Bull. Korean Math. Soc. **59** (2022), No. 1, pp. 73–82
https://doi.org/10.4134/BKMS.b210035
pISSN: 1015-8634 / eISSN: 2234-3016

THE IMAGES OF LOCALLY FINITE $\mathcal{E}$-DERIVATIONS OF POLYNOMIAL ALGEBRAS

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Abstract. Let K be a field of characteristic zero. We first show that images of the linear derivations and the linear $\mathcal{E}$-derivations of the polynomial algebra $K[x]=K[x_1,x_2,\ldots,x_n]$ are ideals if the products of any power of eigenvalues of the matrices according to the linear derivations and the linear $\mathcal{E}$-derivations are not unity. In addition, we prove that the images of D and δ are Mathieu-Zhao spaces of the polynomial algebra $K[x]$ if $D=\sum_{i=1}^{n}(a_ix_i+b_i)\partial_i$ and $\delta=I-\phi$, $\phi(x_i)=\lambda_ix_i+\mu_i$ for $a_i,b_i,\lambda_i,\mu_i\in K$ for $1\le i\le n$. Finally, we prove that the image of an affine $\mathcal{E}$-derivation of the polynomial algebra $K[x_1,x_2]$ is a Mathieu-Zhao space of the polynomial algebra $K[x_1,x_2]$. Hence we give an affirmative answer to the LFED Conjecture for the affine $\mathcal{E}$-derivations of the polynomial algebra $K[x_1,x_2]$.

1. Introduction

Throughout this paper, we will write K for a field of characteristic zero without specific note and $K[x]=K[x_1,x_2,\ldots,x_n]$ for the polynomial algebra over K in n indeterminates. And ∂_i denotes the derivations $\frac{\partial}{\partial x_i}$ for $1\le i\le n$.

A K-linear endomorphism η of $K[x]$ is said to be locally nilpotent if for each $a\in K[x]$ there exists $m\ge 1$ such that $\eta^m(a)=0$, and locally finite if for each $a\in K[x]$ the subspace of $K[x]$ spanned by $\eta^i(a)$ $(i\ge 0)$ over K is finitely generated.

A derivation D of $K[x]$ means a K-linear map $D:K[x]\to K[x]$ that satisfies $D(ab)=D(a)b+aD(b)$ for all $a,b\in K[x]$ and $D(c)=0$ for any $c\in K$. An $\mathcal{E}$-derivation δ of $K[x]$ means a K-linear map $\delta:K[x]\to K[x]$ such that for all $a,b\in K[x]$ the following equation holds:

$$\delta(ab)=\delta(a)b+a\delta(b)-\delta(a)\delta(b).$$

Received January 11, 2021; Revised July 30, 2021; Accepted November 5, 2021.
2010 *Mathematics Subject Classification.* Primary 14R10, 13N15, 13F20.
Key words and phrases. LFED conjecture, locally finite $\mathcal{E}$-derivations, Mathieu-Zhao spaces.
The second author is supported by the NSF of China (Grant No. 11871241; 11601146), the China Scholarship Council and the Construct Program of the Key Discipline in Hunan Province.

It is easy to verify that δ is an $\mathcal{E}$-derivation of $K[x]$ if and only if $\delta = I - \phi$ for some K-algebra endomorphism ϕ of $K[x]$. D is called a linear derivation of $K[x]$ if $D(x_i)$ is a linear form for all $1 \le i \le n$.

The Mathieu-Zhao space was introduced by Zhao in [7] and [8], which is a natural generalization of ideals. We give the definition here for the polynomial algebras. A K-subspace M of $K[x]$ is said to be a Mathieu-Zhao space if for any $a, b \in K[x]$ with $a^m \in M$ for all $m \ge 1$, we have $ba^m \in M$ when $m \gg 0$. The radical of a Mathieu-Zhao space was first introduced in [8], denoted by $\mathfrak{r}(M)$, and

$$\mathfrak{r}(M) = \{a \in K[x] \,|\, a^m \in M \text{ for all } m \gg 0\}.$$

There is an equivalent definition about Mathieu-Zhao space which was proved in Proposition 2.1 of [8]. We only give the equivalent definition here for the polynomial algebras. A K-subspace M of $K[x]$ is said to be a Mathieu-Zhao space if for any $a, b \in K[x]$ with $a \in \mathfrak{r}(M)$, we have $ba^m \in M$ when $m \gg 0$.

In [13], Wenhua Zhao posed the following two conjectures:

Conjecture 1.1 (LFED). *Let K be a field of characteristic zero and $\mathcal{A}$ a K-algebra. Then for every locally finite derivation or $\mathcal{E}$-derivation δ of $\mathcal{A}$, the image* $\operatorname{Im}\delta := \delta(\mathcal{A})$ *of δ is a Mathieu-Zhao space of $\mathcal{A}$.*

Conjecture 1.2 (LNED). *Let K be a field of characteristic zero and $\mathcal{A}$ a K-algebra and δ a locally nilpotent derivation or $\mathcal{E}$-derivation of $\mathcal{A}$. Then for every ideal I of $\mathcal{A}$, the image $\delta(I)$ of I under δ is a Mathieu-Zhao space of $\mathcal{A}$.*

