Commun. Korean Math. Soc. **37** (2022), No. 1, pp. 269–277
https://doi.org/10.4134/CKMS.c200443
pISSN: 1225-1763 / eISSN: 2234-3024

μ-COUNTABLY COMPACTNESS AND $\mu\mathcal{H}$-COUNTABLY COMPACTNESS

Zuhier Altawallbeh and Ibrahim Jawarneh

Abstract. We define and study the notion of μ-countably compact spaces in generalized topology and $\mu\mathcal{H}$-countably compact spaces which are considered with respect to a hereditary class $\mathcal{H}$. Some interesting properties and relations are provided in the paper. Moreover, some preservation of functions properties are studied and investigated.

1. Introduction

A collection μ of subsets of a nonempty set X is a generalized topology (GT) if $\phi \in \mu$ and any union of elements of μ belongs to μ, this notion was introduced by Császár in the sense of [4]. The elements of μ are called μ-open sets and their complements are called μ-closed sets, see [5]. In this paper, we denote the μ-open set by u^μ, the family $\{u^\mu_\alpha : \alpha \in \Delta\}$ is a μ-open cover of the set X if $X = \bigcup\{u^\mu_\alpha : \alpha \in \Delta\}$. In particular, X is a μ-compact space if every μ-open cover of X has a finite subcover, more generalizations can be seen in [3, 10], where some covering spaces are studied in the generalized topology with respect to a hereditary class $\mathcal{H}$, the hereditary class $\phi \neq \mathcal{H}$ on a set X is a collection of subsets of X that satisfies the following property: If $A \in \mathcal{H}$ and $B \subset A$, then $B \in \mathcal{H}$, see [6]. If the hereditary class $\mathcal{H}$ satisfies the additional condition: If $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$, then $\mathcal{H}$ is called an ideal on X, see [8]. We define hereditary generalized topological space $(X, \mu, \mathcal{H})$ as a generalized topological space (X, μ) with hereditary class $\mathcal{H}$. Assume that A is a nonempty subset of X. Then the generalized local function A^* of A with respect to $\mathcal{H}$ and μ is defined as $A^* = \{x \in X : \text{for all } u^\mu \in \mu(x), \text{ we have } u^\mu \cap A \notin \mathcal{H}\}$, $c^*_\mu(A) = A \cup A^*$ and there is a GT $\mu^* = \{A \subset X : X \setminus A = c^*_\mu(X \setminus A)\}$, see [6]. An element of μ^* is called μ^*-open and its complement is μ^*-closed, and $c^*_\mu(A)$ is the intersection of all μ^*-closed supersets of A. Recall that $c_\mu(A)$ is the intersection of all μ-closed supersets of A. Then $c_\mu(A)$ is μ-closed for $A \subset X$ and $x \in c_\mu(A)$ if and only if $x \in u^\mu \in \mu$ implies that $u^\mu \cap A \neq \phi$, see [9]. In this paper, we define μ-countably compact spaces and $\mu\mathcal{H}$-countably compact spaces as

Received November 23, 2020; Revised January 8, 2021; Accepted January 20, 2021.
2010 *Mathematics Subject Classification.* Primary 58B34, 58J42, 81T75.
Key words and phrases. μ-countably compactness, $\mu\mathcal{H}$-countably compactness.

generalizations of μ-compact spaces and $\mu\mathcal{H}$-compact spaces, respectively, with some interesting results regarding (X, μ^*) as a GTS and some relations in the sense of generalized topologies and hereditary classes are presented very well in logical order.

2. μ-countably compact and $\mu\mathcal{H}$-countably compact spaces

In this section, we introduce and study the notion of countably compact spaces in the sense of a GTS and a hereditary class with some interesting properties.

Definition. A GTS (X, μ) is said to be a μ-countably compact space if for every countable μ-open cover $\{u^\mu_\alpha : \alpha \in \Delta\}$ of X, there is a finite subset Λ of Δ such that $X = \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\}$

Definition. Let (X, μ) be a GTS. A subset A of X is said to be a μ-countably compact set if for every countable μ-open cover $\{u^\mu_\alpha : \alpha \in \Delta\}$ of A, there is a finite subset Λ of Δ such that $A \subset \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\}$.

