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RINGS IN WHICH EVERY IDEAL CONTAINED IN THE SET OF ZERO-DIVISORS IS A D-IDEAL

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Abstract. In this paper, we introduce and study the class of rings in which every ideal consisting entirely of zero divisors is a d-ideal, considered as a generalization of strongly duo rings. Some results including the characterization of AA-rings are given in the first section. Further, we examine the stability of these rings in localization and study the possible transfer to direct product and trivial ring extension. In addition, we define the class of d_E-ideals which allows us to characterize von Neumann regular rings.

1. Introduction

Throughout this article, all rings are commutative with identity and all modules are unital. If R is a ring and E is an R-module, $Z(E) = Z_R(E) := \{r \in R \mid re = 0$ for some nonzero element $e \in E\}$, denotes the set of zero-divisors of R on E and $Z(R) := Z_R(R)$, denotes the set of zero-divisors of the ring R; $Nil(R)$ denotes the set of nilpotent elements of R; $T(R) := R_{R\setminus Z(R)}$, the total quotient ring of R; $(N :_E I) := \{e \in E \mid Ie \subseteq N\}$, denotes the residual of a submodule N of E by an ideal I of R; $(J : I) := (J :_R I)$, denotes the residual of an ideal J of R by an ideal I; $Ann_E(I) := (0 :_E I)$, denotes the annihilator of an ideal I of R on E; I^{-1} denotes the set $(R :_{T(R)} I)$; $Ann_R(N) := \{r \in R \mid rN = 0\}$, denotes the annihilator of a submodule N of E on R; $Ann(E) := Ann_R(E)$ denotes the annihilator of E; $Ann^2(x) := Ann(Ann(x))$ denotes the annihilator of the annihilator (or, colloquially speaking, the double-annihilator) of an element x of R. If R is an integral domain, we will usually denote its quotient field by $qf(R)$.

An ideal I of R is called a *d-ideal* if for each $x \in I$, $Ann^2(x) \subseteq I$. These ideals that will be the subject of study in this paper have appeared in various guises with different names. They were first studied by Speed [16] in 1972 in the context of Baer rings (a ring R is called Baer, if for each $r \in R$ there

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exists an idempotent $e \in R$ such that $Ann(r) = Re$). He called them "Baer ideals". In [3], Bernau studied them as z-ideals. In [11], Khabazian, Safaeeyan and Vedadi extended the concept of d-ideals to the category of modules and introduced strongly duo modules as follows: An R-module E is called strongly duo module if $Tr(N, E) := \{\sum \mathrm{Im} f \mid f \in \mathrm{Hom}_R(N, E)\} = N$ for all submodule N of E; and a ring R is said to be a strongly duo ring if it is a strongly duo as an R-module. Additional information about strongly duo modules and rings can be found in the interesting article [15]. In [8, Theorem 1], Jayaram proved that if R is a reduced ring, then R is von Neumann regular if and only if R is strongly duo.

Let A be a ring and E be an A-module. Then $R = A \ltimes E$, the trivial (ring) extension of A by E, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology, such as the idealization.) The basic properties of trivial ring extensions are summarized in the books [6, 7]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [1, 2, 10]).

The aim of this paper is to study rings in which every ideal contained in the set of zero-divisors is a d-ideal that we call AA-rings. In Section 2, we observe that in the context of total ring of quotients, the notion of an AA-ring coincides with the definition of a strongly duo ring. Also, recall from [5] that a ring R is said to be a quasi-regular ring if its classical ring of quotients is von Neumann regular. We show that reduced AA-rings are exactly quasi-regular rings. As a corollary, we provide that every ideal consisting entirely of zero divisors of a hereditary ring is idempotent. Moreover, we introduce and investigate the concept of a d_E-ideal of a ring R for an R-module E. An ideal I of R is said to be a d_E-ideal for an R-module E if for each $x \in I$, $Ann_R(Ann_E(x)) \subseteq I$. From this notion, we prove that a ring R is von Neumann regular if and only if there exists a reduced R-module E such that every ideal of R is a d_E-ideal (according to [12], an R-module E is said to be reduced if for any $e \in E$ and $r \in R$, $re = 0$ implies $Re \cap rE = 0$ and it can be easily verified that E is a reduced R-module if and only if for any $e \in E$ and $r \in R$, $r^2 e = 0$ implies $re = 0$). Finally, in Section 3 we give several results on the transfer of the AA-property to direct product, localization and to various contexts of trivial extension.

