

## LEGENDRE TRAJECTORIES OF TRANS- $S$ -MANIFOLDS

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ABSTRACT. In this paper, we consider Legendre trajectories of trans- $S$ -manifolds. We obtain curvature characterizations of these curves and give a classification theorem. We also investigate Legendre curves whose Frenet frame fields are linearly dependent with certain combination of characteristic vector fields of the trans- $S$ -manifold.

### 1. Introduction

Let  $(M, g)$  be a Riemannian manifold,  $F$  a closed 2-form and let us denote the Lorentz force on  $M$  by  $\Phi$ , which is a  $(1, 1)$ -type tensor field. If  $F$  is associated by the relation

$$(1) \quad g(\Phi X, Y) = F(X, Y), \quad \forall X, Y \in \chi(M),$$

then it is called a *magnetic field* ([1], [3] and [7]). Let  $\nabla$  be the Riemannian connection associated to the Riemannian metric  $g$  and  $\gamma : I \rightarrow M$  a smooth curve. If  $\gamma$  satisfies the Lorentz equation

$$(2) \quad \nabla_{\gamma'(t)} \gamma'(t) = \Phi(\gamma'(t)),$$

then it is called a *magnetic curve* for the magnetic field  $F$ . The Lorentz equation is a generalization of the equation for geodesics. Magnetic curves have constant speed. If the speed of the magnetic curve  $\gamma$  is equal to 1, then it is called a *normal magnetic curve* [8]. For extensive information about almost contact metric manifolds and Sasakian manifolds, we refer to Blair's book [4].

Let  $\gamma(t)$  be a Frenet curve parametrized by the arc-length parameter  $t$  in an almost contact metric manifold  $M$ . The function  $\theta(t)$  defined by  $\cos[\theta(t)] = g(T(t), \xi)$  is called *the contact angle function*. A curve  $\gamma$  is called a *slant curve* if its contact angle is a constant [6]. If a slant curve is with contact angle  $\frac{\pi}{2}$ , then it is called a *Legendre curve* [4]. Likewise, C. Özgür and the present author defined Legendre curves of  $S$ -manifolds in [15]. A curve  $\gamma : I \rightarrow M = (M^{2n+s}, f, \xi_i, \eta_i, g)$  is called a Legendre curve if  $\eta_i(T) = 0$ , for every  $i = 1, \dots, s$ , where  $T$  is the tangent vector field of  $\gamma$ . This definition can be used in trans- $S$ -manifolds.

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Let  $\gamma$  be a curve in an almost contact metric manifold  $(M, \varphi, \xi, \eta, g)$ . In [12], Lee, Suh and Lee introduced the notions of  $C$ -parallel and  $C$ -proper curves in the tangent and normal bundles. A curve  $\gamma$  in an almost contact metric manifold  $(M, \varphi, \xi, \eta, g)$  is defined to be  $C$ -parallel if  $\nabla_T H = \lambda \xi$ ,  $C$ -proper if  $\Delta H = \lambda \xi$ ,  $C$ -parallel in the normal bundle if  $\nabla_T^\perp H = \lambda \xi$ ,  $C$ -proper in the normal bundle if  $\Delta^\perp H = \lambda \xi$ , where  $T$  is the unit tangent vector field of  $\gamma$ ,  $H$  is the mean curvature vector field,  $\Delta$  is the Laplacian,  $\lambda$  is a non-zero differentiable function along the curve  $\gamma$ ,  $\nabla^\perp$  and  $\Delta^\perp$  denote the normal connection and Laplacian in the normal bundle, respectively [12]. The present author and C. Özgür generalized this definition for  $S$ -manifolds in [10]. In the present study, this definition will be used in trans- $S$ -manifolds as well.

An almost contact metric manifold  $M$  is called a *trans-Sasakian manifold* [14] if there exist two functions  $\alpha$  and  $\beta$  on  $M$  such that

$$(3) \quad (\nabla_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X]$$

for any vector fields  $X, Y$  on  $M$ .  $C$ -parallel and  $C$ -proper slant curves of trans-Sasakian manifolds were studied in [9].

