a special $\mathcal{A}$-cover which makes the pullback diagram*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{\lambda} & P_1 & \xrightarrow{\beta} & P & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & T & \longrightarrow & M & \xrightarrow{\pi} & M/T & \longrightarrow & 0, \end{array}$$

then $\alpha : P_1 \to M$ is a special $\mathcal{A}$-cover.

Proof. Because P_1 is w-isomorphic to P, one has $P_1 \in \mathcal{A}$. Set $A := \mathrm{Ker}(\varphi)$. Since $\mathrm{Ker}(\alpha) \cong A$, it follows that $\alpha : P_1 \to M$ is a special $\mathcal{A}$-precover. Let $h : P_1 \to P_1$ be a homomorphism such that $\alpha h = \alpha$. It follows from Proposition 3.12(1) that P is a GV-torsion-free module. Thus h induces a homomorphism $\overline{h} : P \to P$ such that $\varphi\overline{h} = \varphi$. So $\overline{h}$ is an isomorphism. Thus one has the following commutative diagram with two exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & P_1 & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \overline{h}} & & \\ 0 & \longrightarrow & T & \longrightarrow & P_1 & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

So h is an isomorphism. Therefore α is a special $\mathcal{A}$-cover. □

Remark 3.15. Taking $\mathcal{A} := \mathrm{w}\mathcal{P}_w$, by Theorem 3.13 and Proposition 3.14, in order to discuss the existence of a weak w-projective cover of a module, just consider whether the w-module has a weak w-projective cover.

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