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SEMI-SYMMETRIC CUBIC GRAPH OF ORDER $12p^3$

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Abstract. A simple graph is called semi-symmetric if it is regular and edge transitive but not vertex transitive. In this paper we prove that there is no connected cubic semi-symmetric graph of order $12p^3$ for any prime number p.

1. Introduction

In this paper all graphs are finite, undirected and simple, i.e., without loops or multiple edges. A graph is called semi-symmetric if it is regular and edge transitive but not vertex transitive. The class of semi-symmetric graphs was first studied by Folkman [6], who found several infinite families of such graphs and posed eight open problems. In [6], Folkman proved that there are no semi-symmetric graphs of order $2p$ or $2p^2$ for any prime p. In [13] the authors prove that there is no connected cubic semi-symmetric graph of order $2p^3$ for any prime $p > 3$ and that for $p = 3$ the Gray graph (see [1]) is the only connected cubic semi-symmetric graph of order $2p^3$. Also in [5] it is proved that a connected cubic semi-symmetric graph of order $6p^3$ exists if and only if $p - 1$ is divisible by 3. The classification of connected cubic semi-symmetric graphs of order $20p$, any prime p, is achieved in [15]. Also the authors of [4] prove that for any prime p other than 17, there is no connected cubic semi-symmetric graphs of order $34p^3$.

In this paper we consider graphs of order $12p^3$. We prove that there is no connected cubic semi-symmetric graph of order $12p^3$ for all primes p.

2. Preliminaries

In this paper, the cardinality of a finite set A is denoted by $|A|$. The alternating group of degree n and the cyclic group of order n are denoted by $\mathbb{A}_n$ and $\mathbb{Z}_n$, respectively. Other notations about finite simple groups are standard. If G is a group and $H \leq G$, then $Aut(G)$, G', $Z(G)$, $C_G(H)$ and $N_G(H)$ denote,

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respectively, the group of automorphisms of G, the commutator subgroup of G, the center of G, the centralizer and the normalizer of H in G. We also write $H \trianglelefteq^c G$ to denote H is a characteristic subgroup of G. If $H \trianglelefteq^c K \trianglelefteq G$, then $H \trianglelefteq G$. For a prime p dividing the order of finite G, $O_p(G)$ will denote the largest normal p-subgroup of G. It is easy to verify that $O_p(G) \trianglelefteq^c G$. A function f acts on its argument from the left, i.e., we write $f(x)$. The composition fg of two functions f and g is defined as $(fg)(x) = f(g(x))$. For a group G and a nonempty set Ω, an action of G on Ω is a function $(g, \omega) \to g.\omega$ from $G \times \Omega$ to Ω, where $1.\omega = \omega$ and $g.(h.\omega) = (gh).\omega$ for every $g, h \in G$ and every $\omega \in \Omega$. We write $g\omega$ instead of $g.\omega$, if there is no fear of ambiguity. For $\omega \in \Omega$, the stabilizer of ω in G is defined as $G_\omega = \{g \in G : g\omega = \omega\}$. The action is called *semiregular* if the stabilizer of each element in Ω is trivial; it is called *regular* if it is semiregular and transitive.

Let Γ be a graph. For two vertices u and v, we write $u \sim v$ to denote u is adjacent to v. If $u \sim v$, then each of the ordered pairs (u, v) and (v, u) is called an *arc*. The set of all vertices adjacent to u is denoted by $\Gamma(u)$. The degree or valency of a vertex u is $|\Gamma(u)|$. The graph Γ is called *regular* if all of its vertices have the same valency. The vertex set, the edge set, the arc set and the set of all automorphisms of Γ are denoted by $V(\Gamma)$, $E(\Gamma)$, $Arc(\Gamma)$ and $Aut(\Gamma)$, respectively. If Γ is a graph and $N \trianglelefteq Aut\,(\Gamma)$, then Γ_N will denote a simple undirected graph whose vertices are the orbits of N in its action on $V(\Gamma)$, and where two vertices Nu and Nv are adjacent if and only if $u \sim nv$ in Γ for some $n \in N$.