## 2. Preliminaries

Firstly, let us recall framed  $f$ -manifolds. Let  $(M, g)$  be a  $(2n+s)$ -dimensional Riemann manifold. It is called a *framed metric  $f$ -manifold* with a *framed metric  $f$ -structure*  $(f, \xi_i, \eta_i, g)$ ,  $i \in \{1, \dots, s\}$ , if it satisfies the following equations:

$$(4) \quad f^2 = -I + \sum_{i=1}^s \eta_i \otimes \xi_i, \quad \eta_i(\xi_j) = \delta_{ij}, \quad f(\xi_i) = 0, \quad \eta_i \circ f = 0$$

$$(5) \quad g(fX, fY) = g(X, Y) - \sum_{i=1}^s \eta_i(X)\eta_i(Y),$$

$$(6) \quad \eta_i(X) = g(X, \xi_i).$$

Here,  $f$  is a  $(1, 1)$  tensor field of rank  $2n$ ;  $\xi_1, \dots, \xi_s$  are vector fields;  $\eta_1, \dots, \eta_s$  are 1-forms and  $g$  is a Riemannian metric on  $M$ ;  $X, Y \in \chi(M)$  and  $i, j \in \{1, \dots, s\}$  [13].  $(f, \xi_i, \eta_i, g)$  is called  $S$ -structure, when the Nijenhuis tensor of  $\varphi$  is equal to  $-2d\eta^i \otimes \xi_i$  for all  $i \in \{1, \dots, s\}$  [4].

Secondly, the concept of trans- $S$ -manifolds is as follows:

A  $(2n + s)$ -dimensional metric  $f$ -manifold  $M$  is called an *almost trans- $S$ -manifold* if it satisfies

$$(7) \quad (\nabla_X f)Y = \sum_{i=1}^s \left[ \begin{array}{l} \alpha_i \{g(fX, fY)\xi_i + \eta_i(Y)f^2X\} \\ + \beta_i \{g(fX, Y)\xi_i - \eta_i(Y)fX\} \end{array} \right],$$

where  $\alpha_i, \beta_i$  ( $i = 1, \dots, s$ ) are smooth functions and  $X, Y \in \chi(M)$  [2]. If  $M$  is normal, then it is called a *trans- $S$ -manifold*. If  $s = 1$ , a trans- $S$ -manifold

becomes a *trans-Sasakian manifold*. In trans-Sasakian case, the above condition implies normality. But, for  $s \geq 2$ , this statement is no longer valid [2]. Since

$$(8) \quad [f, f](X, Y) + 2 \sum_{i=1}^s d\eta_i(X, Y) \xi_i = \sum_{i,j=1}^s [\eta_j(\nabla_X \xi_i) \eta_j(Y) - \eta_j(\nabla_Y \xi_i) \eta_j(X)] \xi_i,$$

and  $\{\xi_i\}_{i=1}^s$  is  $g$ -orthonormal, it is found that

$$\sum_{j=1}^s [\eta_j(\nabla_X \xi_i) \eta_j(Y) - \eta_j(\nabla_Y \xi_i) \eta_j(X)] = 0$$

for all  $i = 1, \dots, s$ . After calculations, one obtains

$$(9) \quad \nabla_X \xi_i = -\alpha_i fX - \beta_i f^2 X$$

for  $i = 1, \dots, s$  [2].

The notion of a Frenet curve is well-known as below:

Let us consider a unit-speed curve  $\gamma : I \rightarrow M$  in an  $n$ -dimensional Riemannian manifold  $(M, g)$ . If there exist orthonormal vector fields  $E_1, E_2, \dots, E_r$  along  $\gamma$  satisfying

$$(10) \quad \begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ &\dots \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}, \end{aligned}$$

then  $\gamma$  is called a *Frenet curve of osculating order  $r$* , where  $\kappa_1, \dots, \kappa_{r-1}$  are positive functions on  $I$  and  $1 \leq r \leq n$ .

A Frenet curve of osculating order 1 is called a *geodesic*. A Frenet curve of osculating order 2 is a *circle* if  $\kappa_1$  is a non-zero positive constant. A Frenet curve of osculating order  $r \geq 3$  is called a *helix of order  $r$* , when  $\kappa_1, \dots, \kappa_{r-1}$  are non-zero positive constants; a helix of order 3 is simply called a *helix*.

