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A HOMOLOGICAL CHARACTERIZATION OF PRÜFER v-MULTIPLICATION RINGS

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Abstract. Let R be a ring and M an R-module. Then M is said to be regular w-flat provided that the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is a w-monomorphism for any regular ideal I. We distinguish regular w-flat modules from regular flat modules and w-flat modules by idealization constructions. Then we give some characterizations of total quotient rings and Prüfer v-multiplication rings (PvMRs for short) utilizing the homological properties of regular w-flat modules.

1. introduction

Recall from [6, Theorem 2.1] that an integral domain R is a *Prüfer v-multiplication domain* (abbreviated PvMD) provided that any nonzero finitely generated ideal is w-invertible. Obviously, PvMDs can be seen as w-versions of Prüfer domains which are integral domains that any nonzero finitely generated ideal is invertible. In 2015, Wang and Qiao [16, Theorem 3.5] gave a homological characterization of PvMDs which states that an integral domain R is a PvMD if and only if the w-weak global dimension of R is at most 1. Our original motivation for this work is to extend this result to commutative rings with zero divisors. Early in 1980, Huckaba and Papick [8] and Matsuda [11] extended the notion of PvMDs to that of PvMRs by declaring that a commutative ring R is a PvMR provided that any finitely generated regular ideal is w-invertible. Certainly PvMRs are viewed as a w-version of Prüfer rings for which any finitely generated regular ideal is invertible. In 2005, Lucas [10, Theorem 7.8; Theorem 7.12] determined a commutative ring R when the polynomial ring $R[x]$ and the Nagata ring $R(x)$ are PvMRs respectively. In 2014, Yin [20] characterized PvMRs by largely localizing at prime ideals (see [20, Theorem 2.1]). Recently, the author and Zhao [23] characterized ϕ-PvMRs

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using w-ϕ-flat modules. In this work, we will give some homological characterizations of the total quotient rings and PvMRs utilizing regular w-flat modules (see Theorem 4.8).

Throughout this paper, R denotes a commutative ring with identity and $T(R)$ is its total quotient ring. An R-submodule I of $T(R)$ is said to be fractional if there exists a regular element $s \in R$ such that $sI \subseteq R$. If I is a fractional ideal, we denote $I^{-1} = \{r \in T(R) \mid rI \subseteq R\}$.

Now we review some definitions and notations related to the w-operation. A finitely generated ideal J of R is called a *Glaz-Vasconcelos ideal* (GV-*ideal* for short) if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism. The set of GV-ideals is denoted by $\mathrm{GV}(R)$. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) := \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

An R-module M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsion-free module M is called a w-*module* if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$, and the w-*envelope* of M is given by

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective envelope of M. A fractional ideal I is said to be w-invertible if $(II^{-1})_w = R$. A DW *ring* R is a ring over which every module is a w-module, equivalently the only GV-ideal of R is R. A *maximal* w-*ideal* is an ideal of R which is maximal among all w-submodules of R. The set of all maximal w-ideals is denoted by w-$\mathrm{Max}(R)$. By [15, Theorem 6.2.14], any maximal w-ideal is prime.

An R-homomorphism $f : M \to N$ is said to be a w-*monomorphism* (resp., w-*epimorphism*, w-*isomorphism*) if for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$, $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism). Note that f is a w-monomorphism (resp., w-epimorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$) is GV-torsion. A sequence $A \to B \to C$ is said to be w-*exact* if for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$, $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is exact. A class $\mathcal{C}$ of R-modules is said to be closed under w-isomorphisms provided that for any w-isomorphism $f : M \to N$, if one of the modules M and N is in $\mathcal{C}$, so is the other. Following from [14], an R-module M is said to be w-flat if for any w-monomorphism $f : A \to B$, the induced sequence $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is also a w-monomorphism. The class of w-flat modules is closed under w-isomorphisms, see [15, Corollary 6.7.4].

An R-module M is said to be of *finite type* if there exist a finitely generated free module F and a w-epimorphism $g : F \to M$, and it is said to be of *finitely presented type* if there is a w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_0 and F_1 are finitely generated free modules. The classes of finite type and finitely presented type modules are all closed under w-isomorphisms, see [15, Corollary 6.4.4; Corollary 6.4.13]. Following [12], a ring R is said to be w-*coherent* if every finitely generated ideal of R is of finitely presented type. The authors [22, Theorem 2.2] gave a w-version of Chase Theorem to characterize

w-coherent rings as follows: a ring R is w-coherent if and only if any direct product of flat modules is w-flat, if and only if any direct product of R is w-flat.