2. Basic results

We will be using the following definition (which agrees with the classical one if R is a domain). An R-module E is said to be a torsion R-module if, for each $e \in E$, there exists $r \in R \setminus \{0\}$ such that $re = 0$. We will also use the following three standard definitions: A regular element of a ring R is any element of $R \setminus Z(R)$; an R-module E is divisible if, for each $e \in E$ and each regular element r of R, there exists $f \in E$ such that $e = rf$; an R-module E is (a) torsion-free (R-module) if $r \in R, e \in E,\ re = 0$ implies that either $r \in Z(R)$ or $e = 0$.

We shall begin with the following definition:

Definition. A ring R is said to be an AA-ring if every ideal contained in the set of zero-divisors is a d-ideal.

The first main result establishes a characterization of AA-rings:

Theorem 2.1. *Let R be a ring. The following statements are equivalent*:

(1) *R is an AA-ring.*
(2) *If I and J are ideals of R such that $I \subseteq Z(R)$ and $f : I \to J$ is an epimorphism of R-modules, then $J \subseteq I$.*
(3) *$Tr(I, R) = I$ for every ideal I contained in $Z(R)$.*
(4) *For each $x, y \in Z(R)$, $Ann(x) \subseteq Ann(y)$ implies that $y \in Rx$.*
(5) *Every principal ideal contained in $Z(R)$ is a d-ideal.*

Proof. (1) $\Rightarrow$ (2) Let I and J be two ideals of R such that $I \subseteq Z(R)$ and let $f : I \to J$ be an epimorphism of R-modules. It is easy to see that for each $x \in I$, $Ann(x) \subseteq Ann(f(x))$ and so $f(x) \in I$ since I is a d-ideal. Hence $J \subseteq I$.

(2) $\Rightarrow$ (3) It is clear to see that $I \subseteq Tr(I, R)$. For the reverse inclusion, we have by hypothesis $\mathrm{Im} f \subseteq I$ for every $f \in \mathrm{Hom}_R(I, R)$. So $Tr(I, R) = I$.

(3) $\Rightarrow$ (4) It is similar to the proof of [11, Theorem 2.1].

(4) $\Rightarrow$ (5) It is obvious since $Ann(x) \subseteq Ann(ax)$ for each $a \in R$.

(5) $\Rightarrow$ (1) Let I be an ideal of R such that $I \subseteq Z(R)$. Let $x \in I$ and $y \in R$ such that $Ann(x) \subseteq Ann(y)$. Therefore, the hypothesis ensures that Rx is d-ideal and hence $y \in I$. □

According to [11], a ring R is a strongly duo ring if and only if every ideal of R is a d-ideal. The next proposition identifies an important class of AA-rings and then shows that, within the context of total rings, AA-rings coincide with strongly duo rings.

Proposition 2.2. *Let R be a ring. Then R is a strongly duo ring if and only if R is a total AA-ring.*

Proof. Assume that R is a strongly duo ring and let x be a regular element of R. Then the principal ideal Rx is a d-ideal and so $R = Ann^2(x) \subseteq Rx$. Thus $x \in U(R)$ and therefore $R = T(R)$ as desired.

Conversely, let I be an ideal of R. Since R is a total ring, $I \subseteq Z(R)$; and since R is an AA-ring, I is a d-ideal. Thus R is a strongly duo ring. □

Remark 2.3. By the above result, a non strongly duo AA-ring contains necessary a regular element. For example, every domain which is not a field is a non strongly duo AA-ring.

The following corollary is an immediate consequence of Proposition 2.2.

Corollary 2.4. *Let R be a ring in which every maximal ideal is a d-ideal. Then R is a strongly duo ring if and only if R is an AA-ring.*

Proof. It suffices to prove that R is a total ring. Let x be a regular element of R. If x is not a unit of R, then there exists a maximal ideal M of R such that $x \in M$. But since M is a d-ideal, $R = Ann^2(x) \subseteq M$, which is absurd. Thus $x \in U(R)$ and so $R = T(R)$. Now, the result follows immediately by Proposition 2.2. □

Recall from [9], that a finitely generated R-module E is said to be a von Neumann regular module if for each $e \in E$, there exists a weakly idempotent $x \in R$ (i.e., $x - x^2 \in Ann(E)$) such that $Re = xE$.

