RATIONAL HOMOTOPY TYPE OF MAPPING SPACES BETWEEN COMPLEX PROJECTIVE SPACES AND THEIR EVALUATION SUBGROUPS

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Abstract. We use $L_\infty$ models to compute the rational homotopy type of the mapping space of the component of the natural inclusion $i_{n,k}: \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$ between complex projective spaces and show that it has the rational homotopy type of a product of odd dimensional spheres and a complex projective space. We also characterize the mapping aut$_1 \mathbb{C}P^n \to \text{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ and the resulting $G$-sequence.

1. Introduction

Let $f: X \to Y$ be a map between simply connected CW-complexes of finite type. We denote by map$(X,Y; f)$ the path component of $f$ in the space of continuous mappings from $X$ to $Y$. The study of the rational homotopy type of map$(X,Y; f)$ was initiated by Haefliger [10] who describes its Sullivan model. Afterwards there were attempts to find a Quillen model of map$(X,Y; f)$ from either a Sullivan or a Quillen model of $f$. Chain complexes of which the homology coincides with rational homotopy groups of function spaces were investigated [8,12,13]. Those chain complexes were later developed into models of function spaces [2–5].

Following [5] we describe in this paper an $L_\infty$ model of the inclusion $i_{n,k}: \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$. We shall use rational homotopy theory for which the standard reference is [6].

The notion of $L_\infty$-algebra was introduced by Lada [11] and we remind here the definition.

Definition 1. A permutation $\sigma \in S_n$ is called an $(i,n-i)$-shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$, where $i = 1, \ldots, n$. For graded objects $x_1, \ldots, x_n$, the Koszul sign $\epsilon(\sigma)$ is determined by

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma)x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}.$$

It depends not only of the permutation $\sigma$ but also of degrees of $x_1, \ldots, x_n$. 

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We assume that all vector spaces are over the field of rational numbers \( \mathbb{Q} \).

**Definition 2.** An \( L_\infty \)-algebra or a strongly homotopy Lie algebra is a graded vector space \( L = \oplus_{i \geq 0} L_i \) with maps \( \ell_k : L^\otimes k \to L \) of degree \( k-2 \) such that

1. \( \ell_k \) is graded skew symmetric, that is, for a \( k \)-permutation \( \sigma \)
   \[ \ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \text{sgn}(\sigma) \epsilon(\sigma) \ell_k(x_1, \ldots, x_k), \]
   where \( \text{sgn}(\sigma) \) is the sign of \( \sigma \).

2. There are some generalized Jacobi identities
   \[ \sum_{i+j=n+1} \epsilon(\sigma)(-1)^{(i-1)} \ell_j(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \]
   where the summation extends to all \( (i, n-i) \) shuffles of the symmetric group \( S_n \).

If \( \ell_k = 0 \) for \( k \geq 3 \), one retrieves the definition of a graded differential Lie algebra \( (L, d) \) where \( d = \ell_1 \) and \( \ell_2 \) is the Lie bracket.

Let \( (L, \ell_k) \) be an \( L_\infty \) algebra and \( sL \) the suspension of \( L \), and \( C_\infty(L) = (\wedge sL, d) \) the generalized Cartan-Chevalley-Eilenberg functor (see [6, §22]). One gets linear mappings \( d_k : \wedge^k (sL) \to sL \) defined by

\[ d_k(sx_1 \wedge \cdots \wedge sx_k) = (-1)^{\frac{k(k-1)}{2}} \ell_k(x_1, \ldots, x_k), \]

each of which extends into a codifferential on the coalgebra \( \wedge sL \). This gives an equivalence between \( L_\infty \) structures on \( L \) and codifferentials on \( \wedge sL \) [11]. Moreover if \( L \) is of finite type, then \( C_\infty(L) = (\wedge (sL)^\#, d) \) is a commutative differential graded algebra (cdga for short). The differential \( d = d_1 + \cdots + d_k + \cdots \) is defined by

\[ (dv, sx_1 \wedge \cdots \wedge sx_k) = (-1)^{v} (v, \ell_k(x_1, \ldots, x_k)), \]

where \( v \in (sL)^\# \) and \( \epsilon = \sum_{i=1}^{k-1} (k-i)|x_i| \).