Finally, we can define Legendre curves in trans- $S$ -manifolds like:

**Definition.** Let  $M = (M^{2n+s}, f, \xi_i, \eta_i, g)$  be a trans- $S$ -manifold. Consider a unit-speed smooth curve  $\gamma : I \rightarrow M$  and its unit tangential vector field  $T = \gamma'$ . If  $\eta_i(T) = 0$  for all  $i = 1, 2, \dots, s$ , then it is called a *Legendre curve*.

Here are the direct results from the definition:

$$(11) \quad \begin{aligned} f^2 T &= -T, \\ \kappa_1 \eta_i(E_2) + \beta_i &= 0, \\ (\nabla_T f) T &= \sum_{i=1}^s \alpha_i \xi_i, \end{aligned}$$

which gives us

$$\nabla_T f T = (\nabla_T f) T + f(\nabla_T T)$$

$$\begin{aligned}
 (12) \quad &= \sum_{i=1}^s \alpha_i \xi_i + f(\kappa_1 E_2) \\
 &= \sum_{i=1}^s \alpha_i \xi_i + \kappa_1 f E_2.
 \end{aligned}$$

Let us recall what a magnetic curve is and what we mean by trajectory:

Let  $M^{2n+s} = (M^{2n+s}, f, \xi_i, \eta_i, g)$  be a trans- $S$ -manifold and  $\Omega$  the fundamental 2-form of  $M^{2n+s}$  defined by

$$(13) \quad \Omega(X, Y) = g(X, fY),$$

(see [13]). From Proposition 3.1(i) in [2], for a trans- $S$ -manifold,

$$(14) \quad d\Omega = 2\Omega \wedge \sum_{i=1}^s \beta_i \eta_i.$$

When  $\Omega$  is closed, the magnetic field  $F_q$  on  $M^{2n+s}$  can be defined by

$$F_q(X, Y) = q\Omega(X, Y),$$

where  $X$  and  $Y$  are vector fields on  $M^{2n+s}$  and  $q$  is a real constant.  $F_q$  is called the contact magnetic field with strength  $q$  [11]. If  $q = 0$ , then the magnetic curves are geodesics of  $M^{2n+s}$ . Because of this reason one can consider  $q \neq 0$  (see [5] and [8]).

From (1) and (13), the Lorentz force  $\Phi$  associated to the contact magnetic field  $F_q$  can be written as

$$\Phi_q = -qf.$$

So the Lorentz equation (2) can be written as

$$(15) \quad \nabla_T T = -qfT,$$

where  $\gamma : I \subseteq \mathbb{R} \rightarrow M^{2n+s}$  is a smooth unit-speed curve and  $T = \gamma'$  (see [8] and [11]).

From 14, for trans- $S$ -manifolds, notice that  $\Omega$  does not need to be closed in general. But, we can still look for curves satisfying  $\nabla_T T = -q\varphi T$  in a trans- $S$ -manifold, calling them *trajectories*. In this paper, for sake of computations, Legendre trajectories will be considered. The general solution of the problem is in progress.

For the last part of this study, it is necessary to define  $C$ -parallel  $C$ -proper curves as below:

We can generalize the definition from [10] to trans- $S$ -manifolds:

**Definition** ([10]). Let  $\gamma : I \rightarrow (M^{2n+s}, f, \xi_i, \eta_i, g)$  be a unit speed curve in a trans- $S$ -manifold. Then  $\gamma$  is called

i)  $C$ -parallel (in the tangent bundle) if

$$\nabla_T H = \lambda \sum_{i=1}^s \xi_i,$$

ii) *C*-parallel in the normal bundle if

$$\nabla_T^\perp H = \lambda \sum_{i=1}^s \xi_i,$$

iii) *C*-proper (in the tangent bundle) if

$$\Delta H = \lambda \sum_{i=1}^s \xi_i,$$

iv) *C*-proper in the normal bundle if

$$\Delta^\perp H = \lambda \sum_{i=1}^s \xi_i,$$

where  $H$  is the mean curvature field of  $\gamma$ ,  $\lambda$  is a real-valued non-zero differentiable function,  $\nabla$  is the Levi-Civita connection,  $\nabla^\perp$  is the Levi-Civita connection in the normal bundle,  $\Delta$  is the Laplacian and  $\Delta^\perp$  is the Laplacian in the normal bundle.