There are many positive answers to the above two conjectures. In [9], Wenhua Zhao proved that Conjecture 1.1 is true for polynomial algebras in one variable and Conjecture 1.2 is true for polynomial algebras in one variable for derivations and most $\mathcal{E}$-derivations. Arno van den Essen, David Wright, Wenhua Zhao showed that Conjecture 1.1 is true for derivations for polynomial algebras in two variables in [2]. In [10], Wenhua Zhao proved that Conjecture 1.1 is true for Laurent polynomial algebras in one or two variables and Conjecture 1.2 is true for all Laurent polynomial algebras. Wenhua Zhao proved the above two conjectures for algebraic algebras in [11]. In [4], Dayan Liu, Xiaosong Sun showed that Conjecture 1.1 is true for linear locally nilpotent derivations of a polynomial algebras in dimension three. They also proved that Conjecture 1.1 is true for triangular derivations and homogeneous locally nilpotent derivations of a polynomial algebras in dimension three in [6]. Arno van den Essen, Wenhua Zhao showed that Conjecture 1.1 is true for locally integral domains and $K[[x]][x^{-1}]$ in [3].

Note that we call $x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n} < x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ if $l_1 = i_1, \ldots, l_{j-1} = i_{j-1}$, $l_j < i_j$ for some $j \in \{1, 2, \ldots, n\}$. $D(x)$, $\phi(x)$ denote to $(D(x_1), D(x_2), \ldots, D(x_n))^t$ and $(\phi(x_1), \phi(x_2), \ldots, \phi(x_n))^t$, respectively.

In our paper, we prove that images of the linear derivations and the linear $\mathcal{E}$-derivations of the polynomial algebra $K[x] = K[x_1, x_2, \ldots, x_n]$ are ideals if the products of any power of eigenvalues of the matrices according to the linear derivations and the linear $\mathcal{E}$-derivations are not unity. We also prove that the images of D and δ are Mathieu-Zhao spaces of the polynomial algebra $K[x]$ if $D=\sum_{i=1}^{n}(a_ix_i+b_i)\partial_i$ and $\delta=I-\phi$, $\phi(x_i)=\lambda_ix_i+\mu_i$ for $a_i,b_i,\lambda_i,\mu_i\in K$ for $1\le i\le n$ in Section 2. In Section 3, we mainly prove the following result.

Theorem 1.3. *Let $\delta=I-\phi$ be an $\mathcal{E}$-derivation of $K[x]$. If $\phi(x_1)=\lambda_1x_1+x_2$ and $\phi(x_s)=\lambda_{s-1}x_s$ for all $2\le s\le n$, then $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x]$.*

Theorem 1.3 is Theorem 3.2 in Section 3. Similar as Lemma 3.2 in [6], we can assume that K is an algebraically closed field in our paper.

2. The positive answer to Conjecture 1.1 for some derivations and $\mathcal{E}$-derivations

Lemma 2.1. *Let $\delta=I-\phi$ be an $\mathcal{E}$-derivation of $K[x]$ and $\phi(x_i)=\lambda_ix_i+f_i(x_{i+1},\dots,x_n)$ with $f_i(x_{i+1},\dots,x_n)\in K[x_{i+1},\dots,x_n]$ and $f_n\in K$ for $1\le i\le n-1$. If $\lambda_i\neq 1$ for all $1\le i\le n$, then there exists $\sigma\in\operatorname{Aut}(K[x])$ such that $\sigma^{-1}\delta\sigma=I-\tilde{\phi}$ and $\tilde{\phi}(x_i)=\lambda_ix_i+\tilde{f}_i(x_{i+1},\dots,x_n)$, where $\tilde{f}_i(0)=0$ for $1\le i\le n$.*

Proof. Let $\tilde{\delta}=\sigma^{-1}\delta\sigma$ and $\sigma(x_i)=x_i+c_i$ for $1\le i\le n$, where $c_i=(\lambda_i-1)^{-1}f_i(-c_{i+1},\dots,-c_n)$ and $c_n=(\lambda_n-1)^{-1}f_n$ for $1\le i\le n-1$. Then $\tilde{\delta}=\sigma^{-1}\delta\sigma=I-\tilde{\phi}$ and

$$\tilde{\phi}(x_i)=\lambda_ix_i+(1-\lambda_i)c_i+f_i(x_{i+1}-c_{i+1},\dots,x_n-c_n)$$

for $1\le i\le n$. Let $\tilde{f}_i(x_{i+1},\dots,x_n)=f_i(x_{i+1}-c_{i+1},\dots,x_n-c_n)+(1-\lambda_i)c_i$ for $1\le i\le n$. Then the conclusion follows. $\square$

Theorem 2.2. *Let $\delta=I-\phi$ be an $\mathcal{E}$-derivation of $K[x]$ and $\phi(x_i)=\lambda_ix_i+f_i(x_{i+1},\dots,x_n)$, where $f_i\in K[x_{i+1},\dots,x_n]$ and $\lambda_i\in K$ for all $1\le i\le n$. If $\lambda_1^{i_1}\cdots\lambda_n^{i_n}\neq 1$ for all $i_1,\dots,i_n\in\mathbb{N}$, $i_1+\cdots+i_n\ge 1$, then $\operatorname{Im}\delta$ is an ideal generated by $x_1,x_2,\dots,x_n$. In particular, if $\lambda_1=\lambda_2=\cdots=\lambda_n:=\tilde{\lambda}$, then $\operatorname{Im}\delta$ is an ideal generated by $x_1,x_2,\dots,x_n$ in the case that $\tilde{\lambda}$ is not a root of unity.*