**Definition 3.** Two cdga’s \( (A, d) \) and \( (B, d) \) have the same homotopy type if they are linked by a sequence of quasi-isomorphisms

\[ (A, d) = A_0 \to A_1 \leftarrow A_2 \cdots \to A_{n-1} \leftarrow A_n = (B, d). \]

Let \( V \) be a graded vector space. A Sullivan algebra \( (\wedge V, d) \) is the free graded commutative algebra generated by \( V \) together with a filtration \( V(0) \subset V(1) \subset \cdots \subset V \) such that \( dV(i) \subset \wedge^i V(i-1) \). It is called minimal if \( dV \subset \wedge^2 V \). A Sullivan model of a simply connected space \( X \) is a Sullivan algebra \( (\wedge V, d) \) such that there exists a quasi-isomorphism \( \varphi : (\wedge V, d) \to A_{PL}(X) \), where \( A_{PL}(X) \) denotes the cdga of piecewise linear forms of \( X \) [16]. A cdga model of \( X \) is a cdga \( (A, d) \) which has the same homotopy type as \( A_{PL}(X) \).

**Definition 4.** If \( f : X \to Y \) is a map between simply connected spaces of finite type, then there is a cdga map \( \phi : (\wedge V, d) \to (B, d) \), called a model of \( f \), where \( (B, d) \) and \( (\wedge V, d) \) are respective cdga models of \( X \) and \( Y \), respectively.
Definition 5. Let $L$ be an $L_\infty$-algebra of finite type. Then $L$ is called an $L_\infty$ model of a topological space $X$ if $C^\infty(L)$ is a Sullivan model of $X$. It is minimal if $\ell_1 = 0$. In this case $\pi_* (\Omega X) \otimes \mathbb{Q} \cong L$.

In this note, we give another proof of the following result using $L_\infty$ models of function spaces (see [15], Example 3.4).

Theorem 6. The function space map($\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k}$) has the rational homotopy type of $\mathbb{C}P^k \times S^{2k+3} \ldots \times S^{2(n+k)+1}$.

Moreover we study evaluation subgroups of the mapping aut$_1 \mathbb{C}P^n \to \mathbb{C}P^{n+k}$ and prove the following result.

Theorem 7. The $G$-sequence associated with the inclusion
\[ \text{aut}_1 \mathbb{C}P^n \to \text{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k}) \]
is not exact.

2. $L_\infty$-models of function spaces

Definition 8. Let $\phi : (\land V, d) \to (B, d)$ be a morphism of cdga’s. A $\phi$-derivation of degree $k$ is a linear mapping $\theta : (\land V)^n \to B^{n-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{|a|}\phi(a)\theta(b)$. We denote by Der($\land V, B; \phi$) the $\mathbb{Z}$-graded vector space of all $\phi$-derivations. The differential on Der($\land V, B; \phi$) is defined by $\delta\theta = d\theta - (-1)^k\theta d$.

Define $\overline{\text{Der}}(\land V, B; \phi)$ as
\[ \overline{\text{Der}}(\land V, B, \phi)_i = \begin{cases} \text{Der}(\land V, B; \phi)_i, & i > 1, \\ \{ \theta \in \text{Der}(\land V, B; \phi)_1 : \delta\theta = 0 \}, & i = 1. \end{cases} \]

If $\varphi_1, \ldots, \varphi_k \in \overline{\text{Der}}(\land V, B; \phi)$ are $\phi$-derivations of respective degrees $n_1, \ldots, n_k$, define
\[ [\varphi_1, \ldots, \varphi_k](v) = (-1)^{n_1 + \cdots + n_k - 1} \sum_{i_1, \ldots, i_k} \epsilon\phi(v_1 \cdots \hat{v}_{i_1} \cdots \hat{v}_{i_k} \cdots v_m)\varphi_1(v_{i_1}) \cdots \varphi_k(v_{i_k}) \]
where $dv = \sum v_1 \cdots v_m$ and $\epsilon$ is the corresponding Koszul sign of the permutation
\[ (\varphi_1, \ldots, \varphi_k, v_1, \ldots, v_m) \to (v_1, \ldots, \hat{v}_{i_1}, \ldots, \hat{v}_{i_k}, \ldots, v_m, \varphi_1, v_{i_1}, \ldots, \varphi_k, v_{i_k}). \]