Let Γ_c and Γ be two graphs. Then Γ_c is said to be a *covering graph* for Γ if there is a surjection map $f : V(\Gamma_c) \to V(\Gamma)$ which preserves adjacency and for each $u \in V(\Gamma_c)$, the restricted function $f|_{\Gamma_c(u)} : \Gamma_c(u) \to \Gamma(f(u))$ is a one to one correspondence. The function f is called a *covering projection.* Clearly, if Γ is bipartite, then so is Γ_c. For each $u \in V(\Gamma)$, the *fibre* on u is defined as $fib_u = f^{-1}(u)$. The following important set is a subgroup of $Aut\,(\Gamma_c)$ and is called the *group of covering transformations* for f:

$$CT\,(f) = \{\sigma \in Aut\,(\Gamma_c) \mid \forall u \in V(\Gamma), \sigma(fib_u) = fib_u\}.$$

It is known that $K = CT\,(f)$ acts semiregularly on each fibre [11]. If this action is regular, then Γ_c is said to be a *regular K-cover* of Γ.

Let $X \leq Aut\,(\Gamma)$. Then Γ is said to be *X-vertex transitive*, *X-edge transitive* or *X-arc transitive* if X acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $Arc(\Gamma)$, respectively. The graph Γ is called *X-semi-symmetric* if it is regular and X-edge transitive but not X-vertex transitive. Also Γ is called *X-symmetric* if it is X-vertex transitive and X-arc transitive. For $X = Aut(\Gamma)$, we omit X and simply talk about Γ being edge transitive, vertex transitive, symmetric or semi-symmetric.

An X-edge transitive but not X-vertex transitive graph is necessarily bipartite, where the two parties are the orbits of the action of X on $V(\Gamma)$. If Γ is regular, then the two partite sets have equal cardinality. So an X-semi-symmetric graph is bipartite such that X is transitive on each partite but X

carries no vertex from one partite set to the other. A census of all connected semi-symmetric cubic graphs of orders up to 768 is given in [3].

Any minimal normal subgroup of a finite group is the internal direct product of isomorphic copies of a simple group.

A finite simple group G is called a K_n-group if its order has exactly n distinct prime divisors, where $n \in \mathbb{N}$. The following result determines all simple K_3-groups [9].

Theorem 2.1. *If G is a simple K_3-group, then G is one of the following groups*: $\mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, $L_2(2^3)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

Theorem 2.2 ([16]). *If H is a subgroup of a group G, then $C_G(H) \trianglelefteq N_G(H)$ and $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.*

Theorem 2.3 ([14]). *Let G be a finite group and p be a prime. If G has an abelian Sylow p-subgroup, then p does not divide $|G' \cap Z(G)|$.*

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

Theorem 2.4 ([16]). *For any two distinct primes p and q and any two non-negative integers a and b, every finite group of order $p^a q^b$ is solvable.*

The following important theorem limits the order of vertex stabilizers in a cubic semi-symmetric graph.

Theorem 2.5 ([8]). *If Γ is a connected cubic X-semi-symmetric graph, then for each vertex u, the order of the stabilizer X_u is of the form $2^r \cdot 3$ for some $0 \leq r \leq 7$.*

Proposition 2.6 ([18]). *Let Γ be a connected cubic X-semi-symmetric graph for some $X \leq Aut(\Gamma)$ and let $N \trianglelefteq X$. If $|X/N|$ is not divisible by 3, then Γ is also N-semi-symmetric.*

Theorem 2.7 ([12]). *Let Γ be a connected cubic X-semi-symmetric graph. Let $\{U, W\}$ be a bipartition for Γ and assume $N \trianglelefteq X$. If the actions of N on both U and W are intransitive, then N acts semiregularly on both U and W, Γ_N is X/N-semi-symmetric, and Γ is a regular N-covering of Γ_N.*

For every normal subgroup $N \trianglelefteq X$ either N is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of N is divisible by $|U| = |W|$. In the latter case, according to Theorem 2.7, the induced action of N on both U and W is semiregular and hence the order of N divides $|U| = |W|$. So we have the following handy corollary to Theorem 2.7.