Proof. It follows from Lemma 2.1 that we can assume that $f_i(0)=0$ for all $1\le i\le n$. Thus, we have $f_n=0$. We proceed by induction according to the lexicographical order $x_1>x_2>\cdots>x_n$ on $K[x]$. Since $\phi(x_n)=\lambda_nx_n$, we have $\delta(x_n^{i_n})=(1-\lambda_n^{i_n})x_n^{i_n}$. Note that $\lambda_n^{i_n}\neq 1$. We have $x_n^{i_n}\in\operatorname{Im}\delta$ for all $i_n\in\mathbb{N}^*$. Suppose that $x_k^{l_k}x_{k+1}^{l_{k+1}}\cdots x_{n-1}^{l_{n-1}}x_n^{l_n}\in\operatorname{Im}\delta$ for all $x_k^{l_k}x_{k+1}^{l_{k+1}}\cdots x_n^{l_n}<x_k^{i_k}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n}$. Then we have

$$\begin{aligned}\delta(x_k^{i_k}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})&=x_k^{i_k}\cdots x_n^{i_n}-(\lambda_kx_k+f_k(x_{k+1},\dots,x_n))^{i_k}\cdots(\lambda_nx_n)^{i_n}\\&=(1-\lambda_k^{i_k}\cdots\lambda_n^{i_n})x_k^{i_k}\cdots x_n^{i_n}+\tilde{Q}.\end{aligned}$$

By induction hypothesis, we have $\tilde{Q} \in \operatorname{Im}\delta$. Since $\lambda_k^{i_k} \cdots \lambda_n^{i_n} \neq 1$ for any $k \in \{1, 2, \ldots, n\}$, we have $x_k^{i_k} \cdots x_n^{i_n} \in \operatorname{Im}\delta$ for all $i_k, \ldots, i_n \in \mathbb{N}$, $i_k + \cdots + i_n \geq 1$, $1 \leq k \leq n$. Since $1 \notin \operatorname{Im}\delta$, we have that $\operatorname{Im}\delta$ is the ideal generated by $x_1, x_2, \ldots, x_n$. □

Corollary 2.3. *Let $\delta = I - \phi$ be an $\mathcal{E}$-derivation of $K[x]$ and $\phi = Ax$ is a linear polynomial homomorphism of $K[x]$ with $A \in \mathrm{M}_n(K)$. If $\lambda_{11}^{i_1} \cdots \lambda_{nn}^{i_n} \neq 1$ for all $i_1, \ldots, i_n \in \mathbb{N}$, $i_1 + \cdots + i_n \geq 1$, where $\lambda_{11}, \ldots, \lambda_{nn}$ are the eigenvalues of A, then $\operatorname{Im}\delta$ is an ideal of $K[x]$. In particular, if $\lambda_{11} = \cdots = \lambda_{nn} := \lambda$, then $\operatorname{Im}\delta$ is an ideal of $K[x]$ in the case that λ is not a root of unity.*

Proof. Since $\phi = Ax$, there exists $T \in \mathrm{GL}_n(K)$ such that

$$T^{-1}AT = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{nn} \end{pmatrix}.$$

Let $\sigma(x) = Tx$. Then we have $\sigma^{-1}\delta\sigma = I - \sigma^{-1}\phi\sigma$. It suffices to prove that $\operatorname{Im}(\sigma^{-1}\delta\sigma)$ is an ideal of $K[x]$. Let $\tilde{\delta} = \sigma^{-1}\delta\sigma = I - \tilde{\phi}$. Then $\tilde{\phi}(x_i) = \sum_{j=i}^n \lambda_{ij}x_j$ for $1 \leq i \leq n$. Thus, the conclusion follows from Theorem 2.2. □

Proposition 2.4. *Let $\delta = I - \phi$ be an $\mathcal{E}$-derivation of $K[x]$ and $\phi(x_i) = \lambda_i x_i + \mu_i$, where $\lambda_i, \mu_i \in K$ for all $1 \leq i \leq n$. Then $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x]$.*

Proof. If $\lambda_i \neq 1$ for some $i \in \{1, 2, \ldots, n\}$, then we have $\sigma_i^{-1}\phi\sigma_i(x_i) = \lambda_i x_i$ and $\sigma_i^{-1}\phi\sigma_i(x_j) = \lambda_j x_j + \mu_j$, where $\sigma_i(x_i) = x_i + (\lambda_i - 1)^{-1}\mu_i$, $\sigma_i(x_j) = x_j$ for $j \neq i$ for all $1 \leq j \leq n$.

If $\lambda_i = 1$, then $\delta(x_i) = -\mu_i$. If $\mu_i \neq 0$, then $1 \in \operatorname{Im}\delta$. It's easy to check that δ is locally finite, it follows from Proposition 1.4 in [12] that $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x]$. If $\mu_i = 0$, then $\phi(x_i) = \lambda_i x_i$. We assume that $\sigma_i = I$ in this case. Let $\sigma = \sigma_n \circ \cdots \circ \sigma_1 \in \operatorname{Aut}(K[x])$. Then $\sigma^{-1}\delta\sigma = I - \tilde{\phi}$, where $\tilde{\phi}(x_i) = \lambda_i x_i$ for all $1 \leq i \leq n$ or $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x]$. Let $\tilde{\delta} = \sigma^{-1}\delta\sigma$. It follows from Lemma 3.2 and Corollary 3.3 in [1] that $\operatorname{Im}\tilde{\delta}$ is a Mathieu-Zhao space of $K[x]$. Thus, $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x]$. □