From the definition, same direct proposition as in [10] is obtained:

**Proposition 2.1** ([10]). *Let  $\gamma : I \rightarrow (M^{2n+s}, f, \xi_i, \eta_i, g)$  be a unit speed curve in a trans- $S$ -manifold. Then*

i)  $\gamma$  is *C*-parallel (in the tangent bundle) if and only if

$$(16) \quad -\kappa_1^2 T + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{i=1}^s \xi_i,$$

ii)  $\gamma$  is *C*-parallel in the normal bundle if and only if

$$(17) \quad \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{i=1}^s \xi_i,$$

iii)  $\gamma$  is *C*-proper (in the tangent bundle) if and only if

$$(18) \quad 3\kappa_1 \kappa_1' T + (\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa_1'') E_2 - (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{i=1}^s \xi_i,$$

iv)  $\gamma$  is *C*-proper in the normal bundle if and only if

$$(19) \quad (\kappa_1 \kappa_2^2 - \kappa_1'') E_2 - (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{i=1}^s \xi_i.$$

### 3. Main results on Legendre trajectories

Let  $M = (M, f, \xi_i, \eta_i, g)$  be a trans- $S$ -manifold and  $\gamma : I \rightarrow M$  a unit-speed Legendre curve with arc-length parameter  $t$ . Assume that  $\gamma$  satisfies  $\nabla_T T = -qfT$ . Then, using (5) and (10), we have

$$\nabla_T T = -qfT = \kappa_1 E_2$$

and

$$g(fT, fT) = 1.$$

So,

$$fT \neq 0.$$

Using the norm of both sides gives us

$$(20) \quad \kappa_1 = |q|.$$

Thus

$$|q| E_2 = -qfT$$

and

$$(21) \quad fT = \delta E_2,$$

where  $\delta = \text{sgn}(-q)$ . From (11) and (21), we have

$$\beta_i |_\gamma = 0.$$

(21) gives us

$$(22) \quad fE_2 = -\delta T.$$

From (12), (21) and (22), we can write

$$\begin{aligned} \nabla_T fT &= \delta \nabla_T E_2 = \delta(-\kappa_1 T + \kappa_2 E_3) \\ &= \sum_{i=1}^s \alpha_i \xi_i - \kappa_1 \delta T. \end{aligned}$$

As a result, we find

$$(23) \quad \kappa_2 E_3 = \delta \sum_{i=1}^s \alpha_i \xi_i,$$

which gives us

$$(24) \quad \kappa_2 = \sqrt{\sum_{i=1}^s \alpha_i^2}.$$

Then

$$\kappa_2 = 0 \Leftrightarrow \alpha_i |_\gamma = 0.$$

Let  $\kappa_2 \neq 0$ . Notice that  $\text{sgn}(g(E_3, \sum_{i=1}^s \alpha_i \xi_i)) = \delta$ . Using (23) and (24), we find

$$(25) \quad E_3 = \frac{\delta}{\sqrt{\sum_{i=1}^s \alpha_i^2}} \sum_{i=1}^s \alpha_i \xi_i.$$

If we differentiate  $E_3$ , we obtain

$$(26) \quad \kappa_3 E_4 = \delta \sum_{i=1}^s \left( \frac{\alpha_i}{\sqrt{\sum_{i=1}^s \alpha_i^2}} \right)' \xi_i.$$

$$(27) \quad \kappa_3 = \sqrt{\sum_{i=1}^s \left[ \left( \frac{\alpha_i}{\sqrt{\sum_{i=1}^s \alpha_i^2}} \right) \right]^2}.$$

Moreover, if  $\kappa_3 = 0$ , then

$$\frac{\alpha_i}{\sqrt{\sum_{i=1}^s \alpha_i^2}} = c_i = \text{constant}, \forall i.$$