Corollary 2.8. *If Γ is a connected cubic X-semi-symmetric graph with $\{U, W\}$ as a bipartition and $N \trianglelefteq X$, then either $|N|$ divides $|U|$ or $|U|$ divides $|N|$.*

3. Main results

In this section, our goal is to prove the following important result:

Theorem 3.1. *Let p be an arbitrary prime number. Then there is no connected cubic semi-symmetric graph of order $12p^3$.*

This theorem may be stated as follow: If there is a connected cubic edge transitive graph of order $12p^3$, where p is prime, then it will also be vertex transitive.

In order to prove Theorem 3.1, we need few lemmas that we now state and prove.

Lemma 3.2. *For each prime $p > 3$, the group $GL_2(p)$ does not have a subgroup isomorphic to $L_2(p)$.*

Proof. Suppose on the contrary that $L_2(p) \cong K \leq GL_2(p)$. As $SL_2(p) \trianglelefteq GL_2(p)$, we have $K \cap SL_2(p) \trianglelefteq K$ and so $K \cap SL_2(p) = 1$ or K since K is simple. If $K \cap SL_2(p) = 1$, then $SL_2(p)K$ is a subgroup of $GL_2(p)$ of order $|SL_2(p)| \cdot |L_2(p)|$. But this order is divisible by p^2 whereas $|GL_2(p)|$ is not divisible by p^2. Therefore $K \cap SL_2(p) = K$ which implies $K \leq SL_2(p)$ and then $K \trianglelefteq SL_2(p)$ since $|K| = \frac{|SL_2(p)|}{2}$. Take Z to be the center of $SL_2(p)$. Then

$$\begin{aligned} K/(K \cap Z) &\cong KZ/Z \trianglelefteq SL_2(p)/Z \\ &\cong K. \end{aligned}$$

Again because K is simple, this implies $K/(K \cap Z) = 1$ or $|K/(K \cap Z)| = |K|$. In the former case, $K \cap Z = K$ and so $K \leq Z$ which is impossible. In the latter case, $K \cap Z = 1$ and so KZ is a subgroup of $SL_2(p)$ of order $|K| \cdot |Z| = |SL_2(p)|$, implying that $SL_2(p) = KZ$. Now we get $SL_2(p)' = (KZ)' = K' = K$. By using the well-known fact that $SL_2(q)' = SL_2(q)$ for $q > 3$ [10], we obtain $K = SL_2(p)$, a contradiction to $K \cong L_2(p)$. □

Lemma 3.3. *Suppose Γ is a semi-symmetric cubic graph of order $12p^3$, where $p > 7$ is a prime. Let $A = Aut(\Gamma)$. For $0 \leq i \leq 2$ if $|O_p(A)| = p^i$, then A does not have a normal subgroup of order $6p^i$.*

Proof. Let $\{U, W\}$ be the bipartition for Γ. Then $|U| = |W| = 6p^3$. Also if $u \in U$ is an arbitrary vertex, according to Theorem 2.5, $|A_u| = 2^r \cdot 3$ for some $0 \leq r \leq 7$. Due to transitivity of A on U, the equality $[A : A_u] = |U|$ holds which yields $|A| = 2^{r+1} \cdot 3^2 \cdot p^3$.

Let M be a normal subgroup of A of order $6p^i$ for $0 \leq i \leq 2$. Then M is intransitive on the partite sets and according to Theorem 2.7 the quotient graph Γ_M is A/M-semi-symmetric with a bipartition $\{U_M, W_M\}$. We prove that the combination $(|O_p(A)|, |M|) = (p^i, 6p^i)$ leads to contradiction.