2. Regular w-flat modules

Let R be a ring. An ideal I of R is said to be regular if I contains a regular element. An R-module M is said to be *regular flat* provided that the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is a monomorphism for any regular ideal I, equivalently, $\mathrm{Tor}_1^R(R/I, M) = 0$ for any regular ideal I (see [19]). In this section, we introduce and study regular w-flat modules which generalize both regular flat modules and w-flat modules.

Definition 2.1. Let R be a ring. An R-module M is said to be *regular w-flat* provided that the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is a w-monomorphism for any regular ideal I.

Clearly, any w-flat module and regular flat module are regular w-flat. We can obtain some characterizations of regular w-flat modules which is similar to [13, Proposition 1.1].

Theorem 2.2. *Let R be a ring. The following statements are equivalent for an R-module M:*

(1) *M is regular w-flat;*
(2) *$\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion for all regular ideals I of R;*
(3) *$\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion for all finitely generated regular ideals I of R;*
(4) *the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is w-exact for all finitely generated regular ideals I of R;*
(5) *the natural homomorphism $i_I : I \otimes_R M \to IM$ is a w-isomorphism for all regular ideals I of R;*
(6) *the natural homomorphism $i_I : I \otimes_R M \to IM$ is a w-isomorphism for all finitely generated regular ideals I of R.*

Proof. (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6): These implications are trivial.

(1) $\Leftrightarrow$ (2) (resp., (3) $\Leftrightarrow$ (4)): Let I be a (resp., finitely generated) regular ideal of R. The equivalence follows from the exact sequence

$$0 \to \mathrm{Tor}_1^R(R/I, M) \to I \otimes_R M \to R \otimes_R M \to R/I \otimes_R M \to 0.$$

(1) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6): These follow noting that $\mathrm{Im}(f \otimes_R 1) = IM$.

(6) $\Rightarrow$ (5): Let I be a regular ideal of R and s a regular element in I. We just need to show $\ker(i_I)$ is GV-torsion. Suppose that $i_I(\sum_{i=1}^n a_i \otimes x_i) = \sum_{i=1}^n a_i x_i = 0$. Let $K = Ra_1 + \cdots + Ra_n + Rs$. Then the finitely generated

regular ideal K is contained in I. Consider the following commutative diagram

$$\begin{array}{ccc} K \otimes_R M & \xrightarrow{i_K} & KM \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ I \otimes_R M & \xrightarrow{i_I} & IM. \end{array}$$

By (6), i_K is a w-isomorphism. So there is a GV-ideal J such that $J\sum_{i=1}^{n} a_i \otimes x_i = 0$ in $K \otimes_R M$. Since h is a monomorphism, g is a w-monomorphism. Thus there is a GV-ideal J' such that $J'J\sum_{i=1}^{n} a_i \otimes x_i = 0$ in $I \otimes_R M$. Since $J'J \in \mathrm{GV}(R)$, we have $\ker(i_I)$ is GV-torsion. □

Corollary 2.3. *Let R be a ring. The class of regular w-flat R-modules is closed under w-isomorphisms.*

Proof. Let $f : M \to N$ be a w-isomorphism and I a regular ideal. There exist two exact sequences $0 \to T_1 \to M \to L \to 0$ and $0 \to L \to N \to T_2 \to 0$ with T_1 and T_2 GV-torsion. Consider the induced two long exact sequences $\mathrm{Tor}_1^R(R/I, T_1) \to \mathrm{Tor}_1^R(R/I, M) \to \mathrm{Tor}_1^R(R/I, L) \to R/I \otimes T_1$ and $\mathrm{Tor}_2^R(R/I, T_2) \to \mathrm{Tor}_1^R(R/I, L) \to \mathrm{Tor}_1^R(R/I, N) \to \mathrm{Tor}_1^R(R/I, T_2)$. By [15, Theorem 6.7.2], M is regular w-flat if and only if N is regular w-flat. □

Following [19, Definition 3.1], a ring R is said to be *regular coherent* if any finitely generated regular ideal is finitely presented. In order to distinguish regular w-flat modules from regular flat modules and w-flat modules, we generalize regular coherent rings in the sense of the w-operation.

Definition 2.4. A ring R is said to be *regular w-coherent* provided that any finitely generated regular ideal is of finitely presented type.

Obviously, w-coherent rings and regular coherent rings are examples of regular w-coherent rings. Certainly, a DW-ring is regular w-coherent if and only if it is a regular coherent ring, and an integral domain is regular w-coherent if and only if it is a w-coherent domain. We can characterize regular w-coherent rings by regular w-flat modules.