Corollary 2.5. *Let $\delta = I - \phi$ be an $\mathcal{E}$-derivation of the polynomial algebra $K[x_1, x_2, x_3]$, where $\phi(x_1) = x_1 + f_1(x_2, x_3)$, $\phi(x_2) = x_2 + f_2(x_3)$, $\phi(x_3) = x_3 + f_3$ with $f_1(x_2, x_3) \in K[x_2, x_3]$, $f_2(x_3) \in K[x_3]$, $f_3 \in K$. Then $\operatorname{Im}\delta$ is a Mathieu-Zhao space of the polynomial algebra $K[x_1, x_2, x_3]$.*

Proof. Since δ is triangular, it follows from Theorem 2.1 and Corollary 2.4 in [12] that there exists a triangular derivation D such that $\operatorname{Im}\delta = \operatorname{Im}D$. It follows from Corollary 3.10 in [6] that $\operatorname{Im}D$ is a Mathieu-Zhao space of $K[x_1, x_2, x_3]$. Thus, $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x_1, x_2, x_3]$. □

Proposition 2.6. *Let $D = \sum_{i=1}^{n}(a_i x_i + b_i)\partial_i$ be a derivation of $K[x]$ with $a_i, b_i \in K$ for all $1 \leq i \leq n$. Then $\operatorname{Im} D$ is a Mathieu-Zhao space of $K[x]$.*

Proof. If $a_i \neq 0$ for some $i \in \{1, 2, \ldots, n\}$, then we have

$$\sigma_i^{-1} D \sigma_i = a_i x_i \partial_i + \sum_{\substack{1 \leq j \leq n \\ j \neq i}} (a_j x_j + b_j)\partial_j,$$

where $\sigma_i(x_i) = a_i x_i + b_i$, $\sigma_i(x_j) = x_j$ for $j \neq i$ for all $1 \leq j \leq n$.

If $a_i = 0$, then $D(x_i) = b_i$. If $b_i \neq 0$, then $1 \in \operatorname{Im} D$. It follows from Example 9.3.2 in [5] that D is locally finite. Thus, it follows from Proposition 1.4 in [12] that $\operatorname{Im} D$ is a Mathieu-Zhao space of $K[x]$. If $b_i = 0$, then

$$D = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} (a_j x_j + b_j)\partial_j.$$

Hence we have that $\operatorname{Im} D$ is a Mathieu-Zhao space of $K[x]$ or there exists $\sigma \in \operatorname{Aut}(K[x])$ such that $\sigma^{-1} D \sigma = \sum_{j=1}^{n} a_j x_j \partial_j$. It follows from Lemma 3.4 in [2] that $\operatorname{Im}(\sigma^{-1} D \sigma)$ is a Mathieu-Zhao space of $K[x]$. Thus, $\operatorname{Im} D$ is a Mathieu-Zhao space of $K[x]$. $\square$

Proposition 2.7. *Let $D = \sum_{i=1}^{n}(a_i x_i + b_i(x_{i+1}, \ldots, x_n))\partial_i$ be a derivation of $K[x]$ with $a_i \in K$, $b_i \in K[x_{i+1}, \ldots, x_n]$ for all $1 \leq i \leq n$. If $a_i \neq 0$ for some $i \in \{1, 2, \ldots, n\}$, then there exists $\sigma \in \operatorname{Aut}(K[x])$ such that $\sigma^{-1} D \sigma = \sum_{i=1}^{n}(a_i x_i + \tilde{b}_i(x_{i+1}, \ldots, x_n))\partial_i$ and $\tilde{b}_i(0, \ldots, 0) = 0$ for some $i \in \{1, 2, \ldots, n\}$.*

Proof. Since $a_i \neq 0$, we have

$$\sigma^{-1} D \sigma(x_i) = a_i x_i + a_i(b_i(x_{i+1}, \ldots, x_n) - c_i),$$

where $\sigma(x_i) = a_i x_i + c_i$, $\sigma(x_j) = x_j$ for $j \neq i$ for all $1 \leq j \leq n$, $c_i \in K$. Let $c_i = b_i(0, \ldots, 0)$. Then $\sigma^{-1} D \sigma(x_i) = a_i x_i + \tilde{b}_i(x_{i+1}, \ldots, x_n)$ and $\tilde{b}_i(x_{i+1}, \ldots, x_n) = a_i(b_i(x_{i+1}, \ldots, x_n) - b_i(0, \ldots, 0))$. Thus, the conclusion follows. $\square$

Proposition 2.8. *Let $D = \sum_{i=1}^{n}(a_i x_i + b_i(x_{i+1}, \ldots, x_n))\partial_i$ be a derivation of $K[x]$ with $a_i \in K$, $b_i \in K[x_{i+1}, \ldots, x_n]$ for all $1 \leq i \leq n$ and S the set of positive integral solutions of the linear equation $\sum_{i=1}^{n} a_i y_i = 0$. If $S = \emptyset$, then $\operatorname{Im} D$ is an ideal generated by $x_1, \ldots, x_n$. In particular, if $a_1 = a_2 = \cdots = a_n := a \neq 0$, then $\operatorname{Im} D$ is an ideal generated by $x_1, \ldots, x_n$.*

Proof. Since $S = \emptyset$, we have $a_1 a_2 \cdots a_n \neq 0$. It follows from Proposition 2.7 that we can assume that $b_1(0, \ldots, 0) = b_2(0, \ldots, 0) = \cdots = b_n = 0$. We proceed by induction according to the lexicographical order $x_1 > x_2 > \cdots > x_n$ on $K[x]$. Since $x_n = D(a_n^{-1} x_n)$, we have $x_n \in \operatorname{Im} D$. Suppose that $x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} \in \operatorname{Im} D$ for all $x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} < x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. Then we have

$$D(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}) = (i_1 a_1 + i_2 a_2 + \cdots + i_n a_n) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + Q.$$