Hence

$$\sum_{i=1}^s \alpha_i^2 \left( \sum_{i=1}^s c_i^2 - 1 \right) = 0.$$

So,

$$\sum_{i=1}^s \alpha_i^2 = 0 \Leftrightarrow \kappa_2 = 0,$$

or

$$\sum_{i=1}^s c_i^2 = 1.$$

To sum up, if  $\kappa_3 = 0$  and  $\kappa_2 \neq 0$ , we have

$$E_2 = \delta fT,$$

$$E_3 = \delta \sum_{i=1}^s c_i \xi_i,$$

where

$$(28) \quad \alpha_i^2 = c_i^2 \sum_{i=1}^s \alpha_i^2, \forall i,$$

$$(29) \quad c_i = \text{constant such that } \sum_{i=1}^s c_i^2 = 1.$$

Now we can state the following theorem:

**Theorem 3.1.** *Let  $\gamma : I \rightarrow M$  be a Legendre trajectory. Then  $\gamma$  is one of the following:*

1) *a Legendre circle with  $\kappa_1 = |q|$  and the Frenet frame field  $\{T, \delta fT\}$ , where  $\delta = \text{sgn}(-q)$ . In this case,  $\alpha_i = 0, \beta_i = 0, \forall i$ .*

2) *a Legendre curve of osculating order  $r \geq 3$  with*

$$\kappa_1 = |q|, \kappa_2 = \sqrt{\sum_{i=1}^s \alpha_i^2},$$

$\kappa_3$  given in (27) and the Frenet frame field

$$\{T, \delta fT, E_3, E_4, \dots, E_r\},$$

where  $\delta = \text{sgn}(-q)$ ;  $E_3, E_4$  are given in (25) and (26), respectively. In this case,  $\alpha_i \neq 0, \exists i, \beta_i = 0, \forall i$ . Moreover, if  $r = 3$ , equations (28) and (29) are also satisfied and its Frenet frame field is

$$\{T, \delta fT, E_3\}.$$

#### 4. Main results of $C$ -parallel and $C$ -proper Legendre curves

Let  $M^{2n+s}$  be a trans- $S$ -manifold and  $\gamma : I \rightarrow M$  a Legendre curve in  $M$ , i.e.,  $\eta_i(T) = 0, \forall i$ . Now, we will consider four cases.

i)  $C$ -parallel in the tangent bundle:

From Proposition 2.1, we have

$$-\kappa_1^2 T + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{i=1}^s \xi_i.$$

If we apply  $T$  to both sides, we have the following result:

**Theorem 4.1.** *There does not exist a  $C$ -parallel Legendre curve (in the tangent bundle) in a trans- $S$ -manifold.*

ii)  $C$ -parallel in the normal bundle:

From Proposition 2.1, we can write

$$(30) \quad \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{i=1}^s \xi_i.$$

Now, let us consider two subcases  $r = 2$  and  $r \geq 3$ , where  $r$  denotes the osculating order of  $\gamma$ .

a)  $r = 2$ .

In this case, since  $\kappa_2 = 0$ , (30) becomes

$$(31) \quad \kappa_1' E_2 = \lambda \sum_{i=1}^s \xi_i.$$

So, we can state the following theorem:

**Theorem 4.2.** *Let  $r = 2$ . Then  $\gamma$  is  $C$ -parallel in the normal bundle if and only if*

$$\kappa_1 = \mp \sqrt{s} \beta,$$

$$\lambda = -\beta',$$

$$\sum_{i=1}^s \xi_i = \pm \sqrt{s} E_2.$$

In this case,  $\beta_1 = \beta_2 = \dots = \beta_s = \beta$  and  $\sum_{i=1}^s \alpha_i = 0$ .