Proposition 2.5. *Let R be a ring. The following statements are equivalent*:

(1) *R is regular w-coherent*;
(2) *the direct product of flat modules is regular w-flat*;
(3) *the direct product of projective modules is regular w-flat*;
(4) *the direct product of R is regular w-flat.*

Proof. (1) ⇒ (2): Let $\{F_i\}_{i\in I}$ be a family of flat modules. By Theorem 2.2, we just need to show for any finitely generated regular ideal J, the natural homomorphism $i_J : J \otimes_R \prod_{i\in I} F_i \to J\prod_{i\in I} F_i$ is a w-isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccc} J \otimes_R \prod_{i\in I} F_i & \xrightarrow{i_J} & J \prod_{i\in I} F_i \\ \downarrow{\scriptstyle \phi_J} & & \downarrow \\ \prod_{i\in I}(J \otimes_R F_i) & \xrightarrow{\gamma} & \prod_{i\in I}(JF_i). \end{array}$$

Since R is regular w-coherent, J is of finitely presented type. By [22, Proposition 2.12], ϕ_J is a w-monomorphism. Since F_i is flat for any $i \in I$, we have γ is an isomorphism (See [15, Theorem 2.5.6]). Thus i_J is a w-monomorphism. Since i_J is an epimorphism, we have i_J is a w-isomorphism.

(2) ⇒ (3) ⇒ (4): These implications are trivial.

(4) ⇒ (1): We just need to show any finitely generated regular ideal J is of finitely presented type. Consider the following commutative diagram

$$\begin{array}{ccc} J \otimes \prod_{i\in I} R & \xrightarrow{\alpha} & R \otimes \prod_{i\in I} R \\ \downarrow{\scriptstyle i_J} & & \downarrow{\scriptstyle =} \\ \prod_{i\in I}(J) & \xrightarrow{\beta} & \prod_{i\in I}(R). \end{array}$$

Since $\prod_{i\in I} R$ is regular w-flat, α is a w-monomorphism. Thus i_J is a w-monomorphism. By [22, Proposition 2.12], J is of finitely presented type. □

Some non-integral domain examples are provided by the idealization construction $R(+)M$ where M is an R-module (see [7]). We recall this construction. Let $R(+)M = R \oplus M$ as a R-module, and define

(1) $(r, m)+(s, n)=(r + s, m + n)$.
(2) $(r, m)(s, n)=(rs, sm + rn)$.

Under this construction, $R(+)M$ becomes a commutative ring with identity $(1, 0)$.

Proposition 2.6 ([4, Proposition 2.2]). *Let D be an integral domain, K its quotient field and $R = D(+)K$. Then the following statements hold.*

(1) $\mathrm{GV}(R) = \{J(+)K \mid J \in \mathrm{GV}(D)\}$.
(2) *Let T be a D-module. Then T is* GV*-torsion if and only if $T \otimes_D R$ is* GV*-torsion over R.*
(3) *Let M be a D-module. Then M is a w-module if and only if $M \otimes_D R$ is a w-module over R.*
(4) *Let I be a nonzero ideal of D. Then I is finitely generated (resp., finitely presented) if and only if $I \otimes_D R$ ($\cong I(+)K$) is finitely generated (resp., finitely presented) over R.*
(5) *Let M be a D-module. Then M is of finite type if and only if $M \otimes_D R$ is of finite type over R.*
(6) *Let M be a D-module. Then M is of finitely presented type if and only if $M \otimes_D R$ is of finitely presented type over R.*

(7) *D is a coherent domain if and only if R is a regular coherent ring.*

(8) *D is a w-coherent domain if and only if R is a regular w-coherent ring.*

Proof. (1) Since K is divisible over D, every ideal of R is of the form $I(+)K$ or $0(+)L$, where I is a nonzero ideal over D and L is a D-submodule of K (See [1, Corollary 3.4]). Since any ideal of the form $0(+)L$ is not semi-regular, any GV-ideal of R is of the form $I(+)K$, where I is a nonzero D-ideal. Suppose I is a nonzero ideal of D. Since an element (d, m) in R is regular if and only if $d \neq 0$, $I(+)K$ is a regular ideal. Thus $I(+)K$ is a GV-ideal over R if and only if $(I(+)K)^{-1} = R$ by [21]. By [9, Theorem 11(d)], $(I(+)K)^{-1} = I^{-1}(+)K$. Thus $I(+)K$ is a GV-ideal over R if and only if $I^{-1} = D$, if and only if $I \in \mathrm{GV}(D)$.

(2) Assume T is GV-torsion over D. For any $t = \sum_{i=1}^{n} t_i \otimes r_i \in T \otimes_D R$, there exists a GV-ideal J such that $Jt_i = 0$, for each $i = 1, \ldots, n$. Then $J(+)Kt = 0$, and thus $T \otimes_D R$ is GV-torsion over R. Suppose $T \otimes_D R$ is GV-torsion over R. For any $t' \in T$, by (1), there exists a GV-ideal $J(+)K$ such that $J(+)K(t' \otimes 1) = 0$. Then $Jt' = 0$. Thus T is GV-torsion over D.