Theorem 2.5. *Let R be a ring and E be a finitely generated R-module. Then E is a von Neumann regular module if and only if E is a multiplication reduced strongly duo module.*

Proof. Assume that E is a von Neumann regular module. Then, by [4, Proposition 1.1] and [9, Lemma 10], E is a multiplication reduced module. We need only to prove that E is a strongly duo module. Let $e, f \in E$ such that $Ann_R(e) \subseteq Ann_R(f)$. By [9, Lemma 5], there exists a weakly idempotent $x \in R$ such that $Re = xE$. Moreover $e = xm = x^2m$ for some $m \in E$. Thus $(1 - x) \in Ann_R(e)$ and so $f = xf \in xE$. Thus $f \in Re$. Conversely, let $x \in R$. Since E is a reduced module, $Ann_R(x^2e) \subseteq Ann_R(xe)$ for each $e \in E$. Thus $xE = x^2E$ and, by [9, Theorem 2], E is a von Neumann regular module. □

As an immediate consequence, we obtain the following result of Jayaram [8, Theorem 1].

Corollary 2.6. *A ring R is von Neumann regular if and only if R is a reduced strongly duo ring.*

Recall from [8], an ideal of R is a 0-*ideal* if $I = O(S) := \{r \in R \,|\, rs = 0$ for some $s \in S\}$, for some multiplicative subset S of R. A ring R is called quasi-regular if its classical ring of quotients $T(R)$ is a von Neumann regular ring.

Proposition 2.7. *Let R be a ring. Then the following statements are equivalent*:

(1) *R is a reduced AA-ring.*
(2) *For each $x \in Z(R)$, $x \in Rx^2$.*
(3) *R is a quasi-regular ring.*
(4) *R is a reduced ring and every principal ideal contained in $Z(R)$ is a 0-ideal.*

Proof. (1) ⇒ (2) Assume that R is a reduced AA-ring and let $x \in Z(R)$. Then $Ann(x^2) \subseteq Ann(x)$ and therefore $x \in Rx^2$.

(2) $\Rightarrow$ (3) Let $x/s \in T(R)$. If $x \in R \setminus Z(R)$, then x/s is invertible in $T(R)$. If $x \in Z(R)$, by hypothesis, $x \in Rx^2$ and $x/s \in R(x^2/s^2)$. Consequently, R is a quasi-regular ring.

(3) $\Rightarrow$ (4) Since R is a quasi-regular ring, then for each $x \in Z(R)$, there exist $r \in R$ and $d \in R \setminus Z(R)$ such that $dx = rx^2$. Then if $x^2 = 0$, we have $dx = 0$ and so $x = 0$. Thus R is reduced. Let $x \in Z(R)$ and set $S := \{y \in R \mid Ann(x) \subseteq Ann^2(y)\}$. It is easy to see that S is a multiplicative subset of R. On the other hand, $(d - rx) \in Ann(x)$ and also $(d - rx) \in S$. It follows that $Rx = O(S)$.

(4) $\Rightarrow$ (1) R is an AA-ring by Theorem 2.1 since every 0-ideal is a d-ideal. The proof is completed. $\square$

Remark 2.8. In the case of reduced rings, we can now find more examples of AA-rings that are not strongly duo. For instance, every semi-hereditary ring that is not a von Neumann regular ring is a non strongly duo AA-ring.

Recall that a commutative ring is a hereditary ring if every ideal is projective.

Corollary 2.9. *Let R be a hereditary ring. Then every ideal of R contained in $Z(R)$ is an idempotent ideal.*

Proof. Let I be an ideal of R such that $I \subsetneq Z(R)$. Since R is a hereditary ring, R is a quasi-regular ring. By Proposition 2.7, R is an AA-ring. Thus I is a d-ideal and so $I = Tr(I, R)$. Since I is projective, by [17, Result 18.7], $I^2 = Tr(I, R)^2 = Tr(I, R) = I$ and therefore I is an idempotent ideal, as desired. $\square$

Next, we extend the notion of d-ideals to d-ideals with respect to modules by introducing the following definition.