Definition. A HGTS $(X, \mu, \mathcal{H})$ is said to be a $\mu\mathcal{H}$-countably compact space or a μ-countably compact space with respect to a hereditary class $\mathcal{H}$ if for every countable μ-open cover $\{u^\mu_\alpha : \alpha \in \Delta\}$ of X, there is a finite subset Λ of Δ such that $X \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$.

It is noted that $\mu\mathcal{H}$-countably compactness refers to $X \setminus \{u^\mu_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$. As an analogous point of view, μ-countably compactness refers to $X \setminus \{u^\mu_\alpha : \alpha \in \Lambda\} = \phi \in \mathcal{H}$. The assumption of countability remains as it is in these definitions. This kind of analogy together with the notion of hereditary classes motivates us to expect interesting generalizations and analogous properties of topological spaces in the sense of generalized topologies discussed in this paper.

Definition. Let $(X, \mu, \mathcal{H})$ be a HGTS. A subset A of X is said to be a $\mu\mathcal{H}$-countably compact set if for every countable μ-open cover $\{u^\mu_\alpha : \alpha \in \Delta\}$ of A, there is a finite subset Λ of Δ such that $A \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$.

It is clear that if a GTS (X, μ) is a μ-countably compact space, then a HGTS $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space for any hereditary class $\mathcal{H}$. The following example shows that the converse may not be true. But if we use certain hereditary classes, then we can get the converse as in Theorem 2.2.

Example 2.1. Let $\mu = \{u \subseteq \mathbb{R} : u \text{ is uncountable}\} \cup \{\phi\}$ be a GTS on $\mathbb{R}$ and $\mathcal{H} = \{\mathbb{R} \setminus u : u \in \mu\}$ be a hereditary class on $\mathbb{R}$. It is clear that the countable μ-open cover $\{(-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ has no a finite subcover, so $(\mathbb{R}, \mu, \mathcal{H})$ is not a μ-countably compact space. Now, let $\{u^\mu_\alpha : \alpha \in \Delta\}$ be a countable μ-open cover of $\mathbb{R}$ then it is obvious that if we pick $u^\mu_{\alpha_\circ}$ for any $\alpha_\circ \in \Delta$, we get $\mathbb{R} \setminus u^\mu_{\alpha_\circ} \in \mathcal{H}$ which means $(\mathbb{R}, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-countably compact.

The relation between μ-countably compact spaces and $\mu\mathcal{H}_f$-countably compact spaces can be expressed as follows.

Theorem 2.2. *Let $\mathcal{H}_f$ be the set of all finite subsets of a nonempty set X and μ be a generalized topology on X. The (X, μ) is a μ-countably compact space if and only if $(X, \mu, \mathcal{H}_f)$ is a $\mu\mathcal{H}_f$-countably compact space.*

Proof. The first direction is direct. To prove the second direction, let $(X, \mu, \mathcal{H}_f)$ be a $\mu\mathcal{H}_f$-countably compact space and $\{u^\mu_\alpha : \alpha \in \Delta\}$ be a countable μ-open cover of X. From the assumption, there is a finite subset Λ of Δ such that $X \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\} \in \mathcal{H}_f$. So, $X \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\} = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality and for each x_i, $1 \leq x_i \leq n$, choose $u^\mu_{\alpha_i}$ such that $x_i \in u^\mu_{\alpha_i}$.Thus, $X = (\bigcup\limits_{\alpha\in\Lambda} u^\mu_\alpha) \cup (\bigcup\limits_{i=1}^{n} u^\mu_{\alpha_i})$ and so (X, μ) is a μ-countably compact space. □

The proof of these theorems are straightforward and thus omitted.

Theorem 2.3. *Let (X, μ) be a μ-countably compact space. If A is a μ-closed subset of X, then A is μ-countably compact.*

Theorem 2.4. *Let $(X, \mu, \mathcal{H})$ be a $\mu\mathcal{H}$-countably compact space. If A is a μ-closed subset of X, then A is $\mu\mathcal{H}$-countably compact.*

The next theorem is crucial for our considerations.

Theorem 2.5. *A GTS (X, μ) is a μ-countably compact space if and only if for any countable family $\{F_\alpha : \alpha \in \Delta\}$ of μ-closed subsets of X having the property that $\bigcap\{F_\alpha : \alpha \in \Lambda\} \neq \phi$ for every finite subset Λ of Δ, then $\bigcap\{F_\alpha : \alpha \in \Delta\} \neq \phi$.*

An interesting characterization of $\mu\mathcal{H}$-countably compact spaces can be determined by the following theorem.