(3) Since K is flat and divisible over D, for each $i = 0, 1$ and any $J \in \mathrm{GV}(D)$, we have

$$\begin{aligned}\mathrm{Ext}_R^i(R/J(+)K, M \otimes_D R) &\cong \mathrm{Ext}_R^i(D/J, M \otimes_D R)\\ &\cong \mathrm{Ext}_R^i(D/J \otimes_D R, M \otimes_D R)\\ &\cong \mathrm{Ext}_D^i(D/J, M \otimes_D R)\\ &\cong \mathrm{Ext}_D^i(D/J, M) \oplus \mathrm{Ext}_D^i(D/J, M \otimes_D K)\\ &\cong \mathrm{Ext}_D^i(D/J, M).\end{aligned}$$

Consequently, $M \otimes_D R$ is a w-module over R if and only M is a w-module over D by (1).

(4) Since R is a faithful flat D-module, $F_1 \xrightarrow{g} F_0 \xrightarrow{f} I \to 0$ is exact over D if and only if $F_1 \otimes_D R \xrightarrow{g \otimes_D R} F_0 \otimes_D R \xrightarrow{f \otimes_D R} I \otimes_D R \to 0$ is exact over R. Note that F_i is finitely generated free over D if and only if $F_i \otimes_D R$ is finitely generated free over R. One can easily check that (4) holds.

(5) Let M be of finite type over D. Then there is a w-epimorphism $F \xrightarrow{f} M$ with F finitely generated free. Let $\{e_i \mid i = 1, \ldots, n\}$ be the standard basis of F and $\sum_{j=1}^{k}(m_j \otimes r_j) \in M \otimes_D R$. Then for each j, there exists a GV-ideal $J_j \in \mathrm{GV}(R)$ such that $J_j m_j \subseteq \mathrm{Im} f$. Let $J = J_1 \cdots J_k$. Then $J \otimes_D R \sum_{j=1}^{k}(m_j \otimes r_j) \subseteq \mathrm{Im} f \otimes_D R$. Thus $F \otimes_D R \xrightarrow{f \otimes_D R} M \otimes_D R$ is a w-epimorphism over R by [15, Proposition 6.4.2(3)].

Let $L = \langle \sum_{j=1}^{j=k_i} m_{i,j} \otimes r_{i,j} \mid i = 1, \ldots, n\rangle$ be the finitely generated submodule of $M \otimes_D R$ such that $L_w = (M \otimes_D R)_w$. For any $m \in M$, there exists a GV-ideal $J \otimes_D R$ such that $J \otimes_D R(m \otimes 1) \subseteq L$. Set $N := \langle m_{i,j} \mid j = 1, \ldots, k_i; i = 1, \ldots, n\rangle$ be a finitely generated submodule of M. Then $Jm \subseteq N$. Thus M is of finite type over D by [15, Proposition 6.4.2(3)].

(6) Assume M is of finitely presented type over D. Then there is a w-exact sequence $F_1 \xrightarrow{g} F_0 \xrightarrow{f} M \to 0$ with F_0, F_1 finitely generated free. By applying $-\otimes_D R$, we obtain a w-exact sequence $F_1 \otimes_D R \xrightarrow{g\otimes_D R} F_0 \otimes_D R \xrightarrow{f\otimes_D R} M \otimes_D R \to 0$ over D as R is a flat D-module. Similarly to the proof of (5), one can check that it is also a w-exact sequence over R.

Now assume that $M \otimes_D R$ is of finitely presented type over R. Then M is of finite type over D by (5). Let $F_1' \xrightarrow{g'} F_0 \xrightarrow{f} M \to 0$ be a w-exact sequence with F_0 finitely generated free and F_1' free. By applying $-\otimes_D R$, we can also obtain a w-exact sequence $F_1' \otimes_D R \xrightarrow{g'\otimes_D R} F_0 \otimes_D R \xrightarrow{f\otimes_D R} M \otimes_D R \to 0$ over R. Since $M \otimes_D R$ is of finitely presented type, $\mathrm{Im}(g' \otimes_D R) = \mathrm{Im}(g') \otimes_D R$ is of finite type by [15, Theorem 6.4.11]. By (5), $\mathrm{Im}(g')$ is of finite type, and thus M is of finitely presented type over D by [15, Theorem 6.4.11] again.

(7) As in the proof of (1), a regular ideal of R is of the form $I(+)K$, where I is a nonzero ideal of D. Thus (7) follows trivially from (4).

(8) This also follows trivially from (4) and (6). □

Utilizing these results, we give a regular w-coherent ring which is neither w-coherent nor regular coherent.