First let $i = 0$ or 1, and suppose to the contrary, that $|O_p(A)| = p^i$ and $|M| = 6p^i$. Then $|U_M| = |W_M| = p^{3-i}$ and $|A/M| = 2^r \cdot 3 \cdot p^{3-i}$. Let K/M be a minimal normal subgroup of A/M. If K/M is non-solvable, it must be a simple group and by Corollary 2.8 its order is of the form $2^j \cdot 3 \cdot p^{3-i}$ for some j. But there is no simple K_3-group of such order since $3 - i \geq 2$. Therefore K/M is solvable and hence elementary abelian. Whether it is intransitive or transitive on the partite sets, its order must be p^k for some $1 \leq k \leq 3 - i$. Therefore $|K| = 6p^{i+k}$. The Sylow p-subgroup of K is normal in K. So it is characteristic in K and hence normal in A, contradicting the assumption that $|O_p(A)| = p^i$.

Now let $i = 2$ and suppose $|O_p(A)| = p^2$ and $|M| = 6p^2$. In this case $|U_M| = |W_M| = p$ and $|A/M| = 2^r \cdot 3 \cdot p$. Again let K/M be a minimal normal subgroup of A/M. If K/M is non-solvable, it must be a simple group of order $2^j \cdot 3 \cdot p$ for some j and $p > 7$. But there is no such simple K_3-group (Theorem 2.1). On the other hand if K/M is solvable, then by virtue of Corollary 2.8 and the fact that the power of p in $|A/M|$ is just 1, we conclude that $|K/M| = p$ and hence $|K| = 6p^3$. Now if P is a Sylow p-subgroup of K, then $P \trianglelefteq K$. So $P \trianglelefteq^c K \trianglelefteq A$ which implies $P \trianglelefteq A$ contradicting the assumption that $|O_p(A)| = p^2$. □

Proof of Theorem 3.1. For $p = 2, 3$ there is no connected cubic semi-symmetric graph of order $12p^3$ according to [3]. Also for $p = 5, 7$ the order of the graph is respectively 1500 and 4116 which are less than 10000 and we may use the recent result obtained in [2] to conclude that there is no connected cubic semi-symmetric graph of order $12p^3$. So let p be an arbitrary prime greater than 7. We show that there is no connected cubic semi-symmetric graph of order $12p^3$, by proving that the existence of such a graph leads to a contradiction. So assume Γ is a connected cubic semi-symmetric graph of order $12p^3$ with a bipartition $\{U, W\}$. Each of the two partite sets has cardinality $6p^3$ and if $A = Aut(\Gamma)$, then $|A| = 2^{r+1} \cdot 3^2 \cdot p^3$ for some $0 \leq r \leq 7$. Let $N \cong T^k$ be a minimal normal subgroup of A, where T is simple.

If T is nonabelian, then it is a simple K_3-group. According to Corollary 2.8 either $|N|$ divides $|U| = 6p^3$ or $6p^3$ divides $|N|$. In the former case $|T|$ is not divisible by 4 which is impossible as the order of every simple K_3-group, all listed in Theorem 2.1 is divisible by 4 (in general, the order of every nonabelian simple group is divisible by 4). So $6p^3$ must divide $|N|$. Since the power of 3 in $|N|$ is at most 2, k can only be 1 or 2. In both cases since p^3 divides $|N|$, we conclude that p^2 must divide $|T|$. But for $p > 3$, the square of p does not divide the order of any simple K_3-group which are all listed in Theorem 2.1.

Therefore N should be elementary abelian and hence by Corollary 2.8 we have that $|N|$ divides $6p^3$. As a result, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p^i$ for some $1 \leq i \leq 3$.

In the following, M will always denote the normal subgroup $O_p(A) \trianglelefteq A$. Since $|M| \leq p^3$, M is always intransitive on both U and W and hence according to Proposition 2.7, Γ_M is a connected cubic A/M-semi-symmetric graph with the bipartition $\{U_M, W_M\}$. There are four possibilities for $|M|$. We will show that all the possibilities result in contradiction.

Case 1. $M = 1$. In this case, the minimal normal subgroup of A is $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. The graph Γ_N is A/N-semi-symmetric with a bipartition $\{U_N, W_N\}$. Take $K/N \cong T^m$ to be a minimal normal subgroup of A/N where T is simple.