Theorem 2.6. *A HGTS $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space if and only if for any countable family $\{F_\alpha : \alpha \in \Delta\}$ of μ-closed subsets of X having the property that $\bigcap\{F_\alpha : \alpha \in \Lambda\} \notin \mathcal{H}$ for every finite subset Λ of Δ, then $\bigcap\{F_\alpha : \alpha \in \Delta\} \neq \phi$.*

Proof. Assume that $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space and $\{F_\alpha : \alpha \in \Delta\}$ is a countable family of μ-closed subsets of X having the property that $\bigcap\{F_\alpha : \alpha \in \Lambda\} \notin \mathcal{H}$ for every subset Λ of Δ. Now, if $\bigcap\{F_\alpha : \alpha \in \Delta\} = \phi$, then $\{X \setminus F_\alpha : \alpha \in \Delta\}$ is a countable μ-open cover of X. Since $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space, there is a finite subset Λ of Δ such that $X \setminus \bigcup\{X \setminus F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$ and so $\bigcap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$ which contradicts the assumption. Thus $\bigcap\{F_\alpha : \alpha \in \Delta\} \neq \phi$. Conversely, let $\{u^\mu_\alpha : \alpha \in \Delta\}$ be a countable μ-open cover of X. Assume that for any finite subset Λ of Δ, we have $X \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\} \notin \mathcal{H}$, which means that $\{X \setminus u^\mu_\alpha : \alpha \in \Lambda\}$ is a countable family of μ-closed subsets of X where $\bigcap\{X \setminus u^\mu_\alpha : \alpha \in \Lambda\} \notin \mathcal{H}$, and by the assumption $\bigcap\{X \setminus u^\mu_\alpha : \alpha \in \Delta\} \neq \phi$, this is contradiction to the fact that $\{u^\mu_\alpha : \alpha \in \Delta\}$ is a cover of X. Thus $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space. □

The proof of the following theorem is similar to the above theorem and thus omitted.

Theorem 2.7. *Let $(X, \mu, \mathcal{H})$ be a HGTS. A subset A of X is a $\mu\mathcal{H}$-countably compact set if and only if for any countable family $\{F_\alpha : \alpha \in \Delta\}$ of μ-closed subsets of X having the property that $(\bigcap\{F_\alpha : \alpha \in \Lambda\}) \cap A \notin \mathcal{H}$ for every finite subset Λ of Δ, then $(\bigcap\{F_\alpha : \alpha \in \Delta\}) \cap A \neq \phi$.*

The next definition of θ_μ-accumulation point can be used to get more results in this research.

Definition ([13])**.** Let (X, μ) be a GTS and A be a subset of X. A point $x \in X$ is called a θ_μ-accumulation point of A if $c_\mu(u^\mu) \cap A \neq \phi$ for every μ-open subset u^μ of X that contains x. The set of all θ_μ-accumulation points of A is called the θ_μ-closure of A and its denoted by $(c_\mu)_\theta(A)$. Moreover, A is said to be μ_θ-closed if $(c_\mu)_\theta(A) = A$. The complement of a μ_θ-closed is called μ_θ-open.

It is observed that, if $x \in X$ is an accumulation (limit) point of $A \subseteq X$, then x is a θ_μ-accumulation point of A. Also, since $(c_\mu)_\theta(A)$ is the set of all θ_μ-accumulation points of A and $u^\mu \subseteq c_\mu(u^\mu)$, we deduce that $c_\mu(A) \subseteq (c_\mu)_\theta(A)$ for any $A \subseteq X$.

Lemma 2.8 ([13])**.** *Let (X, μ) be a GTS. A subset A of X is μ_θ-open if and only if for each $x \in A$, there exists a μ-open set u^μ such that $x \in u^\mu \subseteq c_\mu(u^\mu) \subseteq A$.*

Proof. Let A be a μ_θ-open subset of X and $x \in A$. That means, A is a μ_θ-closed. So, there is $u^\mu \in \mu$ and $x \in u^\mu$ such that $c_\mu(u^\mu) \cap A = \phi$. Thus, $x \in u^\mu \subseteq c_\mu(u^\mu) \subseteq A$. Conversely, let $x \notin (X \setminus A)$. That means, $x \in A$, and by the assumption there is $u^\mu \in \mu$ and $x \in u^\mu$ such that $x \in u^\mu \subseteq c_\mu(u^\mu) \subseteq A$. So, $c_\mu(u^\mu) \cap (X \setminus A) = \phi$. From the definition of θ_μ-closure, we get $x \notin (c_\mu)_\theta(X \setminus A)$. Thus, $(c_\mu)_\theta(X \setminus A) \subseteq X \setminus A$. This gives the fact that, $X \setminus A$ is μ_θ-closed and so A is μ_θ-open. □