Example 2.7. Let D be a non-coherent w-coherent domain (see [15, Example 9.1.18] for example), K its quotient field. Then $R = D(+)K$ is a regular w-coherent ring. However, it is neither w-coherent nor regular coherent.

Proof. Since D is a non-coherent w-coherent domain, R is a non-regular coherent regular w-coherent ring by Proposition 2.6. We will show R is not w-coherent as well. Note that $(0,1)R$ is a finitely generated ideal over R. Consider the natural $0 \to L \to R \to (0,1)R \to 0$. Then $L = \mathrm{Nil}(R) = 0(+)K$. Since D is not a field, the w-module K is not a finitely generated module over D. Thus K is not of finite type over D. By [2, Lemma 2.2], the w-ideal $\mathrm{Nil}(R)$ is not of finite type over R. Thus $(0,1)R$ is not of finitely presented type over R. □

Now we are ready to give an example of a regular w-flat module which is neither w-flat nor regular flat.

Example 2.8. Let R be a regular w-coherent which is neither w-coherent nor regular coherent (See Example 2.7). By comparing [22, Theorem 2.2] and [19, Theorem 3.2], we can find a regular w-flat module $F = \prod_{i\in I} F_i$ with each F_i flat neither w-flat nor regular flat.

3. On the homological dimension of regular w-flat modules

Let R be a ring and M an R-module. Following [16], the w-flat dimension $w\text{-fd}_R(M)$ of an R-module M is defined as the length of the shortest w-flat w-resolution of M and the w-weak global dimension $w\text{-w.gl.dim}(R)$ of R is the supremum of the w-flat dimensions of all R-modules. We now introduce the notion of the regular w-flat dimension of an R-module M as follows.

Definition 3.1. Let R be a ring and M an R-module. We write reg-w-$\mathrm{fd}_R(M) \leq n$ (reg-w-fd abbreviates *regular w-flat dimension*) if there is a w-exact sequence of R-modules

$$(\diamondsuit) \qquad 0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

with each F_i w-flat ($i = 0, \ldots, n-1$) and F_n regular w-flat. The w-exact sequence ($\diamondsuit$) is said to be a regular w-flat w-resolution of length n of M. The regular w-flat dimension reg-w-$\mathrm{fd}_R(M)$ is defined to be the length of the shortest regular w-flat w-resolution of M. If such finite w-resolution ($\diamondsuit$) does not exist, then we say reg-w-$\mathrm{fd}_R(M) = \infty$.

It is obvious that an R-module M is regular w-flat if and only if reg-w-$\mathrm{fd}_R(M) = 0$ and reg-w-$\mathrm{fd}_R(N) \leq w$-$\mathrm{fd}_R(N)$ for any R-module N.

Lemma 3.2 ([16, Lemma 2.2])**.** *Let N be an R-module and $0 \to A \to F \to C \to 0$ a w-exact sequence of R-modules with F w-flat. Then for any $n > 0$, the induced map $\mathrm{Tor}^R_{n+1}(C, N) \to \mathrm{Tor}^R_n(A, N)$ is a w-isomorphism. Hence, $\mathrm{Tor}^R_{n+1}(C, N)$ is* GV*-torsion if and only if so is $\mathrm{Tor}^R_n(A, N)$.*

Proposition 3.3. *Let R be a ring. The following statements are equivalent for an R-module M:*

(1) *reg-w-$fd_R(M) \leq n$;*
(2) *$\mathrm{Tor}^R_{n+1}(M, R/I)$ is* GV*-torsion for all regular ideals I;*
(3) *$\mathrm{Tor}^R_{n+1}(M, R/I)$ is* GV*-torsion for all finitely generated regular ideals I;*
(4) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules, then F_n is regular w-flat;*
(5) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an w-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are w-flat R-modules, then F_n is regular w-flat;*
(6) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are w-flat R-modules, then F_n is regular w-flat;*
(7) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an w-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules, then F_n is regular w-flat.*

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on n. For the case $n = 0$, (2) holds by Theorem 2.2 as M is a regular w-flat module. If $n > 0$, then there is a w-exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i w-flat ($i = 0, \ldots, n-1$) and F_n is regular w-flat. Let $K_0 = \ker(F_0 \to M)$. We have two w-exact sequences $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to K_0 \to 0$. Note that reg-w-$\mathrm{fd}_R(K_0) \leq n-1$. By induction, $\mathrm{Tor}^R_n(K_0, R/I)$ is GV-torsion for all regular ideals I. It follows from Lemma 3.2 that $\mathrm{Tor}^R_{n+1}(M, R/I)$ is GV-torsion.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (4): Let $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence. Set $K_0 = \ker(F_0 \to M)$ and $K_i = \ker(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Since all $F_0, F_1, \ldots, F_{n-1}$ are flat, $\mathrm{Tor}_1^R(F_n, R/I) \cong \mathrm{Tor}_{n+1}^R(M, R/I)$ is GV-torsion for all finitely generated regular ideals I. Hence F_n is a regular w-flat module by Theorem 2.2.