If $N \cong \mathbb{Z}_2$, then $|U_N| = |W_N| = 3p^3$ and $|A/N| = 2^r \cdot 3^2 \cdot p^3$. If K/N is non-solvable, then by Theorem 2.4 and Corollary 2.8 its order must be divisible by $3p^3$. So T is a simple K_3-group and by considering the power of 3, it follows that $m = 1$ or 2. Therefore the power of $p > 3$ in $|T|$ must be at least 2. But there is no such simple K_3-group. On the other hand if K/N is elementary abelian, then by Corollary 2.8 its order divides $3p^3$ and hence $K/N \cong \mathbb{Z}_3$ or $\mathbb{Z}_p^i$ for $1 \leq i \leq 3$. If $K/N \cong \mathbb{Z}_3$, then K is a normal subgroup of A of order 6 which is not possible according to Lemma 3.3. On the other hand if $K/N \cong \mathbb{Z}_p^i$, then K is a normal subgroup of A of order $6p^i$. Now K should have a normal Sylow p-subgroup which would also be normal in A, contradicting the assumption that $M = 1$.

Now suppose $N \cong \mathbb{Z}_3$. Then $|U_N| = |W_N| = 2p^3$ and $|A/N| = 2^{r+1} \cdot 3 \cdot p^3$. In this case if K/N is non-solvable, then $m = 1$ and $N \cong T$ is a simple K_3-group. Again since $|N|$ cannot divide $|U_N| = 2p^3$, the order of N must be divisible by $2p^3$ which is not possible for any K_3-group. So K/N is elementary abelian and its order divides $2p^3$. This subcase leads to a contradiction exactly as in the previous case where we had $N \cong \mathbb{Z}_2$.

Case 2. $|M| = p$. In this case $|U_M| = |W_M| = 6p^2$ and $|A/M| = 2^{r+1} \cdot 3^2 \cdot p^2$. Take L/M to be a minimal normal subgroup of A/M. We consider two cases of solvability and non-solvability for L/M and show that both lead to contradictions.

(a) If $L/M \cong T^m$ is non-solvable, then T would be a simple K_3-group and hence $|T|$ would be divisible by 4. Hence $|L/M|$ would not divide $6p^2$. So by Corollary 2.8 the order of L/M is divisible by $6p^2$. Now m cannot equal 1 since the order of any simple K_3-group is not divisible by the square of a prime greater than 3. Therefore $m = 2$ and since $|T|^2$ must divide $|A/M|$, it follows that the power of 3 in $|T|$ equals 1. So according to Theorem 2.1 we have $T \cong \mathbb{A}_5$ or $L_2(7)$. But this is not possible as we have assumed $p > 7$.

(b) Now assume L/M is solvable and hence elementary abelian. By Corollary 2.8 the order of L/M should divide $|U_M| = 6p^2$ and so $L/M \cong \mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p^i$ for some $1 \leq i \leq 2$. The isomorphism $L/M \cong \mathbb{Z}_p^i$ results in $|L| = p^{i+1}$ which contradicts the assumption that $|M| = p$. We discuss the other two cases. First suppose $L/M \cong \mathbb{Z}_2$. Then $|L| = 2p$. The normal subgroup $L \trianglelefteq A$ is intransitive on both U and W due to its order and so we can consider the graph Γ_L which is connected cubic A/L-semi-symmetric (Theorem 2.7) with the bipartition $\{U_L, W_L\}$, where $|U_L| = |W_L| = 3p^2$ with $|A/L| = 2^r \cdot 3^2 \cdot p^2$.