Definition. Let R be a ring and E be an R-module. An ideal I of R is said to be a d_E-ideal if for each $x \in I$, $Ann_R(Ann_E(x)) \subseteq I$.

Note that an ideal I of a ring R is a d-ideal if and only if I is a d_R-ideal. Furthermore, if R is a ring and E is an R-module, then it is easy to show that a proper ideal I of R is a d_E-ideal implies that $Ann(E) \subseteq I$ and $I \subseteq Z(E)$.

Proposition 2.10. *Let R be a ring and E be an R-module. An ideal I of R is a d_E-ideal if and only if for any $x, y \in R$, $Ann_E(x) \subseteq Ann_E(y)$ and $x \in I$ imply that $y \in I$.*

Proof. Suppose that I is a d_E-ideal of R and let $x \in I$ and $y \in R$ such that $Ann_E(x) \subseteq Ann_E(y)$. Then $y \in Ann_R(Ann_E(x)) \subseteq I$ and so $y \in I$.

Conversely, assume that whenever $Ann_E(x) \subseteq Ann_E(y)$ and $x \in I$. Then $y \in I$. Let $x \in I$ and $y \in Ann_R(Ann_E(x))$. Then $yAnn_E(x) = 0$ and so $Ann_E(x) \subseteq Ann_E(y)$. By the assumption, $y \in I$, and therefore $Ann_R(Ann_E(x)) \subseteq I$, as desired. $\square$

Proposition 2.11. *Let R be a ring and E be an R-module. Then*:

(1) *$Ann(E)$ is a d_E-ideal of R.*
(2) *The zero ideal of R is a d_E-ideal if and only if $Ann(E) = 0$.*
(3) *The intersection of all d_E-ideals is a d_E-ideal.*

Proof. Straightforward. □

Example 2.12. Let $R := \mathbb{Z}$. Consider the $\mathbb{Z}$-module $E := \bigoplus_{k=1}^{\infty} \mathbb{Z}/2^k\mathbb{Z}$. Then every ideal of R which has the form $I = 2^n\mathbb{Z}$ is a d_E-ideal.

Proof. One checks easily that $Z(E) = 2\mathbb{Z}$ and $Ann(E) = 0$. Let I be an ideal of R such that $I = 2^n\mathbb{Z}$ for some integer $n \geq 1$. Therefore, for each $x \in I$, $Ann_E(x) = \bigoplus_{k=1}^{\infty} N_k$ with $N_k = \mathbb{Z}/2^k\mathbb{Z}$ if $k \leq n$ and $N_k = 2^{k-n}\mathbb{Z}/2^k\mathbb{Z}$ if $k > n$. Hence $Ann_R(Ann_E(x)) = 2^n\mathbb{Z}$ for each $x \in I$ and so I is a d_E-ideal of R. □

Proposition 2.13. *Let R be a ring and E be an R-module. Let I be a d_E-ideal of R. Then the following statements hold*:

(1) *For every ideal J of R, $(I : J)$ is a d_E-ideal of R.*
(2) *Every minimal prime over I is a d_E-ideal of R.*
(3) *$\sqrt{I}$ is a d_E-ideal of R.*

We need the following lemma before proving Proposition 2.13.

Lemma 2.14. *Let R be a ring and E be an R-module. If $x, y \in R$ and $Ann_E(x) \subseteq Ann_E(y)$, then $Ann_E(x^n) \subseteq Ann_E(y^n)$ for each $n \geq 2$.*

Proof. We proceed by induction on n. For $n = 2$, let $e \in Ann_E(x^2)$. Then $x^2e = 0$ and hence $yxe = 0$. Therefore $ye \in Ann_E(x) \subseteq Ann_E(y)$. Thus $y^2e = 0$, and therefore $Ann_E(x^2) \subseteq Ann_E(y^2)$. Assume that $n \geq 3$ and the induction hypothesis for $n - 1$. Let $e \in Ann_E(x^n)$. Then $x^ne = 0$ and so $x^{n-1}e \in Ann_E(x) \subseteq Ann_E(y)$. Thus $yx^{n-1}e = 0$ and so $ye \in Ann_E(x^{n-1}) \subseteq Ann_E(y^{n-1})$. Hence $y^ne = 0$ and therefore $e \in Ann_E(y^n)$. It follows that $Ann_E(x^n) \subseteq Ann_E(y^n)$, as desired. □

Proof of Proposition 2.13. (1) Let $x \in (I : J)$ and $y \in R$ such that $Ann_E(x) \subseteq Ann_E(y)$. Then for each $a \in J$, $Ann_E(xa) \subseteq Ann_E(ya)$. Since I is a d_E-ideal and $xa \in I$, by Proposition 2.10, $ya \in I$ and so $y \in (I : J)$. Again, by Proposition 2.10, $(I : J)$ is a d_E-ideal.