(3) $\Rightarrow$ (5): Let $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ be a w-exact sequence. Set $L_n = F_n$ and $L_i = \mathrm{Im}(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Then both $0 \to L_{i+1} \to F_i \to L_i \to 0$ and $0 \to L_1 \to F_0 \to M \to 0$ are w-exact sequences. By using Lemma 3.2 repeatedly, we can obtain that $\mathrm{Tor}_1^R(F_n, R/I)$ is GV-torsion for all finitely generated regular ideals I. Thus F_n is regular w-flat by Theorem 2.2.

(4) $\Rightarrow$ (1), (5) $\Rightarrow$ (6) $\Rightarrow$ (4) and (5) $\Rightarrow$ (7) $\Rightarrow$ (4): These implications are trivial. □

Definition 3.4. The *reg-w-weak global dimension* of a ring R is defined by

$$reg\text{-}w\text{-w.gl.dim}(R) = \sup\{\text{reg-}w\text{-fd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Obviously, by definition, reg-w-w.gl.dim$(R) \leq w$-w.gl.dim(R) for any ring R. Following from Proposition 3.3, we have the following result.

Proposition 3.5. *The following statements are equivalent for a ring R.*

(1) *reg-w-w.gl.dim*$(R) \leq n$;
(2) *reg-w-fd*$_R(M) \leq n$ *for all R-modules M*;
(3) $\mathrm{Tor}_{n+1}^R(M, R/I)$ *is* GV*-torsion for all R-modules M and all regular ideals I of R*;
(4) $\mathrm{Tor}_{n+1}^R(M, R/I)$ *is* GV*-torsion for all R-modules M and all finitely generated regular ideals I of R.*

4. Rings with regular w-weak global dimension at most one

Recall from [5, Definition 2.1.1] that a ring R is said to be a total quotient ring provided that any regular element is a unit, i.e., $T(R) = R$. Recently, Xiao [19, Theorem 2.13] shows that a ring R is a total quotient ring if and only if any R-module is regular flat.

Theorem 4.1. *Let R be a ring. The following statements are equivalent*:

(1) *R is a total quotient ring*;
(2) *every R-module is regular flat*;
(3) *every R-module is regular w-flat*;
(4) *reg-w-w.gl.dim*$(R) = 0$;
(5) $a \in (a^2)_w$ *for any regular element $a \in R$.*

Proof. (1) $\Leftrightarrow$ (2): See [19, Theorem 2.13].

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Leftrightarrow$ (4): This follows from Definition 3.4.

(4) $\Rightarrow$ (5): Let a be a regular element in R. Then Ra is a regular ideal of R. It follows that $\mathrm{Tor}_1^R(R/Ra, R/Ra)$ is GV-torsion since R/Ra is torsion and

regular w-flat. That is, Ra/Ra^2 is GV-torsion, and thus $a \in Ra \subseteq (Ra)_w = (Ra^2)_w$.

(5) $\Rightarrow$ (1): Let a be a regular element in R. There exists a GV-ideal J such that $Ja \subseteq (a^2)$. Suppose J is generated by $j_1, \ldots, j_n$. There exist $r_1, \ldots, r_n$ such that $j_i a = a^2 r_i$ for each $i = 1, \ldots, n$. Since a is regular, $j_i = ar_i$. Let I be generated by $r_1, \ldots, r_n$. Then $J = aI \subseteq (a)$. Since J is a GV-ideal, $R = J_w \subseteq (a)_w \subseteq R$ by [15, Exercise 6.10]. Thus $(a)_w = R$, and then a is a unit by [15, Exercise 6.11]. So R is a total quotient ring. □

Recall that an integral domain R is called a *Prüfer v-multiplication domain* (PvMD for short) if any nonzero finitely generated ideal is w-invertible (see [6, Theorem 2.1]). In 1980, Huckaba and Papick [8] and Matsuda [11] extended this notion to commutative rings with zero divisors.

Definition 4.2 ([5, Definition 2.5.9])**.** A ring R is said to be a *Prüfer v-multiplication ring* (PvMR for short) provided that any finitely generated regular ideal is w-invertible.

Proposition 4.3. *Let D be an integral domain, K its quotient field and $R = D(+)K$. The following assertions hold.*

(1) *R is a total quotient ring if and only if D is a field*;
(2) *$I(+)K$ is w-invertible over R if and only if I is w-invertible over R*;
(3) *R is a* PvMR *if and only if D is a Prüfer v-multiplication domain.*

Proof. (1) Just note that an element (d, m) in R is regular (resp., a unit) if and only if d is nonzero (resp., a unit) by [1, Theorem 3.5] (resp., [1, Theorem 3.7]).