Let K/L be a minimal normal subgroup of A/L. If K/L is solvable, then it follows from Corollary 2.8 that $K/L \cong \mathbb{Z}_3$ or $\mathbb{Z}_p^j$ for $j = 1$ or 2. If $K/L \cong \mathbb{Z}_3$, then $|K| = 6p$. But according to Lemma 3.3, A cannot have a normal subgroup of order $6p$. On the other hand if $K/L \cong \mathbb{Z}_p^j$, then $|K| = 2p^{j+1}$ and a Sylow

p-subgroup of K would be normal in K and hence, also normal in A, which contradicts the assumption on $|M|$. Now if $K/L \cong T^m$ is non-solvable, then it follows from Corollary 2.8 and Theorem 2.4 that $3p^2$ divides $|K/L|$. So T is a simple $\{2, 3, p\}$-group and $m = 1$ or 2. Since the square of any prime greater than 3 does not divide the order of any simple K_3-group, m has to be 2. It follows that the power of 3 in $|T|$ is 1, and hence $T \cong \mathbb{A}_5$ or $L_2(7)$. Therefore p must be 5 or 7 which do not satisfy our assumption on p.

Case 3. $|M| = p^2$. In this case $|U_M| = |W_M| = 6p$ and $|A/M| = 2^{r+1} \cdot 3^2 \cdot p$. Again take L/M to be a minimal normal subgroup of A/M.

(a) If $L/M \cong T^m$ is non-solvable, then T is a simple K_3-group. Since the power of 3 in $|T|$ is at most 2, according to Theorem 2.1 we must have $T \cong \mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, $L_2(8)$ or $L_2(17)$. Since we have assumed $p > 7$, it follows that $T \cong L_2(17)$ and hence $p = 17$. As 3^2 divides the order of $L_2(17)$, m should equal 1 and so $L/M \cong L_2(17)$. Now 3 does not divide the order of $A/L \cong (A/M)/(L/M)$ and therefore by Proposition 2.6, Γ is L-semi-symmetric. Since L/M is nonabelian simple, M is a maximal normal subgroup of L. By Theorem 2.2 we have $C_L(M) \trianglelefteq N_L(M) = L$. Also M of order 17^2 is abelian and therefore $M \leq C_L(M) \trianglelefteq L$ from which it follows that $C_L(M) = M$ or L.

If $C_L(M) = M$, then according to Theorem 2.2, we have $L/M \leq Aut(M)$. There are two possible cases for M. Either $M \cong \mathbb{Z}_{17^2}$ or $M \cong \mathbb{Z}_{17} \times \mathbb{Z}_{17}$. In the former case $Aut(M)$ is cyclic of order $\varphi(17^2)$ and does not have a subgroup isomorphic to $L_2(17)$. In the latter case $Aut(M) \cong GL_2(17)$. But according to Lemma 3.2, $GL_2(17)$ does not have a subgroup isomorphic to $L_2(17)$ either.

On the other hand if $C_L(M) = L$, then $M \leq Z(L)$. It follows that $Z(L) = M$ or L since M is a maximal normal subgroup of L. As L is not abelian, the equality $Z(L) = L$ is not possible and hence $Z(L) = M$. Since L/M is nonabelian simple, $L'M/M = (L/M)' = L/M$ and so $L'M = L$ from which it follows that $|L| = \frac{|L'|\cdot|M|}{|L'\cap M|}$. The order of $L_2(17) \cong L/M$ and hence the order of L is divisible by 2^4. Therefore 2^4 divides $|L'| \cdot |M|$ and so divides $|L'|$. So $|L'|$ does not divide $|U| = 6 \cdot 17^3$. Consequently according to Corollary 2.8 we have $6 \cdot 17^3$ divides $|L'|$. In the rest of this paragraph we write p instead of 17. This will help better understand the discussion. Since the power of p in $|A|$ is 3, it follows from the equality $|L| = |L'| \cdot \frac{|M|}{|L'\cap M|}$ that $\frac{|M|}{|L'\cap M|}$ is not divisible by p. As M is a p-group, it follows that $L' \cap M = M$ and hence $M \leq L'$. According to Sylow theorems M is contained in a Sylow p-subgroup of L. Assume $M \leq P$ where P is a Sylow p-subgroup of L. Each element of $M = Z(L)$ commutes with every element of P. Therefore $M \leq Z(P)$ and hence $|Z(P)| \geq p^2$. We claim P is abelian. In fact if P is not abelian, then $p^2 \leq |Z(P)| < |P| \leq p^3$ from which it follows that $|P| = p^3$ and $|Z(P)| = p^2$. Now the quotient $P/Z(P)$ is of order p and so cyclic. But it is a well-known fact that for a nonabelian group G, the quotient $G/Z(G)$ cannot be cyclic. Therefore P is abelian and so according to Theorem 2.3 the order of $L' \cap Z(L) = M$ is not divisible by p, a contradiction.