By induction hypothesis, we have $Q \in \operatorname{Im} D$. Hence we have $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in \operatorname{Im} D$ for all $i_1 + i_2 + \cdots + i_n \geq 1$. Note that $1 \notin \operatorname{Im} D$. Then the conclusion follows. $\square$

Corollary 2.9. *Let D be a linear derivation of $K[x]$ and $D(x) = Bx$ with $B \in M_n(K)$ and S the set of positive integral solutions of the linear equation $\sum_{i=1}^n \mu_{ii} y_i = 0$, where $\mu_{11}, \ldots, \mu_{nn}$ are the eigenvalues of B. If $S = \emptyset$, then $\operatorname{Im} D$ is an ideal of $K[x]$. In particular, if $\mu_{11} = \mu_{22} = \cdots = \mu_{nn} := \mu \neq 0$, then $\operatorname{Im} D$ is an ideal of $K[x]$.*

Proof. Since $D(x) = Bx$, there exists $\tilde{T} \in \mathrm{GL}_n(K)$ such that

$$\tilde{T}^{-1} B \tilde{T} = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ 0 & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{nn} \end{pmatrix} := B_\mu.$$

Let $\tilde{\sigma}(x) = \tilde{T}x$. Then $\tilde{\sigma}^{-1} D \tilde{\sigma}(x) = B_\mu x$. Thus, the conclusion follows from Proposition 2.8. $\square$

3. The positive answer to Conjecture 1.1 for $\mathcal{E}$-derivations with ϕ affine polynomial homomorphisms

Lemma 3.1. *Let $\delta = I - \phi$ be an $\mathcal{E}$-derivation of $K[x]$. If $\phi(x_{2i-1}) = \lambda_i x_{2i-1} + x_{2i}$, $\phi(x_{2i}) = \lambda_i x_{2i}$ for all $1 \leq i \leq t$ and $\phi(x_s) = \lambda_{s-t} x_s$ for all $2t+1 \leq s \leq n$, where $1 \leq t \leq [\frac{n}{2}]$, $t \in \mathbb{N}^*$, then $x_2^{i_2} x_4^{i_4} \cdots x_{2t-2}^{i_{2t-2}} x_{2t-1}^{i_{2t-1}} x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n} \in \operatorname{Im} \delta$ for all $i_{2t} \geq 1$.*

Proof. Note that

$$\begin{aligned} &\delta(x_2^{i_2} x_4^{i_4} \cdots x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n}) \\ &= (1 - \lambda_1^{i_2} \lambda_2^{i_4} \cdots \lambda_t^{i_{2t}} \lambda_{t+1}^{i_{2t+1}} \cdots \lambda_{n-t}^{i_n}) x_2^{i_2} x_4^{i_4} \cdots x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n}. \end{aligned}$$

If $\lambda_1^{i_2} \lambda_2^{i_4} \cdots \lambda_t^{i_{2t}} \lambda_{t+1}^{i_{2t+1}} \cdots \lambda_{n-t}^{i_n} \neq 1$ and $i_2 + i_4 + \cdots + i_{2t} \geq 1$, then

$$x_2^{i_2} x_4^{i_4} \cdots x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n} \in \operatorname{Im} \delta.$$

If $\lambda_1^{i_2} \lambda_2^{i_4} \cdots \lambda_t^{i_{2t}} \lambda_{t+1}^{i_{2t+1}} \cdots \lambda_{n-t}^{i_n} = 1$ and $i_2 + i_4 + \cdots + i_{2t} \geq 1$, then, without loss of generality, we can assume $i_2 \geq 1$. Thus, we have

$$\begin{aligned} &\delta(x_1 x_2^{i_2-1} x_4^{i_4} \cdots x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n}) \\ &= -\lambda_1^{i_2-1} \lambda_2^{i_4} \cdots \lambda_t^{i_{2t}} \lambda_{t+1}^{i_{2t+1}} \cdots \lambda_{n-t}^{i_n} x_2^{i_2} x_4^{i_4} \cdots x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n}. \end{aligned}$$

Hence we have $x_2^{i_2} x_4^{i_4} \cdots x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n} \in \operatorname{Im} \delta$, whence $x_2^{i_2} x_4^{i_4} \cdots x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n} \in \operatorname{Im} \delta$ for all $i_2 + \cdots + i_{2t} \geq 1$.

Suppose that $x_2^{i_2} x_4^{i_4} \cdots x_{2t-2}^{i_{2t-2}} x_{2t-1}^{l_{2t-1}} x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n} \in \operatorname{Im} \delta$ for $l_{2t-1} < i_{2t-1}$ and $i_{2t} \geq 1$. Then we have

$$\delta(x_2^{i_2} x_4^{i_4} \cdots x_{2t-2}^{i_{2t-2}} x_{2t-1}^{i_{2t-1}} x_{2t}^{i_{2t}} x_{2t+1}^{i_{2t+1}} \cdots x_n^{i_n})$$

$$= (1-\lambda_1^{i_2}\lambda_2^{i_4}\cdots\lambda_{t-1}^{i_{2t-2}}\lambda_t^{i_{2t-1}+i_{2t}}\lambda_{t+1}^{i_{2t+1}}\cdots\lambda_{n-t}^{i_n})x_2^{i_2}x_4^{i_4}\cdots x_{2t-2}^{i_{2t-2}}x_{2t-1}^{i_{2t-1}}x_{2t}^{i_{2t}}x_{2t+1}^{i_{2t+1}}\cdots x_n^{i_n}+Q_1(x_1,\ldots,x_n).$$