We note that $[\varphi_1, \ldots, \varphi_k]$ is of degree $n_1 + \cdots + n_k - 1$. Now define linear maps $\ell_k$ of degree $k - 2$ on $s^{-1}\overline{\text{Der}}(\land V, B, \phi)$ by
\[ \ell_1(s^{-1}\varphi) = -s^{-1}\delta\varphi, \quad \ell_k(s^{-1}\varphi_1, \ldots, s^{-1}\varphi_k) = (-1)^k s^{-1} [\varphi_1, \ldots, \varphi_k], \]
where $\epsilon_k = \sum_{i=1}^{k-1} (k - i)|\varphi_i|$.

Proposition 9 (Lemma 3.3,[5]). If $\phi : (\land V, d) \to (B, d)$ is a Sullivan model of a mapping $f : X \to Y$ between simply connected spaces and $V$ is finite dimensional, then $(s^{-1}\overline{\text{Der}}(\land V, B; \phi), \ell_k)$ is an $L_\infty$ model of map($X, Y; f$).
3. Component of the inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+k}$

Recall that the minimal Sullivan model of $\mathbb{C}P^n$ is given by $(\wedge(x_2, x_{2n+1}), d)$ where $dx_2 = 0, dx_{2n+1} = x_2^{n+1}$. Our objective is to compute an $L_\infty$ model of the component of the inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+k}$. For $k = 0$, one gets a model of $\operatorname{aut} \mathbb{C}P^n = \operatorname{map}(\mathbb{C}P^n, \mathbb{C}P^n; \operatorname{Id})$ from the differential Lie algebra $(L, \delta)$ of derivations of $(\wedge(x_2, x_{2n+1}), d)$, of which $H_*(L, \delta)$ is spanned by $\{z_2, z_5, \ldots, z_{2n+1}\}$ [7, §3]. Therefore $\operatorname{aut} \mathbb{C}P^n$ has the rational homotopy type of the product $S^2 \times S^2 \times \cdots \times S^{2n+1}$. This result was also proved by Møller and Raussen using another method [15, Example 3.4].

Let $f : (\wedge, d) \to (B, d)$ be a morphism of differential graded algebras. For $v \in V$ and $b \in B$ we denote by $(v, b)$ the unique $f$-derivation $\theta$ such that $\theta(v) = b$ and zero on the remaining generators of $\wedge$.

From now on we assume that $k \geq 1$. A model of the inclusion $i_{n,k} : \mathbb{C}P^n \to \mathbb{C}P^{n+k}$ is given by

$$\psi : (A, d) = (\wedge(x_2, x_{2n+2k+1}), d) = (B, d),$$

where $\psi(x_2) = y_2, \psi(x_{2n+2k+1}) = y_2^k y_{2n+1}$. We consider the composition

$$\phi : A = (\wedge(x_2, x_{2n+2k+1})) \xrightarrow{\psi} (\wedge(y_2, y_{2n+1}), d) = B \simeq (\wedge(y_2)/\langle y_2^{n+1} \rangle, 0).$$

Hence $\phi(x_2) = y_2$ and $\phi(x_{2n+2k+1}) = 0$. The induced map

$$\operatorname{Der}(A, B; \psi, \delta) \to \operatorname{Der}(A, H^*(B); \phi, \delta)$$

is a quasi-isomorphism [1]. In the sequel we compute

$$\operatorname{Der}(\wedge(x_2, x_{2n+2k+1}), \wedge(y_2)/\langle y_2^{n+1} \rangle; \phi)$$

and determine its brackets. As a vector space

$$\operatorname{Der}(\wedge(x_2, x_{2n+2k+1}), \wedge(y_2)/\langle y_2^{n+1} \rangle; \phi)$$

is spanned by

$$\{\beta_2, \alpha_{2k+2i-1}, i = 1, \ldots, n+1\},$$

where $\alpha_{2k+2i-1} = (x_{2n+2k+1}, y_2^{n+i+1})$ and $\beta_2 = (x_2, 1)$. Note that $|\beta_2| = 2$ and $|\alpha_{2k+2i-1}| = 2k + 2i - 1$. Computations show that the only non zero brackets are given by $k+i \choose \beta_2, \ldots, \beta_2 = \alpha_{2k+2i-1}$ for $i = 1, \ldots, n+1$.