(2) Suppose that $P \in Min(I)$. Let $x \in P$ and $y \in R$ such that $Ann_E(x) \subseteq Ann_E(y)$. Then, there exist $a \in R \setminus P$ and $n \in \mathbb{N}$ such that $ax^n \in I$. By Lemma 2.14, $Ann_E(ax^n) \subseteq Ann_E(ay^n)$ which in turns implies that $ay^n \in I$ since I is a d_E-ideal. It follows that $y \in P$.

(3) It follows from Proposition 2.11 since $\sqrt{I}$ is the intersection of all minimal prime ideals over I which are d_E-ideals by (2) as desired. □

Corollary 2.15. *Let R be a ring and E be an R-module such that $Ann(E) = 0$. Then the following conditions hold*:

(1) *Every annihilator ideal is a d_E-ideal of R.*

(2) *Every minimal prime ideal of R is a d_E-ideal of R.*

Proof. The fact that $Ann(E) = 0$ shows that the zero ideal is a d_E-ideal of R. We obtain by part (1) of Proposition 2.13 that $Ann(J) = (0 : J)$ is a d_E-ideal of R for every ideal J, and by part (2) of Proposition 2.13 that every minimal prime of R is a d_E-ideal. □

Recall that two ideals I and J of a ring R are co-prime if $I + J = R$.

Proposition 2.16. *Let R be a ring and E be an R-module. Let $I_1, \ldots, I_n$ be ideals of R such that for each $i \neq j$, I_i and I_j are co-prime. Then $\bigcap_{k=1}^{n} I_k$ is a d_E-ideal of R if and only if I_i is a d_E-ideal for each $i \in \{1, \ldots, n\}$.*

Proof. It is enough to show the converse. Suppose that $\bigcap_{k=1}^{n} I_k$ is a d_E-ideal of R. Fix $i \in \{1, \ldots, n\}$ and let $x \in I_i$ and $y \in R$ such that $Ann_E(x) \subseteq Ann_E(y)$. By hypothesis, for each $j \neq i$, I_i and I_j are co-prime of R and so I_i and $\bigcap_{k=1, k\neq i}^{n} I_k$ are co-prime. Then, $1 = a + b$ for some $a \in I_i$ and $b \in \bigcap_{k=1, k\neq i}^{n} I_k$. Therefore $y = ya + yb$ and $Ann_E(xb) \subseteq Ann_E(yb)$. Since $xb \in \bigcap_{k=1}^{n} I_k$ and $\bigcap_{k=1}^{n} I_k$ is a d_E-ideal of R, then $yb \in \bigcap_{k=1}^{n} I_k$. Thus $y \in I_i$ since $ya \in I_i$. □

Corollary 2.17. *Let R be an Artinian ring and E be an R-module such that $Ann(E) = 0$. Then every maximal ideal of R is a d_E-ideal of R.*

Proof. Since R is an Artinian ring, R has a finite number of maximal ideals, say $M_1, \ldots, M_n$. Moreover, $Nil(R) = J(R)$ and $Nil(R)$ is a d_E-ideal of R. We know that M_i are pair-wise co-prime which implies by Proposition 2.16 that each M_i is a d_E-ideal. □

Theorem 2.18. *Let R be a ring and I be a nonzero ideal of R. Then I is a d-ideal of R if and only if I is a d_E-ideal for every R-module E which has an element with zero annihilator.*

Proof. Assume that I is a d-ideal and let E be an R-module which has an element with zero annihilator. Let $x \in I$ and $y \in Ann_R(Ann_E(x))$. Then, $y \in Ann^2(x)$. In fact, let $a \in Ann(x)$. By hypothesis, there exists an element e of E such that $Ann_R(e) = 0$, thus $ae \in Ann_E(x)$ and so $ae \in Ann_E(y)$. It follows that $a \in Ann(y)$. However I is a d-ideal of R, we obtain finally that $y \in I$ and hence I is a d_E-ideal of R. The converse is clear. □

The class of d-ideals and the class of d_E-ideal are not necessarily comparable, as is illustrated by the following example.