By induction hypothesis, every monomial of $Q_1(x_1,\ldots,x_n)$ is in $\operatorname{Im}\delta$. If $\lambda_1^{i_2}\lambda_2^{i_4}\cdots\lambda_{t-1}^{i_{2t-2}}\lambda_t^{i_{2t-1}+i_{2t}}\lambda_{t+1}^{i_{2t+1}}\cdots\lambda_{n-t}^{i_n}\neq 1$, then

$$x_2^{i_2}x_4^{i_4}\cdots x_{2t-2}^{i_{2t-2}}x_{2t-1}^{i_{2t-1}}x_{2t}^{i_{2t}}x_{2t+1}^{i_{2t+1}}\cdots x_n^{i_n}\in\operatorname{Im}\delta.$$

If $\lambda_1^{i_2}\lambda_2^{i_4}\cdots\lambda_{t-1}^{i_{2t-2}}\lambda_t^{i_{2t-1}+i_{2t}}\lambda_{t+1}^{i_{2t+1}}\cdots\lambda_{n-t}^{i_n}=1$, then we have

$$\begin{aligned}
&\delta(x_2^{i_2}x_4^{i_4}\cdots x_{2t-2}^{i_{2t-2}}x_{2t-1}^{i_{2t-1}+1}x_{2t}^{i_{2t}-1}x_{2t+1}^{i_{2t+1}}\cdots x_n^{i_n})\\
&= x_2^{i_2}x_4^{i_4}\cdots x_{2t-2}^{i_{2t-2}}x_{2t-1}^{i_{2t-1}}x_{2t}^{i_{2t}}x_{2t+1}^{i_{2t+1}}\cdots x_n^{i_n}-(\lambda_1x_2)^{i_2}(\lambda_2x_4)^{i_4}\\
&\quad\cdots(\lambda_{t-1}x_{2t-2})^{i_{2t-2}}(\lambda_tx_{2t-1}+x_{2t})^{i_{2t-1}+1}(\lambda_tx_{2t})^{i_{2t}-1}\\
&\quad\cdot(\lambda_{t+1}x_{2t+1})^{i_{2t+1}}\cdots(\lambda_{n-t}x_n)^{i_n}\\
&= -(i_{2t-1}+1)\lambda_1^{i_2}\lambda_2^{i_4}\cdots\lambda_{t-1}^{i_{2t-2}}\lambda_t^{i_{2t-1}+i_{2t}-1}\lambda_{t+1}^{i_{2t+1}}\cdots\lambda_{n-t}^{i_n}x_2^{i_2}x_4^{i_4}\\
&\quad\cdots x_{2t-2}^{i_{2t-2}}x_{2t-1}^{i_{2t-1}}x_{2t}^{i_{2t}}x_{2t+1}^{i_{2t+1}}\cdots x_n^{i_n}+Q_2(x_1,\ldots,x_n).
\end{aligned}$$

By induction hypothesis, every monomial of $Q_2(x_1,\ldots,x_n)$ is in $\operatorname{Im}\delta$. Thus, we have $x_2^{i_2}x_4^{i_4}\cdots x_{2t-2}^{i_{2t-2}}x_{2t-1}^{i_{2t-1}}x_{2t}^{i_{2t}}x_{2t+1}^{i_{2t+1}}\cdots x_n^{i_n}\in\operatorname{Im}\delta$ for all $i_{2t}\geq 1$. □

Theorem 3.2. *Let $\delta = I-\phi$ be an $\mathcal{E}$-derivation of $K[x]$. If $\phi(x_1)=\lambda_1x_1+x_2$ and $\phi(x_s)=\lambda_{s-1}x_s$ for all $2\leq s\leq n$, then $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x]$.*

Proof. It follows from Lemma 3.1 that $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\in\operatorname{Im}\delta$ for all $i_2\geq 1$. Hence the ideal I generated by x_2 is contained in $\operatorname{Im}\delta$.

Note that

$$\begin{aligned}
\delta(x_1^{i_1}x_3^{i_3}\cdots x_n^{i_n}) &= (1-\lambda_1^{i_1}\lambda_2^{i_3}\cdots\lambda_{n-1}^{i_n})x_1^{i_1}x_3^{i_3}\cdots x_{2t+1}^{i_{2t+1}}x_{2t+2}^{i_{2t+2}}\cdots x_n^{i_n}\bmod I\\
&= \hat{\delta}(x_1^{i_1}x_3^{i_3}\cdots x_n^{i_n})
\end{aligned}$$

for all $i_1,\ i_3,\ldots,i_n\in\mathbb{N}$, where $\hat{\delta}=I-\hat{\phi}$ is an $\mathcal{E}$-derivation of the polynomial algebra $K[x_1,x_3,\ldots,x_n]$ and $\hat{\phi}(x_1)=\lambda_1x_1$ and $\hat{\phi}(x_s)=\lambda_{s-1}x_s$ for all $2\leq s\leq n$. Thus, we have $\operatorname{Im}\delta/I=\operatorname{Im}\hat{\delta}$. It follows from Lemma 3.2 and Corollary 3.3 in [1] that $\operatorname{Im}\hat{\delta}$ is a Mathieu-Zhao space of the polynomial algebra $K[x_1,x_3,\ldots,x_n]$. Then it follows from Proposition 2.7 in [8] that $\operatorname{Im}\delta$ is a Mathieu-Zhao space of $K[x]$. □