*Proof.* If we apply  $\xi_j$  to (31), using the fact that  $g(\xi_j, \xi_i) = \eta_j(\xi_i) = \delta_{ji}$ , we have

$$(32) \quad \kappa'_1 \eta_j(E_2) = \lambda, \forall j.$$

Also, (11) gives us

$$(33) \quad \eta_j(E_2) = \frac{-\beta_j}{\kappa_1}.$$

From (32) and (33), we can write

$$(34) \quad \lambda = \frac{-\kappa'_1}{\kappa_1} \beta_j, \forall j.$$

This gives us  $\beta_1 = \beta_2 = \dots = \beta_s = \beta$  for some function  $\beta$ . On the other hand, (31) shows that  $\sum_{i=1}^s \xi_i$  and  $E_2$  are parallel. Since

$$\left\| \sum_{i=1}^s \xi_i \right\| = \sqrt{g\left(\sum_{i=1}^s \xi_i, \sum_{i=1}^s \xi_i\right)} = \sqrt{s} \text{ and } \|E_2\| = 1,$$

we obtain

$$(35) \quad \sum_{i=1}^s \xi_i = \pm \sqrt{s} E_2,$$

depending on their directions. From (9), replacing  $X = T$ , for a Legendre curve, one may obtain

$$(36) \quad \begin{aligned} \nabla_T \xi_i &= -\alpha_i fT - \beta_i f^2 T \\ &= -\alpha_i fT + \beta_i T. \end{aligned}$$

Since  $r = 2$ , i.e.,  $\kappa_2 = 0$ , differentiating (35) and using (36), we find

$$(37) \quad -\left(\sum_{i=1}^s \alpha_i\right) fT + \left(\sum_{i=1}^s \beta_i\right) T = \pm \sqrt{s} (-\kappa_1 T).$$

Then,  $fT \perp T$  gives us  $\sum_{i=1}^s \alpha_i = 0$ . We also have  $\sum_{i=1}^s \beta_i = s\beta$ . From (37), we get

$$(38) \quad \kappa_1 = \mp \sqrt{s} \beta.$$

If we consider (34) and (38) together, we can write  $\lambda = -\beta'$ . □

b)  $r \geq 3$ .

In this case, for a smooth function  $w = w(t)$ , we have

$$(39) \quad \sum_{i=1}^s \xi_i = \sqrt{s} (\cos w E_2 + \sin w E_3).$$

If we differentiate the above equation, we have

$$(40) \quad \sum_{i=1}^s \beta_i = -\sqrt{s}\kappa_1 \cos w$$

and

$$(41) \quad \kappa_2 = \pm \frac{1}{\sqrt{s}} \sum_{i=1}^s \alpha_i - w'.$$

We also have

$$(42) \quad \lambda = \frac{-\kappa_1 \kappa_1'}{\sum_{i=1}^s \beta_i}.$$

Since  $fT \perp E_1$ , we can write

$$(43) \quad fT = \pm (\sin w E_2 - \cos w E_3).$$

Thus, we have the following result:

**Theorem 4.3.** *Let  $r \geq 3$ . Then  $\gamma$  is  $C$ -parallel in the normal bundle if and only if equations (39), (40), (41), (42) and (43) are satisfied.*

iii)  $C$ -proper in the tangent bundle:

In this case, Proposition 2.1 gives us

$$3\kappa_1 \kappa_1' T + (\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa_1'') E_2 - (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{i=1}^s \xi_i.$$

If we apply  $T$ , we directly have  $\kappa_1 = \text{constant}$ . Then the equation reduces to

$$\kappa_1 (\kappa_1^2 + \kappa_2^2) E_2 - \kappa_1 \kappa_2' E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{i=1}^s \xi_i.$$

Applying  $E_2$ , we get

$$\kappa_1^2 (\kappa_1^2 + \kappa_2^2) = -\lambda \sum_{i=1}^s \beta_i.$$

Now, we will consider three subcases  $r = 2$ ,  $r = 3$  and  $r \geq 4$ :

a)  $r = 2$ .