Example 2.19. Let (R, M) be a local domain which is not a field and E be an R-module with $ME = 0$. Then:

(1) The zero ideal is a d-ideal of R, but it is not a d_E-ideal.
(2) M is a d_E-ideal of R which is not a d-ideal.

Proof. (1) It is obvious because $Ann(E) \neq 0$.

(2) Since M is a proper ideal which is not contained in $Z(R)$, M is not a d-ideal of R. On the other hand, $ME = 0$ implies that M is a d_E-ideal. □

Proposition 2.20. *Let R be a ring and E be an R-module. Then the following statements are equivalent*:

1. *Every ideal I of R such that $I \subseteq Z(E)$ is a d_E-ideal.*
2. *For each $x, y \in Z(E)$, $Ann_E(x) \subseteq Ann_E(y)$ implies that $y \in Rx$.*

Proof. This is a routine argument. □

According to [13], an ideal of R is called invertible if $II^{-1} = R$.

Example 2.21. Let R be a domain which is not a field, $K := qf(R)$ and $E := K/R$. Then every ideal of R contained in $Z(E)$ is a d_E-ideal.

Proof. Note that $Ann(E) = 0$. In fact, let $x \in Ann(E)$, then $x \notin U(R)$. Assume that $x \neq 0$, it follows that $x(1/x^2) \in R$ and so $x \in Rx^2$, that implies $x \in U(R)$, contradiction. On the other hand, we have $Z(E) = R \setminus U(R)$ since $x(1/x) \in R$ for each $x \in R \setminus U(R)$, with $1/x$ is a nonzero element of E. Now, we show that every ideal of R contained in $Z(E)$ is a d_E-ideal. Without loss of generality, let $x, y \in Z(E) \setminus \{0\}$ such that $Ann_E(x) \subseteq Ann_E(y)$ and set $I := Rx$ and $J := Ry$. One can see that $Ann_E(x) = I^{-1}$ and $Ann_E(y) = J^{-1}$. By [13, Lemma 3], I and J are invertible ideals. We obtain that $J = II^{-1}J \subseteq IJ^{-1}J = IR = I$, as desired. □

The following proposition gives a new characterization for von Neumann regular rings.

Proposition 2.22. *Let R be a ring. Then R is a von Neumann regular ring if and only if there exists a reduced R-module E such that every ideal of R is a d_E-ideal.*

Proof. If R is a von Neumann regular ring, then R is a strongly duo ring and so R is reduced and every ideal of R is a d-ideal. Conversely, let E be a reduced R-module such every ideal of R is a d_E-ideal. Then, $Ann_E(x^2) \subseteq Ann_E(x)$ for each $x \in R$ and hence $x \in Rx^2$, as desired. □

Corollary 2.23. *Let R be a ring. Then R is a reduced AA-ring if and only if there exists a reduced module E over $T(R)$ such that every ideal of $T(R)$ is a d_E-ideal.*

3. AA-property for specific rings

In this section, we study the transfer of the AA-property to some several classes of rings. We start by giving necessary and sufficient conditions for a direct product of rings to be an AA-ring.

Proposition 3.1. *Let $(R_i)_{i=1,\dots,n}$ be a family of rings and let $R := \prod_{i=1}^{n} R_i$. Then, the following statements are equivalent*:

1. *R is an AA-ring.*
2. *R_i is a strongly duo ring for each $i \in \{1, \dots, n\}$.*
3. *R is a strongly duo ring.*

Proof. By induction on n, it suffices to prove the assertion for $n = 2$.

(2) $\Leftrightarrow$ (3) By [11, Proposition 4.4].

(3) $\Rightarrow$ (1) This is clear.

(1) $\Rightarrow$ (2) Assume that $R := R_1 \times R_2$ is an AA-ring. Let $x \in R_1 \setminus Z(R_1)$. Then, the ideal generated by $(x, 0)$ is a d-ideal, otherwise $x \in U(R_1)$ since $Ann^2(x, 0) = R_1 \times 0$. Consequently, R_1 is a total ring. On the other hand, let $x \in Z(R_1)$ and $r \in R_1$ such that $Ann(x) \subseteq Ann(r)$. So, we have $Ann(x, 0) \subseteq Ann(r, 0)$. Then, $r \in R_1 x$. Thus, R_1 is an AA-ring and by Proposition 2.2, R_1 is a strongly duo ring. By similar arguments, we obtain that R_2 is a strongly duo ring. □

Next, we study the stability of AA-property in localization.