Proposition 3.3. *Let $\delta=I-\phi$ be an $\mathcal{E}$-derivation of the polynomial algebra $K[x_1,x_2]$. If ϕ is a linear polynomial homomorphism of the polynomial algebra $K[x_1,x_2]$, then $\operatorname{Im}\delta$ is a Mathieu-Zhao space of the polynomial algebra $K[x_1,x_2]$.*

Proof. Since ϕ is a linear polynomial homomorphism, we have that

$$\begin{pmatrix}\phi(x_1)\\ \phi(x_2)\end{pmatrix} = A\begin{pmatrix}x_1\\ x_2\end{pmatrix},$$

where $A \in M_2(K)$. Hence there exists $T \in \mathrm{GL}_2(K)$ such that

$$T^{-1}AT = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \text{ or } \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix},$$

where $\lambda_1 \neq \lambda_2$. Let $(\sigma(x_1), \sigma(x_2))^t = T(x_1, x_2)^t$. Then we have $\sigma^{-1}\delta\sigma = I - \sigma^{-1}\phi\sigma$. It suffices to prove that $\mathrm{Im}(\sigma^{-1}\delta\sigma)$ is a Mathieu-Zhao space of $K[x_1, x_2]$. Let $\tilde{\delta} = \sigma^{-1}\delta\sigma = I - \tilde{\phi}$. Then $\tilde{\phi}(x_1) = \lambda_1 x_1$, $\tilde{\phi}(x_2) = \lambda_2 x_2$ or $\tilde{\phi}(x_1) = \lambda x_1 + x_2$, $\tilde{\phi}(x_2) = \lambda x_2$.

(1) If $\tilde{\phi}(x_1) = \lambda_1 x_1$, $\tilde{\phi}(x_2) = \lambda_2 x_2$, then it follows from Lemma 3.2 and Corollary 3.3 in [1] that $\mathrm{Im}\,\tilde{\delta}$ is a Mathieu-Zhao space of $K[x_1, x_2]$.

(2) If $\tilde{\phi}(x_1) = \lambda x_1 + x_2$, $\tilde{\phi}(x_2) = \lambda x_2$, then it follows from Theorem 3.2 that $\mathrm{Im}\,\tilde{\delta}$ is a Mathieu-Zhao space of $K[x_1, x_2]$. Then the conclusion follows. □

Corollary 3.4. *Let $\delta = I - \phi$ be an $\mathcal{E}$-derivation of the polynomial algebra $K[x_1, x_2]$. If $\phi(x_1) = \lambda x_1 + x_2$, $\phi(x_2) = \lambda x_2$, then $\mathrm{Im}\,\delta$ is an ideal or $\mathfrak{r}(\mathrm{Im}\,\delta)$ is an ideal of the polynomial algebra $K[x_1, x_2]$.*

Proof. (1) If λ is not a root of unity, then it follows from Corollary 2.3 that $\mathrm{Im}\,\delta$ is an ideal of $K[x_1, x_2]$.

(2) If λ is a root of unity, then it follows from the proof of Theorem 3.2 that $x_1^{i_1}x_2^{i_2} \in \mathrm{Im}\,\delta$ for all $i_1 \in \mathbb{N}$, $i_2 \in \mathbb{N}^*$ and $x_1^{i_1} \in \mathrm{Im}\,\delta$ for all $i_1 \neq ds$, $d \in \mathbb{N}$, where s is the least positive integer such that $\lambda^s = 1$. That is, $x_1^{ds} \notin \mathrm{Im}\,\delta$ for all $d \in \mathbb{N}$. Next we prove that $\mathfrak{r}(\mathrm{Im}\,\delta)$ is the ideal generated by x_2. Clearly, the ideal generated by x_2 is contained in $\mathfrak{r}(\mathrm{Im}\,\delta)$. Let $G(x_1, x_2) = x_2G_1(x_1, x_2) + G_2(x_1) \in \mathfrak{r}(\mathrm{Im}\,\delta)$ and $G_2(x_1) \in K[x_1]$. We claim that $G_2(x_1) = 0$. Otherwise, we have $G^m \in \mathrm{Im}\,\delta$ for all $m \gg 0$. Thus, we have $G_2^m \in \mathrm{Im}\,\delta$ for all $m \gg 0$. In particular, $G_2^{ds} \in \mathrm{Im}\,\delta$ for all $d \gg 0$. Suppose that $x_1^{\hat{t}}$ is the leading monomial of $G_2(x_1)$. Since $\mathrm{Im}\,\delta$ is a homogeneous K-subspace of $K[x_1, x_2]$, we have $x_1^{\hat{t}ds} \in \mathrm{Im}\,\delta$ for all $d \gg 0$, which is a contradiction. Thus, we have $G_2(x_1) = 0$. Therefore, G belongs to the ideal generated by x_2. Then the conclusion follows. □

Proposition 3.5. *Let $\delta = I - \phi$ be an $\mathcal{E}$-derivation of $K[x_1, x_2]$. If ϕ is an affine polynomial homomorphism of $K[x_1, x_2]$, then $\mathrm{Im}\,\delta$ is a Mathieu-Zhao space of $K[x_1, x_2]$.*