We deduce the following result (see [15] for a different proof).

**Proposition 10.** The function space $\operatorname{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ has a Sullivan model of the form

$$(\wedge(z_2, z_{2k+1}, \ldots, z_{2k+2n+1}), d),$$

where $dz_2 = 0, dz_{2k+1} = z_2^{k+1}, \ldots, dz_{2k+2n+1} = z_2^{k+n+1}.$
Proof. An $L_\infty$ model $(L,\ell_k)$ of map$(\mathbb{C}P(n), \mathbb{C}P(n+k); i_{n,k})$ is spanned by
$$\langle s^{-1}\beta_2, s^{-1}\alpha_{2k+2i-1}, i = 1, \ldots, n+1 \rangle.$$ Moreover $\ell_j = 0$ for $j = 1, \ldots, k$ and $\ell_{k+1}(s^{-1}\beta_2, \ldots, s^{-1}\beta_2) = s^{-1}\alpha_{2k+2i-1}$, for $i = 1, \ldots, n + 1$. Therefore
$$C^\infty(L) = \wedge(z_2, z_{2k+1}, z_{2k+3}, \ldots, z_{2k+2n+1}, d), \quad dz_2 = 0, \quad dz_{2k+2i+1} = z_{2k+2i+1},$$
where $0 \leq i \leq n$.

**Theorem 11.** The function space map$(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ has the rational homotopy type of $\mathbb{C}P^k \times S^{2k+3} \times \cdots \times S^{2(n+k)+1}$. 

**Proof.** By the above result, a Sullivan model of map$(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ is given by
$$\langle (x_2, x_{2k+1}, x_{2k+3}, \ldots, x_{2n+2k+1}) \rangle,$$
where $dx_2 = 0, dx_{2i+1} = x_{2i+1}^{k+1}, i = k, k+1, \ldots, k+n$. We consider the relative Sullivan model
$$\langle (x_2, x_{2k+1}) , d \rangle \to \langle (x_2, x_{2k+1}) \otimes \wedge x_{2k+3}, D \rangle,$$
where
$$dx_2 = 0, dx_{2k+1} = x_2^{k+1}, D x_2 = dx_2, D x_{2k+1} = dx_{2k+1}, D x_{2k+3} = x_2^{k+2}. $$
It is a Sullivan model of the fibration $S^{2k+3} \to E P\mathbb{C}P^k$, where $p$ is classified by a map $f : \mathbb{C}P^k \to B \text{aut}_1 S^{2k+3}$. Using the algebra of derivations on the minimal Sullivan model of $S^{2k+3}$ [16], it is easily seen that $B \text{aut}_1 S^{2k+3}$ has the rational homotopy type of $K(Q, 2k+4)$ [7, Proposition 2.1].

Moreover equivalence classes
$$[\mathbb{C}P^k, K(Q, 2k+4)]$$
are in a bijective correspondence with $H^{2k+1}(\mathbb{C}P^k, Q) = \{0\}$. Therefore the classifying map $f$ is rationally trivial. So we deduce that the fibration is trivial. Hence the cdga
$$(A,d) = \langle (x_2, x_{2k+1}, x_{2k+3}) , d \rangle, \quad dx_2 = 0, \quad dx_{2k+1} = x_2^{k+1}, \quad dx_{2k+3} = x_2^{k+2}$$
and
$$\langle (x_2, x_{2k+1}) \otimes \wedge x_{2k+3}, d \rangle, \quad dx_2 = 0, \quad dx_{2k+1} = x_2^{k+1}, \quad dx_{2k+3} = 0$$
are isomorphic. We deduce that the cdga $(A,d)$ is a Sullivan model of $\mathbb{C}P^k \times S^{2k+3}$. It follows from an induction argument that map$(\mathbb{C}P^k, \mathbb{C}P^{n+k}; i_{n,k})$ has the rational homotopy type of $\mathbb{C}P^k \times S^{2k+3} \times \cdots \times S^{2(n+k)+1}$. 