Proposition 3.2. *Let R be an AA-ring and S be a multiplicative subset of R. Suppose that at least one of the following conditions holds*:

1. *$S \subseteq R \setminus Z(R)$.*
2. *R is a reduced ring.*
3. *For each $x \in R$, $Ann(x)$ is a finitely generated ideal of R.*

Then R_S is an AA-ring.

Proof. Assume that the condition (1) holds. Let I_S be an ideal of R_S such that $I_S \subseteq Z(R_S)$. Then $I \subseteq Z(R)$. Furthermore, let $x/s \in I_S$. Since $S \subseteq R \setminus Z(R)$, one checks easily that $Ann\,(x/s) = (Ann(x))_S$ and $Ann^2\,(x/s) = (Ann^2(x))_S$. It follows that I_S is a d-ideal of R_S. Next, assume that R is a reduced ring, then R is quasi-regular and so R_S is quasi-regular by [5, Proposition 2] which implies that R_S is an AA-ring. Now, suppose that the condition (3) holds. Let $(x/s) \in Z(R_S)$ and $(r/t) \in R_S$ such that $Ann(x/s) \subseteq Ann(r/t)$. Replacing x/s, r/t with (xt/st), (rs/st) respectively, we can suppose that $s = t$. By hypothesis, there exists a finite subset $\{x_1, \ldots, x_n\}$ of R such that $Ann(x) = \sum_{i=1}^{n} Rx_i$. Then, $x_i/1 \in Ann(x/s)$ and so $x_i s_i r = 0$ for some $s_i \in S$. Hence, $Ann(x) \subseteq Ann(ur)$, with $u := s_1 s_2 \cdots s_n \in S$. On the other hand, as $x/s \in Z(R_S)$, we have $x \in Z(R)$. Furthermore, $ur \in Rx$ and so $r/s \in R_S(x/s)$, as desired. □

Remark 3.3. This remark shows that if R_S is an AA-ring for some multiplicative subset $S \subseteq R \setminus Z(R)$, then R is not necessarily an AA-ring. Indeed, let R be an AA-ring which contains a regular element. By Proposition 3.2, we obtain that $T(R)$ is a strongly duo ring. Thus, $T(R \times R)$ is an AA-ring, however $R \times R$ is not an AA-ring.

Theorem 3.4. *Let R be a ring. Consider the following conditions*:

(a) *R is an AA-ring.*

(b) *R_P is an AA-ring for each $P \in Spec(R)$.*

(c) *R_M is an AA-ring for each $M \in Max(R)$.*

Then:

(1) (b)⇒(c)⇒(a).
(2) *If R is reduced, coherent or $Z(R) = Nil(R)$, then the three conditions are equivalent.*

Proof. (1) (b)⇒(c) It is obvious.

(c)⇒(a) Let $x \in Z(R)$ and $r \in R$ such that $Ann(x) \subseteq Ann(r)$, and set $J := (Rx : Rr)$. We will prove that $J = R$. Assume that J is a proper ideal of R. Hence, J is contained in a maximal ideal M of R. It is easy to see that $Ann(x/1) \subseteq Ann(r/1)$ and $x/1 \in Z(R_M)$, we must have $(r/1) \in R_M(x/1)$, whence $rst \in Rx$ for some $s, t \in R \setminus M$. Consequently, $st \in J \setminus M$, the desired contradiction.

(2) By Proposition 3.2(2), if R is a reduced ring, then the AA-property is locally. Next, suppose that R is a coherent ring, then for each $x \in R$, $Ann(x)$ is a finitely generated ideal of R and it is an immediate application of part (3) of Proposition 3.2. Next, assume that every zero-divisor is a nilpotent element, then $Z(R) \subseteq P$ for every prime ideal P of R. By Proposition 3.2(1), we obtain that R_P is an AA-ring. □

We investigate the possible transfer of the AA-property to various trivial extension contexts. Our results generate new families of examples of AA-rings which are not strongly duo rings.