Proof. Since ϕ is an affine polynomial homomorphism, we have that

$$\begin{pmatrix}\phi(x_1)\\ \phi(x_2)\end{pmatrix} = A\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}c_1\\ c_2\end{pmatrix},$$

where $A \in M_2(K)$ and $(c_1, c_2)^t \in K^2$. Hence there exists $T \in \mathrm{GL}_2(K)$ such that

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where $\lambda_1 \neq \lambda_2$. Let $(\sigma(x_1), \sigma(x_2))^t = T(x_1, x_2)^t$. Then we have $\sigma^{-1}\delta\sigma = I - \sigma^{-1}\phi\sigma$. It suffices to prove that $\mathrm{Im}(\sigma^{-1}\delta\sigma)$ is a Mathieu-Zhao space of $K[x_1, x_2]$. Let $\tilde{\delta} = \sigma^{-1}\delta\sigma = I - \tilde{\phi}$. Then $\tilde{\phi}(x_1) = \lambda_1 x_1 + \mu_1$, $\tilde{\phi}(x_2) = \lambda_2 x_2 + \mu_2$ or $\tilde{\phi}(x_1) = \lambda x_1 + x_2 + \mu_1$, $\tilde{\phi}(x_2) = \lambda x_2 + \mu_2$, where $(\mu_1, \mu_2)^t = T(c_1, c_2)^t$.

(1) If $\lambda_1 \neq 1$, $\lambda_2 \neq 1$ and $\lambda \neq 1$, then it follows from Lemma 2.1 that there exists $\sigma \in \mathrm{Aut}(K[x_1, x_2])$ such that $\sigma^{-1}\tilde{\delta}\sigma = I - \bar{\phi}$, where $\bar{\phi}$ is a linear polynomial homomorphism. Then it follows from Proposition 3.3 that $\mathrm{Im}(\sigma^{-1}\tilde{\delta}\sigma)$ is a Mathieu-Zhao space of $K[x_1, x_2]$. Since σ is a polynomial automorphism, we have that $\mathrm{Im}\,\delta$ is a Mathieu-Zhao space of $K[x_1, x_2]$.

(2) If $\lambda_1 = 1$, then $\tilde{\phi}(x_1) = x_1 + \mu_1$, $\tilde{\phi}(x_2) = \lambda_2 x_2 + \mu_2$. Thus, we have $\tilde{\delta}(x_1) = -\mu_1$. If $\mu_1 \neq 0$, then $1 \in \mathrm{Im}\,\tilde{\delta}$. It's easy to check that $\tilde{\delta}$ is locally finite. It follows from Proposition 1.4 in [12] that $\mathrm{Im}\,\tilde{\delta}$ is a Mathieu-Zhao space of $K[x_1, x_2]$. If $\mu_1 = 0$, then $\tilde{\delta}(x_1^{i_1}) = 0$ for all $i_1 \in \mathbb{N}$. Since $\lambda_2 \neq \lambda_1$, there exists $\tau \in \mathrm{Aut}(K[x_1, x_2])$ such that $\hat{\delta} := \tau^{-1}\tilde{\delta}\tau = I - \hat{\phi}$, where $\hat{\phi}(x_1) = x_1$, $\hat{\phi}(x_2) = \lambda_2 x_2$. Then it follows from Proposition 3.3 that $\mathrm{Im}\,\tilde{\delta}$ is a Mathieu-Zhao space of $K[x_1, x_2]$. Thus, $\mathrm{Im}\,\tilde{\delta}$ is a Mathieu-Zhao space of $K[x_1, x_2]$.

(3) If $\lambda_2 = 1$, then we have that $\mathrm{Im}\,\tilde{\delta}$ is a Mathieu-Zhao space of $K[x_1, x_2]$ by following the arguments of Proposition 3.5(2).

(4) If $\lambda = 1$, then $\tilde{\phi}(x_1) = x_1 + x_2 + \mu_1$, $\tilde{\phi}(x_2) = x_2 + \mu_2$. Thus, we have $\tilde{\delta}(x_2) = -\mu_2$. If $\mu_2 \neq 0$, then $1 \in \mathrm{Im}\,\tilde{\delta}$. Since $\tilde{\delta}$ is locally finite, it follows from Proposition 1.4 in [12] that $\mathrm{Im}\,\tilde{\delta}$ is a Mathieu-Zhao space of $K[x_1, x_2]$. If $\mu_2 = 0$, then $\tilde{\delta}(x_2^{i_2}) = 0$ for all $i_2 \in \mathbb{N}$. Thus, we have

$$\tilde{\delta}(x_1^{i_1} x_2^{i_2}) = -(x_2 + \mu_1)\left(\sum_{j=0}^{i_1-1} x_1^{i_1-j-1}(x_1 + x_2 + \mu_1)^j\right) x_2^{i_2}$$

for $i_1 \in \mathbb{N}^*$, $i_2 \in \mathbb{N}$. It's easy to check that $(x_2 + \mu_1)x_1^{i_1}x_2^{i_2} \in \mathrm{Im}\,\tilde{\delta}$ for all $i_1, i_2 \in \mathbb{N}$. Since $1 \notin \mathrm{Im}\,\tilde{\delta}$, we have that $\mathrm{Im}\,\tilde{\delta}$ is the ideal generated by $x_2 + \mu_1$. Then the conclusion follows. $\square$

Acknowledgement. The second author is very grateful to professor Wenhua Zhao for personal communications about the Mathieu-Zhao spaces. She is also grateful to the Department of Mathematics of Illinois State University, where this paper was partially finished, for hospitality during her stay as a visiting scholar. The authors are very grateful to the referee for some useful suggestions.

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