In this case, we have

$$(44) \quad \kappa_1^3 E_2 = \lambda \sum_{i=1}^s \xi_i.$$

If we apply  $\xi_j$ , we find

$$\kappa_1^3 \eta_j (E_2) = \lambda, \forall j.$$

If we denote

$$\beta_1 = \beta_2 = \cdots = \beta_s = \beta,$$

we get

$$\lambda = -s\beta^3 = \text{constant}.$$

If we differentiate (44), it is easy to see that

$$\sum_{i=1}^s \alpha_i = 0.$$

As a result, we have

$$\kappa_1 = \mp \sqrt{s}\beta = \text{constant},$$

i.e.,  $\gamma$  is a circle. Hence, we obtain the following theorem:

**Theorem 4.4.** *Let  $r = 2$ . Then  $\gamma$  is  $C$ -proper in the tangent bundle if and only if it is a circle with*

$$\kappa_1 = \mp \sqrt{s}\beta = \text{constant}$$

and the Frenet frame field

$$\left\{ T, \frac{\pm 1}{\sqrt{s}} \sum_{i=1}^s \xi_i \right\}.$$

In this case,  $\beta_1 = \beta_2 = \dots = \beta_s = \beta$ ,  $\lambda = -s\beta^3 = \text{constant}$  and  $\sum_{i=1}^s \alpha_i = 0$ .

b)  $r = 3$ .

In this case, from Proposition 2.1, we can see that

$$\kappa_1 (\kappa_1^2 + \kappa_2^2) E_2 - \kappa_1 \kappa_2' E_3 = \lambda \sum_{i=1}^s \xi_i.$$

So,  $\sum_{i=1}^s \xi_i \in \text{sp}\{E_2, E_3\}$ . It can be written as

$$(45) \quad \sum_{i=1}^s \xi_i = \sqrt{s} (\cos w E_2 + \sin w E_3)$$

for a smooth function  $w = w(t)$ . If we differentiate this equation and apply  $T$ , we find

$$(46) \quad \sum_{i=1}^s \beta_i = -\sqrt{s} \kappa_1 \cos w,$$

and

$$(47) \quad \kappa_2 = \pm \frac{1}{\sqrt{s}} \sum_{i=1}^s \alpha_i - w'.$$

We also have

$$(48) \quad fT = \pm (\sin w E_2 - \cos w E_3)$$

and

$$(49) \quad \lambda = \frac{-\kappa_1^2 (\kappa_1^2 + \kappa_2^2)}{\sum_{i=1}^s \beta_i}.$$

To sum up, we can state:

**Theorem 4.5.** *Let  $r = 3$ . Then  $\gamma$  is  $C$ -proper in the tangent bundle if and only if equations (45), (46), (47), (48) and (49) are satisfied.*

c)  $r \geq 4$ .

In this case,  $\sum_{i=1}^s \xi_i \in sp\{E_2, E_3, E_4\}$ , consequently,  $fT \in sp\{E_2, E_3, E_4, E_5\}$ .

Let us write

$$(50) \quad \sum_{i=1}^s \xi_i = \sqrt{s} (\cos w E_2 + \sin w \cos \varphi E_3 + \sin w \sin \varphi E_4)$$

for some smooth functions  $w = w(t)$ ,  $\varphi = \varphi(t)$ . As a result, the curve must satisfy

$$\begin{aligned} \kappa_1 &= \text{constant}, \\ \sum_{i=1}^s \beta_i &= -\sqrt{s} \kappa_1 \cos w, \\ \lambda \sqrt{s} \cos w &= \kappa_1^2 (\kappa_1^2 + \kappa_2^2), \\ \lambda \sqrt{s} \sin w \cos \varphi &= -\kappa_1 \kappa_2', \\ \lambda \sqrt{s} \sin w \sin \varphi &= -\kappa_1 \kappa_2 \kappa_3. \end{aligned}$$

Differentiating (50), we also have

$$\kappa_4 = \frac{-\left(\sum_{i=1}^s \alpha_i\right) \cdot g(fT, E_5)}{\sin w \sin \varphi}.$$

Finally, we have the following theorem:

**Theorem 4.6.** *Let  $r \geq 4$ . Then  $\gamma$  is  $C$ -proper in the tangent bundle if and only if it satisfies the last six equations.*

iv)  $C$ -proper in the normal bundle:

From Proposition 2.1, we get

$$(\kappa_1 \kappa_2'' - \kappa_1'') E_2 - (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{i=1}^s \xi_i.$$

In this case, again  $\sum_{i=1}^s \xi_i \in sp\{E_2\}$ ,  $\sum_{i=1}^s \xi_i \in sp\{E_2, E_3\}$  or  $\sum_{i=1}^s \xi_i \in sp\{E_2, E_3, E_4\}$  depending on the osculating order  $r$ . We can follow the above procedure to get results for  $r = 2$  and  $r = 3$ . The case  $r \geq 4$  is similar to case iii) c) with minor changes in equations.