Recall that a Sullivan algebra $(\wedge V, d)$ is called formal if there is a quasi-isomorphism $(\wedge V, d) \to H^*(\wedge V, d)$. Spheres and complex projective spaces are formal. Moreover a product of formal spaces is also formal. We deduce that:

**Corollary 12.** The function space map$(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ is formal.
4. Evaluation subgroups of the inclusion \( i_{n,k} : \mathbb{C}P^n \to \mathbb{C}P^{n+k} \)

We consider the inclusion \( i_{n,k} : \mathbb{C}P^n \to \mathbb{C}P^{n+k} \) and the corresponding Sullivan model \( \phi \) of the previous section given by the composition

\[
\phi : A = (\wedge(x_2, x_{2n+2k+1}), d) \xrightarrow{\phi} (y_2, y_{2n+1}), d) = B \xrightarrow{\gamma} H^*(B).
\]

Forgetting the desuspension, a model of the inclusion \( (i_{n,k})_* : \text{aut}_1 \mathbb{C}P^n \to \text{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k}) \) is given by

\[
\phi^* : (\text{Der}(B, H^*(B); \gamma), \delta) \to (\text{Der}(A, H^*(B); \phi), \delta).
\]

We now characterize the map \( \phi^* \) when \( k > n \).

**Theorem 13.** If \( k > n \), then the induced map

\[
\phi^* : (\text{Der}(B, H^*(B); \gamma), \delta) \to (\text{Der}(A, H^*(B); \phi), \delta)
\]

is homotopy trivial.

**Proof.** We note that \( L = \text{Der}(B, H^*(B); \gamma) \) is spanned by

\[
\{\delta_2, \theta_1, \theta_3, \ldots, \theta_{2n+1}\},
\]

where \( \delta_2 = (y_2, 1), \theta_{2i+1} = (y_{2n+1}, y_2) \), \( i = 0, \ldots, n \). The differential is given by \( \delta \delta_2 = (n+1)\theta_1 \) and zero otherwise. Therefore

\[
\pi_\ast(\text{aut}_1 \mathbb{C}P^n) \otimes \mathbb{Q} = H_\ast(L, \delta) = ([\theta_3], \ldots, [\theta_{2n+1}]).
\]

Hence \( \text{aut}_1 \mathbb{C}P^n \) has the rational homotopy type of \( S^3 \times S^3 \times \cdots \newline \times S^{2n+1} \). Let

\[
L' = (\text{Der}(A, H^*(B), \phi), \delta) = ([\beta_2, \alpha_{2k+1}, \ldots, \alpha_{2n+2k+1}], \delta).
\]

The mapping \( \phi^* : L \to L' \) is defined by \( \phi^*(\delta_2), \phi^*(\theta_{2i+1}) = 0 \) for \( i < k \), and \( \phi^*(\theta_{2i+1}) = \alpha_{2i+1} \) for \( i \geq k \). If \( k > n \), then \( \phi^*(\delta_2) = \beta_2 \) and zero otherwise. Moreover

\[
C^\infty(s^{-1}L) = (\wedge(x_2, y_1, \ldots, y_{2i-1}, \ldots, y_{2n+1}), d),
\]

where \( dx_2 = 0 \) and \( dy_{2i-1} = x_2^i \). In particular \( dy_1 = x_2 \). In the same way

\[
C^\infty(s^{-1}L') = (\wedge(u_2, v_{2k+1}, \ldots, v_{2n+2k+1}), d),
\]

where \( du_2 = 0 \), \( dv_{2i+1} = u_2^{i+1} \). Hence

\[
\Phi = C^\infty(\phi^*) : C^\infty(s^{-1}L') \to C^\infty(s^{-1}L)
\]

is defined by \( \Phi(u_2) = x_2 \) and vanishes on other generators. As \( C^\infty(s^{-1}L') \) is quasi-isomorphic to

\[
(\wedge(w_2, w_{2k+1}), d) \otimes (\wedge(w_{2k+3}, \ldots, w_{2n+2k+1}), 0),
\]

where \( dw_2 = 0 \), \( dw_{2i+1} = w_2^{i+1} \) and, \( C^\infty(s^{-1}L) \) is quasi-isomorphic to

\[
(\wedge(z_3, \ldots, z_{2n+1}), 0),
\]

then induced map

\[
\tilde{\Phi} : (\wedge(w_2, w_{2k+1}, w_{2k+3}, \ldots, w_{2n+2k+1}), d) \to (\wedge(z_3, \ldots, z_{2n+1}), 0)
\]
between minimal Sullivan models is zero. □