Theorem 2.9. *If a HGTS $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space, then for every countable cover of μ_θ-open sets $\{u^\mu : \alpha \in \Delta\}$ of X there is a finite subset Λ of Δ such that $X \setminus \bigcup\{u^\mu : \alpha \in \Lambda\} \in \mathcal{H}$.*

Proof. Assume that $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space and $\{u^\mu : \alpha \in \Delta\}$ is a countable cover of X by μ_θ-open sets. Then for each $x \in X$, there is $\alpha_x \in \Delta$ such that $x \in u^\mu_{\alpha_x}$. Applying Lemma 2.8, there is a μ-open set $v^\mu_{\alpha_x}$ such that $x \in v^\mu_{\alpha_x} \subseteq c_\mu(v^\mu_{\alpha_x}) \subseteq u^\mu_{\alpha_x}$. Since $v^\mu_{\alpha_x}$ depends on $u^\mu_{\alpha_x}$ for every $\alpha_x \in \Delta$, then the family $\{v^\mu_{\alpha_x} : x \in X\}$ is a countable cover of X by μ-open sets. From the assumption, the space $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-countably compact space and so there is a finite subset Λ of Δ such that $X \setminus \bigcup\{v^\mu_{\alpha_x} : \alpha_x \in \Lambda\} \in \mathcal{H}$. So, $v^\mu_{\alpha_x} \subseteq u^\mu_{\alpha_x}$ for every $\alpha_x \in \Lambda$ which means that $X \setminus \bigcup\{u^\mu_{\alpha_x} : \alpha_x \in \Lambda\} \subseteq X \setminus \bigcup\{v^\mu_{\alpha_x} : \alpha_x \in \Lambda\} \in \mathcal{H}$. Since, $\mathcal{H}$ is a hereditary class, we get $X \setminus \bigcup\{u^\mu_{\alpha_x} : \alpha_x \in \Lambda\} \in \mathcal{H}$ and this completes the proof. □

Theorem 2.10. *If a GTS (X, μ) is a μ-countably compact space, then for every countable cover of μ_θ-open sets $\{u^\mu_\alpha : \alpha \in \Delta\}$ of X there is a finite subcover.*

Theorem 2.11. *Let $(X, \mu, \mathcal{H})$ be a $\mu\mathcal{H}$-countably compact space. If A is a μ_θ-closed subset of X, then A is $\mu\mathcal{H}$-countably compact.*

In the following theorems we see that, if one of the spaces $(X, \mu, \mathcal{H})$ and $(X, \mu^*, \mathcal{H})$ is countably compact in the sense of μ and μ^* respectively, then the other one is countably compact.

Corollary 2.12. *If a HGTS $(X, \mu^*, \mathcal{H})$ is a $\mu^*\mathcal{H}$-countably compact space, then $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-countably compact.*

Proof. Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of μ-closed subsets of X such that $\bigcap\{F_\alpha : \alpha \in \Delta\} = \phi$. Since $A^* \subseteq c^*_\mu(A)$, we deduce that $\{F_\alpha : \alpha \in \Delta\}$ is a countable family of μ^*-closed sets. From the assumption and by using contrapositive in Theorem 2.6 there is a finite subset Λ of Δ such that $\bigcap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$ which means that $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-countably compact. □

By following the same technique of the proof of Corollary 2.12 and applying Theorem 2.5, we get the following result.