(2) Let I be a nonzero ideal over D, $I' = I(+)K$ and $I'' = I^{-1}(+)K$. By [9, Theorem 11], $I'' = I'^{-1}$ and thus I' is w-invertible if and only if $(I'I'')_w = R$. That is, there is a GV-ideal $J' = J(+)K$ such that $J' \subseteq I'I''$. This is equivalent to say $J \subseteq II^{-1}$, i.e., I is w-invertible over D.

(3) This follows immediately from (2). □

Recall from [14] that an R-module M is said to be a *w-projective module* if $\mathrm{Ext}^1_R((M/\mathrm{Tor}_{\mathrm{GV}}(M))_w, N)$ is a GV-torsion module for any torsion-free w-module N. The following proposition is an extension of [17, Theorem 2.7] to commutative rings with zero divisors. Its proof which is very similar to that of [17, Theorem 2.7] is given for completeness.

Proposition 4.4. *Let R be a ring and I a regular fractional ideal of R. Then I is w-invertible if and only if I is w-projective.*

Proof. Assume that I is a w-projective fractional w-ideal, $s = a/b$ is a regular element in I with b regular in R. There exist a GV-ideal $J = (d_1, d_2, \ldots, d_n) \in \mathrm{GV}(R)$ and elements $\{x_i\}_{i \in I}$ such that for each $k \in \{1, 2, \ldots, n\}$, there exist R-homomorphisms $\{f_{ki} \in I^* \mid i \in \Gamma\}$ such that almost all $f_{ki}(x) = 0$ and

$d_k x = \sum_i f_{ki}(x)x_i$ for any $x \in I$. For any $t = c/d \in I$, we have

$$bdf_{ki}(\frac{ac}{bd}) = df_{ik}(\frac{ac}{d}) = adf_{ik}(\frac{c}{d}) = adf_{ik}(t).$$

Thus $f_{ki}(\frac{ac}{bd}) = \frac{a}{b}f_{ik}(t)$, i.e., $f_{ki}(st) = sf(t)$. Similarly, $f_{ki}(st) = tf(s)$. Thus $sf(t) = tf(s)$. Let $x_{ki} = \frac{f_{ki}(s)}{s}$ for any $k = 1, 2, \ldots, n$. Then we have $Ix_{ki} \subseteq R$, and thus $x_{ki} \in I^{-1}$. Note that $sd_k = \sum_{i=1}^{m_k} f_{ki}(s)x_i = s\sum_{i=1}^{m_k} x_{ki}x_i$. Since s is regular, $d_k = \sum_{i=1}^{m_k} x_{ki}x_i \in II^{-1}$. So $J \subseteq II^{-1}$. Therefore $(II^{-1})_w = R$ by [15, Exercise 6.10(2)].

Conversely, assume that $(II^{-1})_w = R$. Without loss of generality, we can also assume that I is a fractional w-ideal of R. Then there exists $J = (b_1, \ldots, b_n) \in \mathrm{GV}(R)$ such that b_k can be expressed as $b_k = \sum_{i=1}^{m_k} a_{ki}x_{ki}$ for any $k = 1, \ldots, n$, where $a_{ki} \in I$, $x_{ki} \in I^{-1}$. Define $\phi : I^{-1} \to \mathrm{Hom}_R(I, R) = I^*$ by $\phi(x)(y) = yx$, where $x \in I^{-1}, y \in I$. By [15, Corollary 6.6.9(1)], ϕ is an isomorphism. Set $f_{ki} = \phi(x_{ki})$. Then $f_{ki} \in I^*$ and $f_{ki}(a) = \phi(x_{ki})(a) = ax_{ki}$ for any $a \in I$. So $b_k s = \sum_{i=1}^{m_k} as_{ki}x_{ki} = \sum_{i=1}^{m_k} f_{ki}(s)x_{ki}$. By [17, Theorem 2.2], I is w-projective. □

In 2015, Wang and Kim [14] introduced the *w-Nagata ring*, $R\{x\}$, of R. It is a localization of $R[X]$ at the multiplicative closed set

$$S_w = \{f \in R[x] \mid c(f) \in \mathrm{GV}(R)\},$$

where $c(f)$ is the ideal of R generated by the coefficients of f. Similarly, the *w-Nagata module* $M\{x\}$ of an R-module M is defined as $M\{x\} = M[x]_{S_w} \cong M \bigotimes_R R\{x\}$.