**Definition 14.** Let \( X \) be a topological space. We say \( \alpha \in \pi_n(X) \) is a Gottlieb element if the map: \( f \vee 1_X : S^n \vee X \to X \) extends to \( S^n \times X \), where \( f \) represents the homotopy class \( \alpha \) [9].

Gottlieb elements form a subgroup of \( \pi_*(X) \) which will be denoted by \( G_*(X) \). It comes from the definition that \( G_*(X) \) is the image of \( \pi_*(ev) : \pi_*(\text{aut}_1 X, 1_X) \to \pi_*(X, x_0) \), where \( ev \) is the evaluation map at \( x_0 \). If \( f : X \to Y \), then \( G_*(Y, X; f) \) is the image of \( \pi_*(ev) \) where \( ev : \text{map}(X, Y; f) \to Y \) is the evaluation map at the base point.

Let \( (\wedge V, d) \) be the minimal Sullivan model of a simply connected space \( X \). Define the Gottlieb group of \( (\wedge V, d) \)

\[
G_n(\wedge V, d) = \{ [\theta] \in H_n(\text{Der} \wedge V, \delta) : \theta(v) = 1, \ v \in V^n \}
\]

Hence \( G_*(\wedge V, d) \cong \text{im} H_n(\epsilon_*) \), where \( \epsilon_* : \text{Der} \wedge V \to \text{Der}(\wedge V, \mathbb{Q}; \epsilon) \) is the post composition with the augmentation map \( \epsilon : \wedge V \to \mathbb{Q} \). Then \( G_n(\wedge V) \cong G_n(X_\mathbb{Q}) \), where \( h : X \to X_\mathbb{Q} \) is the rationalization [6, Proposition 29.8]. There are also relative Gottlieb groups \( G_*^{rel}(Y, X; f) \) and a \( G \)-sequence

\[
\cdots \to G_{n+1}^{rel}(Y, X; f) \to G_n(X) \to G_n(Y, X; f) \to \cdots
\]

which was introduced by Lee and Woo. The sequence is exact in some cases, for instance if \( f \) has a left homotopy inverse [17]. We follow the description of rational evaluation homotopy groups as given by Lupton and Smith [12].

Using augmentation maps we obtain the commutative diagram.

\[
\begin{array}{ccc}
\text{Der}(B, H^*(B); \gamma) & \xrightarrow{\phi^*} & \text{Der}(A, H^*(B); \phi) \\
\downarrow{\epsilon_*} & & \downarrow{\epsilon_*} \\
\text{Der}(B, \mathbb{Q}; \epsilon) & \xrightarrow{\bar{\phi}^*} & \text{Der}(A, \mathbb{Q}; \epsilon)
\end{array}
\]

In the same way we define \( G_*(A, H^*(B); \phi) \) as the image of \( H_*(\epsilon_*) \) in \( H_*(\text{Der}(A, \mathbb{Q}, \epsilon)) \).

In order to define relative rational Gottlieb groups, we recall that if \( \phi : (C, d_C) \to (C', d_{C'}) \) is a map of chain complexes, the mapping cone of \( \phi \), denoted by Rel(\( \phi \)), is the complex of which the underlying graded vector space is \( sC \oplus C' \) and the differential is given by \( D(sx, y) = (-sdc(x), \phi(x) + d_{C'} y) \) [12] or [14, p. 46]. Define chain maps \( J : C'_n \to \text{Rel}_n(\phi) \) and \( P : \text{Rel}_n(\phi) \to C_{n-1} \) by \( J(y) = (0, y) \) and \( P(sz, y) = x \). This yields an exact sequence of chain complexes