Theorem 3.5. *Let A be an integral domain, E be a divisible A-module and $R := A \ltimes E$. Then:*

(1) *R is an AA-ring if and only if every ideal I of A contained in $Z(E)$ is a d_E-ideal and E is a strongly duo module.*
(2) *If $A \setminus (Z(E) \cup U(A)) \neq \emptyset$, then R is never a strongly duo ring.*

Proof. (1) By [1, Corollary 3.4], every ideal of R has the form $I \ltimes E$ or $0 \ltimes N$ for some ideal I of A or submodule N of E. Now, let I be an ideal of A such that $I \subseteq Z(E)$. The ideal $J := I \ltimes E$ is a d-ideal of R if and only if for each $x \in I$, $Ann_A(Ann_E(x)) \subseteq I$. In fact, the necessity is clear since $Ann^2(x, 0) = Ann_A(Ann_E(x)) \ltimes E$, for any element $x \in I$. Conversely, let $(x, e) \in J$. If $x = 0$, either $Ann^2(0, e) = 0 \ltimes Ann_E(Ann_A(e))$ or $Ann^2(0, e) = 0 \ltimes E$. If $x \neq 0$, then $Ann_E(x) \neq 0$ (since $I \subseteq Z(E)$) and so $Ann^2(x, e) = Ann_A(Ann_E(x)) \ltimes E$. In both cases $Ann^2(x, e) \subseteq J$. Thus, J is a d-ideal of R. Next, for every A-submodule N of E, $0 \ltimes N$ is a d-ideal of R if and only if E is a strongly duo module. Indeed, let N and N' be two A-submodules of E such that N' is a homomorphic image of N. There exists an epimorphism of R-modules $0 \ltimes N \to 0 \ltimes N'$. It follows that $N \subseteq N'$ since $0 \ltimes N \subseteq Z(R)$ and R is an AA-ring. Similarly we can prove the converse.

(2) Let $x \in A \setminus (Z(E) \cup U(A))$ and set $I := Ax$ the principal ideal generated by x. Then, I is not contained in $Z(E)$ and so is not a d_E-ideal of A. Consequently, $J := I \ltimes E$ is not a d-ideal of R. □

If A is a principal ideal domain (PID) and E is a divisible A-module, Theorem 3.5(1) specializes to the following result.

Corollary 3.6. *Let A be a PID, E be a divisible A-module and $R := A \ltimes E$. Then, R is an AA-ring if and only if every ideal I of A contained in $Z(E)$ is a d_E-ideal.*

Proof. It is enough to show the converse. Since A is a PID, then E is injective and hence E is strongly duo by [11, Proposition 2.7] and [14, Examples p. 1685]. Thus R is an AA-ring, as desired. □

Corollary 3.7. *Let A be an integral domain and E be a divisible A-module which is torsion-free. Then, $R := A \ltimes E$ is an AA-ring if and only if E is a simple module.*

Proof. Assume that R is an AA-ring. Then, E is a strongly duo module which implies that E is simple since it is torsion-free. For the converse, one can see that $0 \ltimes E$ is the unique nonzero ideal of R contained in $Z(R)$ and it is a d-ideal. □

Using Example 2.21 and Theorem 3.5, we can construct a non-trivial example of an AA-ring.

Example 3.8. Let A be a Dedekind domain, $K := qf(A)$ and $E := K/A$. Then $R := A \ltimes E$ is an AA-ring.

Proof. By hypothesis, E is an injective module. Then E is both a divisible module and a strongly duo module. Moreover, every ideal of A contained in $Z(E)$ is a d_E-ideal, Theorem 3.5 gives that R is an AA-ring. This completes the proof. □

Proposition 3.9. *Let (A, M) be a local ring which is not a field, E be a nonzero A-module with $ME = 0$, and $R := A \ltimes E$ the trivial ring extension of A by E. Then R is never an AA-ring.*

Proof. Let I be a proper ideal of A, then because $ME = 0$, $I \ltimes 0$ is an ideal of R contained in $Z(R)$. Let $x \in I$. It obvious to show that $Ann(x, 0) = Ann(x) \ltimes E$ and so $Ann^2(x, 0) = Ann^2(x) \ltimes E$. Thus, $I \ltimes 0$ is not a d-ideal. □

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