Lemma 4.5. *Every Prüfer v-multiplication ring is regular w-coherent.*

Proof. Let I be a finitely generated regular ideal of R. Then I is w-invertible, and thus w-projective by Proposition 4.4. Thus $I\{x\}$ is finitely generated projective $R\{x\}$-module by [15, Theorem 6.7.18]. So I is of finitely presented type by [14, Theorem 3.9] and R is regular w-coherent. □

It is well known that Prüfer v-multiplication domains are w-coherent domains. However, every Prüfer v-multiplication ring is not w-coherent.

Example 4.6. Let F be a field and V an infinite dimensional vector space over F. Denote $R = F(+)V$. By [1, Theorem 3.5; Theorem 3.7], R is a total quotient ring, and thus a PvMR. One can show R is not w-coherent by the similar proof of Example 2.7.

Proposition 4.7. *Let R be a ring and I a regular fractional ideal of finite type over R. If I is w-flat, then I is w-invertible.*

Proof. Let $\mathfrak{m}$ be a maximal w-ideal of R. Since I is w-flat of finite type over R, $I_\mathfrak{m}$ is a finitely generated flat $R_\mathfrak{m}$-module. Thus $I_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module with finite rank. Since I is a fractional ideal over R, it follows that $I_\mathfrak{m}$ is a fractional ideal over $R_\mathfrak{m}$ as $\mathrm{T}(R_\mathfrak{m}) \subseteq \mathrm{T}(R)_\mathfrak{m}$. Consequently, $I_\mathfrak{m}$ has rank ≤ 1 over $R_\mathfrak{m}$.

Since I is regular and $I_{\mathfrak{m}}$ is free, $I_{\mathfrak{m}}$ is generated by a regular element. Thus $I_{\mathfrak{m}} \cong R_{\mathfrak{m}}$. Consequently, $I_{\mathfrak{m}}$ has rank 1 over $R_{\mathfrak{m}}$. Since I is of finite type, I is w-invertible by [14, Theorem 4.13]. □

Theorem 4.8. *The following statements are equivalent for a ring R:*

(1) *R is a* PvMR;
(2) *every finitely generated regular ideal is w-projective*;
(3) *every finite type regular ideal is w-projective*;
(4) *any submodule of a regular w-flat module is regular w-flat*;
(5) *any submodule of a flat module is regular w-flat*;
(6) *any ideal of R is regular w-flat*;
(7) *any regular ideal of R is w-flat*;
(8) *reg-w-w.gl.dim$(R) \leq 1$.*

Proof. (1) ⇒ (3) and (2) ⇒ (1): These follow from Proposition 4.4.

(3) ⇒ (2): This is trivial.

(2) ⇒ (4): Let I be a finitely generated regular ideal of R. Then I is w-invertible, and thus w-projective by Proposition 4.4. Let $0 \to N \to M \to L \to 0$ be an exact sequence with M regular w-flat. Consider the following exact sequence

$$\cdots \to \mathrm{Tor}_2^R(R/I, L) \to \mathrm{Tor}_1^R(R/I, N) \to \mathrm{Tor}_1^R(R/I, M) \to \cdots .$$

Since I is w-projective, I is w-flat by [15, Theorem 6.7.11]. Thus $\mathrm{Tor}_2^R(R/I, L) \cong \mathrm{Tor}_1^R(I, L)$ is GV-torsion. Because $\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion, we have $\mathrm{Tor}_1^R(R/I, N)$ is GV-torsion. Thus N is regular w-flat.

(4) ⇒ (5) ⇒ (6): These are trivial.

(6) ⇒ (7): Let J be a regular ideal of R. For any ideal I of R, we have $\mathrm{Tor}_1^R(R/J, I) \cong \mathrm{Tor}_2^R(R/J, R/I) \cong \mathrm{Tor}_1^R(R/I, J)$ is GV-torsion. Thus J is a w-flat ideal.

(7) ⇒ (2): Let I be a finitely generated regular ideal. Then I is w-flat, and thus w-invertible by Proposition 4.7.

(5) ⇔ (8): This follows from Proposition 3.5. □

Remark 4.9. By Theorem 4.8, a commutative ring with w-w.gl.dim$(R) \leq 1$ is a PvMR, and an integral domain R is a PvMD if and only if *reg*-w-w.gl.dim$(R) \leq 1$, if and only if w-w.gl.dim$(R) \leq 1$. However, PvMRs need not have w-w.gl.dim(R) at most 1. Let R be a local Gaussian ring with nilpotent maximal ideal. Then R is a Prüfer ring, and thus a PvMR. By [3, Proposition 5.3] and [18, Theorem 3.2], every Gaussian ring is a DW-ring. Thus w-w.gl.dim(R)=w.gl.dim$(R) = \infty$ by [3, Proposition 6.3].

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