(b) Now assume L/M is solvable and hence elementary abelian. By Corollary 2.8 the order of L/M should divide $|U_M| = 6p$ and so $L/M \cong \mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. Certainly $L/M \cong \mathbb{Z}_p$ results in $|L| = p^3$ which contradicts the current assumption on $|M|$. We now discuss the two cases $L/M \cong \mathbb{Z}_2$, $\mathbb{Z}_3$.

(b1) If $L/M \cong \mathbb{Z}_2$, then $|L| = 2p^2$. The normal subgroup $L \trianglelefteq A$ is intransitive on the two partite sets of Γ and so by Theorem 2.7 the graph Γ_L is A/L-semi-symmetric with the bipartition $\{U_L, W_L\}$ where $|U_L| = |W_L| = 3p$ and where $|A/L| = 2^r \cdot 3^2 \cdot p$. Let $K/L \cong T^m$ be a minimal normal subgroup of A/L. If it is solvable and hence elementary abelian, its order should divide $3p$ and so $K/L \cong \mathbb{Z}_3$ or $\mathbb{Z}_p$. Like before $K/L \cong \mathbb{Z}_p$ will contradict the assumption that $|M| = p^2$, and also $K/L \cong \mathbb{Z}_3$ is impossible according to Lemma 3.3. Now if K/L is not solvable, then T is a simple K_3-group where the power of 3 in $|T|$ is only 1 or 2. Therefore according to Theorem 2.1 we have $T \cong \mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, $L_2(8)$ or $L_2(17)$. Since $p > 7$, the only possibility will be $T \cong L_2(17)$ and hence $p = 17$. The order of $L_2(17)$ is divisible by 3^2 and hence $m = 1$. We conclude that $G = K/L \cong L_2(17)$. Now 3 does not divide the order of $(A/L)/G$ and therefore by Proposition 2.6, Γ_L is G-semi-symmetric. It follows that G is transitive on U_L with $3 \cdot 17$ points. For a vertex $u \in U_L$, the stabilizer G_u is of order $|G_u| = \frac{|L_2(17)|}{3 \cdot 17} = 2^4 \cdot 3$. For every prime power q, subgroups of $L_2(q)$ have been completely classified (see Chapter 3 of [16]). It can be verified that the group $L_2(17)$ has no subgroup of order $2^4 \cdot 3$. This shows that the assumption of non-solvability of K/L leads to a contradiction.

(b2) If $L/M \cong \mathbb{Z}_3$, then $|L| = 3p^2$. Like before the graph Γ_L is A/L-semi-symmetric with the bipartition $\{U_L, W_L\}$, where in this case $|U_L| = |W_L| = 2p$ and $|A/L| = 2^{r+1} \cdot 3 \cdot p$. Let $K/L \cong T^m$ be a minimal normal subgroup of A/L. There are two cases. If K/L is solvable and hence elementary abelian, it follows from Corollary 2.8 that $K/L \cong \mathbb{Z}_2$ or $\mathbb{Z}_p$. Again $K/L \cong \mathbb{Z}_p$ will contradict the assumption on $|M|$ and $K/L \cong \mathbb{Z}_2$ is not possible by Lemma 3.3. On the other hand if K/L is not solvable, then T is a simple K_3-group where the power of 3 in $|T|$ is only 1. So according to Theorem 2.1, $T \cong \mathbb{A}_5$ or $L_2(7)$ which are not possible since we have assumed $p > 7$.

So the case $|M| = p^2$ is impossible.