Corollary 2.13. *If a GTS (X, μ^*) is a μ^*-countably compact space, then (X, μ) is μ-countably compact.*

Theorem 2.14. *Let a HGTS $(X, \mu, \mathcal{H})$ be a $\mu\mathcal{H}$-countably compact space and the hereditary class $\mathcal{H}$ is an ideal. Then the space $(X, \mu^*, \mathcal{H})$ is $\mu^*\mathcal{H}$-countably compact.*

Proof. Let $\{u^{\mu^*}_\alpha : \alpha \in \Delta\}$ be a countable μ^*-covering of X. So for an arbitrary element $x \in X$, there is a μ^*-open subset $u^{\mu^*}_{\alpha_x} \in \mu^*$ such that $x \in u^{\mu^*}_{\alpha_x}$. By the definition of μ^*, there is such $v^\mu_{\alpha_x} \in \mu$ such that $x \in v^\mu_{\alpha_x}$ and $v^\mu_{\alpha_x} \cap (X \setminus u^{\mu^*}_{\alpha_x}) \in \mathcal{H}$. Setting $R_{\alpha_x} = v^\mu_{\alpha_x} \cap (X \setminus u^{\mu^*}_{\alpha_x})$ we get $v^\mu_{\alpha_x} \cap (X \setminus R_{\alpha_x}) = v^\mu_{\alpha_x} \cap \Big(X \setminus \big(v^\mu_{\alpha_x} \cap (X \setminus u^{\mu^*}_{\alpha_x})\big)\Big) = v^\mu_{\alpha_x} \cap u^{\mu^*}_{\alpha_x} \subseteq u^{\mu^*}_{\alpha_x}$. Since the family $\{v^\mu_{\alpha_x} : \alpha_x \in \Delta\}$ is a countable μ-covering of X and $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-countably compact, there is a finite subset Λ of Δ such that $(X \setminus \bigcup\{v^\mu_{\alpha_x} : \alpha_x \in \Lambda\}) \in \mathcal{H}$. From the fact that, $v^\mu_{\alpha_x} \cap (X \setminus R_{\alpha_x}) \subseteq u^{\mu^*}_{\alpha_x}$, we get $X \setminus u^{\mu^*}_{\alpha_x} \subseteq (X \setminus v^\mu_{\alpha_x}) \cup R_{\alpha_x}$. So, $(X \setminus \bigcup\big\{u^{\mu^*}_{\alpha_x} : \alpha_x \in \Lambda\big\}) \subseteq (X \setminus \bigcup\big\{v^\mu_{\alpha_x} : \alpha_x \in \Lambda\big\}) \cup (\bigcup\big\{R_{\alpha_x} : \alpha_x \in \Lambda\big\})$. Since $\mathcal{H}$ is closed under finite union and $R_{\alpha_x} \in \mathcal{H}$, we have $(X \setminus \bigcup\big\{v^\mu_{\alpha_x} : \alpha_x \in \Lambda\big\}) \cup (\bigcup\big\{R_{\alpha_x} : \alpha_x \in \Lambda\big\}) \in \mathcal{H}$. Since $\mathcal{H}$ is a hereditary class, we get $(X \setminus \bigcup\big\{u^{\mu^*}_{\alpha_x} : \alpha_x \in \Lambda\big\}) \in \mathcal{H}$. That means $(X, \mu^*, \mathcal{H})$ is $\mu^*\mathcal{H}$-countably compact.

□

The following example shows that the closedness under finite union of a given hereditary class $\mathcal{H}$ is a necessary condition in the above theorem.

Example 2.15. Let μ=$\{A \subseteq \mathbb{R} : A$ is infinite set$\} \cup \{\phi\}$ be a generalized topology on $\mathbb{R}$ and $\mathcal{H}$ =$\{A \subseteq \mathbb{R} : \mathbb{R} \backslash A \in \mu\}$ be a hereditary class on $\mathbb{R}$. It is clear that $\mathcal{H}$ is not an ideal and so it is not closed under finite union. Let $\{u^\mu_\alpha : \alpha \in \Delta\}$ be a countable μ-open cover of $\mathbb{R}$. For any finite subfamily $\{u^\mu_\alpha : \alpha \in \Delta_\circ \subseteq \Delta\}$, we get $(X \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Delta_\circ\}) \in \mathcal{H}$. So $(\mathbb{R}, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-countably compact. Moreover, for any $n \in \mathbb{N}$, we have $(\mathbb{R} \setminus (-n, n))^* \subseteq (\mathbb{R} \setminus (-n, n))$ and so $\{(-n, n) : n \in \mathbb{N}\}$ is a countable μ^*-covering of $\mathbb{R}$. Now, if there is a finite index set $\Delta' \subseteq \mathbb{N}$, then $(\mathbb{R} \setminus \bigcup\{(-n, n) : n \in \Delta'\}) = \big(\mathbb{R} \setminus -(\max\{n : n \in \Delta'\}, \max\{n : n \in \Delta'\})\big)$ which is an infinite set. That means $(\mathbb{R} \setminus \bigcup\{(-n, n) : n \in \Delta'\}) \notin \mathcal{H}$. Thus $(\mathbb{R}, \mu, \mathcal{H})$ is not a $\mu^*\mathcal{H}$-countably compact space.