\[
0 \to C'_* \xrightarrow{J} \text{Rel}_n(\phi) \xrightarrow{P} C_{n-1} \to 0,
\]

which induces a long exact sequence in homology [14, Proposition 4.3]. We consider the mapping cone Rel(\( \phi^* \)) of

\[
\phi^* : (\text{Der}(B, H^*(B), \gamma), \delta) \to (\text{Der}(A, H^*(B), \phi), \delta),
\]
Rel(ϕ*) the mapping cone of ϕ* : Der(B, Q; e) → Der(A, Q; e) and the induced map (εs, εt) : Rel(ϕ*) → Rel(ϕ*). The relative Gottlieb group Grel*(A, B; ϕ) is the image of Hi(εs, εt). From the tower

\[
\begin{array}{c}
0 \longrightarrow \text{Der}(A, H^*(B); \phi) \xrightarrow{J} \text{Rel}(\phi^*) \xrightarrow{P} \text{Der}(B, H^*(B); \gamma) \longrightarrow 0 \\
0 \longrightarrow \text{Der}(A, Q; e) \xrightarrow{J} \text{Rel}(\phi^*) \xrightarrow{P} \text{Der}(B, Q; e) \longrightarrow 0
\end{array}
\]

one gets a sequence

\[
\cdots \rightarrow G_{k+1}(B, H^*(B), \gamma) \rightarrow G_k(A, H^*(B), \phi^*) \rightarrow Grel^*(A, H^*(B), \phi^*) \rightarrow \cdots
\]
called G-sequence of ϕ.

**Proposition 15.** The G-sequence associated to the inclusion aut1 CP^n → aut(CP^n, CP^{n+k}; i_{n,k}) is not exact.

**Proof.** Clearly G*(B, H^*(B); γ) = {[(y_{2n+1}, 1)]} and similarly

\[G_s(A, H^*(B), \phi) = {[(x_2, 1)], [(x_{2n+2k+1}, 1)]}.

We consider first the case where k > n. Then the only non zero differential on Rel(ϕ*) = (sL ⊕ L', d) is given by

\[d(sθ_2, 0) = (-sθ_1, 0) + (0, ϕ*(δ_2)) = (-sθ_1, 0) + (0, β_2).

Similarly the only non zero differential on

\[\text{Rel}(\hat{ϕ}^*) = \{(s\gamma^2_2, 0), (s\gamma^2_{2n+1}, 0), (0, x_2^2), (0, x_{2n+2k+1}^2)\}

is \(d(s\gamma^2_2, 0) = (0, x_2^2)\). We conclude that

\[Grel^*(A, H^*(B), \phi) = {[(s\gamma^2_{2n+1}, 0), (0, x_{2n+2k+1}^2)]} \cong sG_*(\text{CP}^n) \oplus G_*(\text{CP}^{n+k}).

Hence in the G-sequence reduces to fragments

\[0 \rightarrow Grel^*_{2n+2}(A, H^*(B); \phi^*) \cong G_{2n+1}(B, H^*(B); γ) \rightarrow 0,
\]

\[0 \rightarrow G_{2n+2k+1}(A, H^*(B); \phi^*) \cong G_{2n+2k+1}^rel(A, H^*(B); \phi^*) \rightarrow 0
\]

and terminates with

\[0 \rightarrow G_2(A, H^*(B); \phi^*) \rightarrow 0.
\]

As G_2(A, H^*(B); ϕ*) \cong Q, we conclude that the last fragment of the G-sequence is not exact.

If k ≤ n, then ϕ*(θ_{2n+1}) = α_{2n+1}, hence \(d(sθ_{2n+1}, 0) = (0, α_{2n+1})\), therefore \([(s\gamma^2_{2n+1}, 0)] \in H_*(\text{Rel}(\hat{ϕ}^*))\) is not in the image of \(H_*(ε_s, ε_t)\). The only change in the G-sequence is the fragment

\[0 \rightarrow Grel^*_{2n+2}(A, H^*(B); \phi^*) \rightarrow 0,
\]

which in not exact as well, as \(Grel^*_{2n+2}(A, H^*(B)) \cong Q\). □
References


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