Case 4. $|M| = p^3$. In this case according to Theorem 2.7, Γ is a regular M-covering of Γ_M which is itself cubic A/M-semi-symmetric of order 12. So Γ_M is A/M-edge transitive and hence edge transitive. Now if Γ_M is not vertex transitive, then it must be semi-symmetric, but there is no semi-symmetric graph of order 12 according to Theorem 5 of [6]. On the other hand if Γ_M is vertex transitive, then it will be symmetric since according to [17] a cubic vertex and edge transitive graph is necessarily symmetric. But according to [7] there is no symmetric cubic graphs of order 12.

As every assumption on $|M|$ leads to contradictions, we conclude that there is no connected semi-symmetric cubic graph of order $12p^3$ for any prime number p. □

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References

[1] I. Z. Bouwer, *An edge but not vertex transitive cubic graph*, Canad. Math. Bull. **11** (1968), 533–535. https://doi.org/10.4153/CMB-1968-063-0

[2] M. Conder, *Summary of all semi-symmetric cubic graphs on up to 10000 vertices*, https://www.math.auckland.ac.nz/∼conder/SemisymmCubic10000.txt.

[3] M. Conder, A. Malnič, D. Marušič, and P. Potočnik, *A census of semisymmetric cubic graphs on up to 768 vertices*, J. Algebraic Combin. **23** (2006), no. 3, 255–294. https://doi.org/10.1007/s10801-006-7397-3

[4] M. R. Darafsheh and M. Shahsavaran, *Semisymmetric cubic graphs of order* $34p^3$, Bull. Korean Math. Soc. **57** (2020), no. 3, 739–750. https://doi.org/10.4134/BKMS.b190458

[5] Y.-Q. Feng, M. Ghasemi, and C. Wang, *Cubic semisymmetric graphs of order* $6p^3$, Discrete Math. **310** (2010), no. 17-18, 2345–2355. https://doi.org/10.1016/j.disc.2010.05.018

[6] J. Folkman, *Regular line-symmetric graphs*, J. Combinatorial Theory **3** (1967), 215–232.

[7] R. M. Foster, *The Foster Census*, Charles Babbage Research Centre, Winnipeg, MB, 1988.

[8] D. M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. (2) **111** (1980), no. 2, 377–406. https://doi.org/10.2307/1971203

[9] M. Herzog, *On finite simple groups of order divisible by three primes only*, J. Algebra **10** (1968), 383–388. https://doi.org/10.1016/0021-8693(68)90088-4

[10] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[11] J. H. Kwak and R. Nedela, *Graphs and their coverings*, Lecture Notes Series, **17**, 2007.

[12] Z. Lu, C. Wang, and M. Xu, *On semisymmetric cubic graphs of order* $6p^2$, Sci. China Ser. A **47** (2004), no. 1, 1–17. https://doi.org/10.1360/02ys0241

[13] A. Malnič, D. Marušič, and C. Wang, *Cubic edge-transitive graphs of order* $2p^3$, Discrete Math. **274** (2004), no. 1-3, 187–198. https://doi.org/10.1016/S0012-365X(03)00088-8

[14] J. S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.

[15] M. Shahsavaran and M. R. Darafsheh, *Classifying semisymmetric cubic graphs of order* $20p$, Turkish J. Math. **43** (2019), no. 6, 2755–2766. https://doi.org/10.3906/mat-1904-165

[16] M. Suzuki, *Group theory. II*, translated from the Japanese, Grundlehren der Mathematischen Wissenschaften, 248, Springer-Verlag, New York, 1986. https://doi.org/10.1007/978-3-642-86885-6

[17] W. T. Tutte, *Connectivity in graphs*, Mathematical Expositions, No. 15, University of Toronto Press, Toronto, ON, 1966.

[18] C. Q. Wang and T. S. Chen, *Semisymmetric cubic graphs as regular covers of* $K_{3,3}$, Acta Math. Sin. (Engl. Ser.) **24** (2008), no. 3, 405–416. https://doi.org/10.1007/s10114-007-0998-5

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