*Remark.* For sake of shortness,  $\alpha_i|_\gamma$  and  $\beta_i|_\gamma$  are written as  $\alpha_i$  and  $\beta_i$  where possible. This means the equations are not necessarily satisfied globally. But instead, they are satisfied along the curve  $\gamma$ .

### References

- [1] T. Adachi, *Curvature bound and trajectories for magnetic fields on a Hadamard surface*, Tsukuba J. Math. **20** (1996), no. 1, 225–230. <https://doi.org/10.21099/tkbjm/1496162994>
- [2] P. Alegre, L. M. Fernández, and A. Prieto-Martín, *A new class of metric  $f$ -manifolds*, Carpathian J. Math. **34** (2018), no. 2, 123–134.
- [3] M. Barros, A. Romero, J. L. Cabrerizo, and M. Fernández, *The Gauss-Landau-Hall problem on Riemannian surfaces*, J. Math. Phys. **46** (2005), no. 11, 112905, 15 pp. <https://doi.org/10.1063/1.2136215>
- [4] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, second edition, Progress in Mathematics, 203, Birkhäuser Boston, Ltd., Boston, MA, 2010. <https://doi.org/10.1007/978-0-8176-4959-3>
- [5] J. L. Cabrerizo, M. Fernández, and J. S. Gómez, *On the existence of almost contact structure and the contact magnetic field*, Acta Math. Hungar. **125** (2009), no. 1-2, 191–199. <https://doi.org/10.1007/s10474-009-9005-1>
- [6] J. T. Cho, J. Inoguchi, and J.-E. Lee, *On slant curves in Sasakian 3-manifolds*, Bull. Austral. Math. Soc. **74** (2006), no. 3, 359–367. <https://doi.org/10.1017/S0004972700040429>
- [7] A. Comtet, *On the Landau levels on the hyperbolic plane*, Ann. Physics **173** (1987), no. 1, 185–209. [https://doi.org/10.1016/0003-4916\(87\)90098-4](https://doi.org/10.1016/0003-4916(87)90098-4)
- [8] S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu, and A. I. Nistor, *Magnetic curves in Sasakian manifolds*, J. Nonlinear Math. Phys. **22** (2015), no. 3, 428–447. <https://doi.org/10.1080/14029251.2015.1079426>
- [9] Ş. Güvenç and C. Özgür, *On slant curves in trans-Sasakian manifolds*, Rev. Un. Mat. Argentina **55** (2014), no. 2, 81–100.
- [10] Ş. Güvenç and C. Özgür,  *$C$ -parallel and  $C$ -proper slant curves of  $S$ -manifolds*, Filomat **33** (2019), no. 19, 6305–6313.
- [11] M. Jleli, M. I. Munteanu, and A. I. Nistor, *Magnetic trajectories in an almost contact metric manifold  $\mathbb{R}^{2N+1}$* , Results Math. **67** (2015), no. 1-2, 125–134. <https://doi.org/10.1007/s00025-014-0398-y>
- [12] J.-E. Lee, Y. J. Suh, and H. Lee,  *$C$ -parallel mean curvature vector fields along slant curves in Sasakian 3-manifolds*, Kyungpook Math. J. **52** (2012), no. 1, 49–59. <https://doi.org/10.5666/KMJ.2012.52.1.49>
- [13] H. Nakagawa, *On framed  $f$ -manifolds*, Kodai Math. Sem. Rep. **18** (1966), 293–306. <http://projecteuclid.org/euclid.kmj/1138845274>
- [14] J. A. Oubiña, *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), no. 3-4, 187–193.
- [15] C. Özgür and Ş. Güvenç, *On biharmonic Legendre curves in  $S$ -space forms*, Turkish J. Math. **38** (2014), no. 3, 454–461. <https://doi.org/10.3906/mat-1207-8>

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