Theorem 2.16. *Let (X, μ) be a GTS and $\mathcal{H}$ be an ideal on X. Then the union of any two $\mu\mathcal{H}$-countably compact sets of X is $\mu\mathcal{H}$-countably compact.*

Proof. Let A and B be two $\mu\mathcal{H}$-countably compact sets of X and $\{u^\mu_\alpha : \alpha \in \Delta\}$ be any countable μ-covering of $A \cup B$. Since A and B are $\mu\mathcal{H}$-countably compact sets of X, then there are two finite subfamilies $\{u^\mu_\alpha : \alpha \in \Delta_1 \subseteq \Delta\}$ and $\{u^\mu_\alpha : \alpha \in \Delta_2 \subseteq \Delta\}$ such that $(A \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Delta_1\}) \in \mathcal{H}$ and $(B \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Delta_2\}) \in \mathcal{H}$. Setting $\Lambda = \Delta_1 \cup \Delta_2$ and since $\Big((A \cup B) \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\}\Big) \subseteq \Big(A \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Delta_1\}\Big) \cup \Big(B \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Delta_2\}\Big)$ and $\mathcal{H}$ is an ideal with the fact that $A \bigcup B$ is finite, we get $\Big((A \cup B) \setminus \bigcup\{u^\mu_\alpha : \alpha \in \Lambda\}\Big) \in \mathcal{H}$. That means $A \cup B$ is $\mu\mathcal{H}$-countably compact. □

The proof of the following theorem is straightforward and thus omitted.

Theorem 2.17. *Let (X, μ) be a GTS. Then the union of any two μ-countably compact sets of X is μ-countably compact.*

Theorem 2.18. *Let $(X, \mu, \mathcal{H}_1 \cap \mathcal{H}_2)$ be a $\mu(\mathcal{H}_1 \cap \mathcal{H}_2)$-countably compact space for given two hereditary classes $\mathcal{H}_1$ and $\mathcal{H}_2$. Then $(X, \mu, \mathcal{H}_1)$ is a $\mu\mathcal{H}_1$-countably compact space and $(X, \mu, \mathcal{H}_2)$ is a $\mu\mathcal{H}_2$-countably compact space.*

The proof is straightforward if we take in account that the intersection of any two hereditary classes of a set is again a hereditary class of that set.

3. Functions on μ-countably compact and $\mu\mathcal{H}$-countably compact spaces

We study the effect of functions in the sense of generalized topologies on μ-countably compact spaces and $\mu\mathcal{H}$-countably compact spaces.

Definition ([12])**.** Let (X, μ_1) and (Y, μ_2) be two GTSs. Then a function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is said to be (μ_1, μ_2)-continuous if for every $V \in \mu_2$ then $f^{-1}(V) \in \mu_1$.

Definition ([11])**.** Let (X, μ_1) and (Y, μ_2) be two GTSs. Then a function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is said to be (μ_1, μ_2)-open if for every $U \in \mu_1$, then $f(U) \in \mu_2$.

The proof of the following theorem is straightforward and thus omitted.

Theorem 3.1. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be a (μ_1, μ_2)-continuous surjection function. If (X, μ_1) is a μ_1-countably compact space, then (Y, μ_2) is a μ_2-countably compact space.*

Theorem 3.2. *Let $(X, \mu_1, \mathcal{H})$ be a HGTS and (Y, μ_2) be a GTS. If*

$$f : (X, \mu_1, \mathcal{H}) \longrightarrow (Y, \mu_2)$$

is a function, then $f(\mathcal{H}) = \{f(A) : A \in \mathcal{H}\}$ is a hereditary class of Y.

The following corollary is a direct result from Theorem 3.2.

Corollary 3.3. *Let $f : (X, \mu_1, \mathcal{H}) \longrightarrow (Y, \mu_2)$ be a (μ_1, μ_2)-continuous surjection function. If $(X, \mu_1, \mathcal{H})$ is a $\mu_1\mathcal{H}$-countably compact space, then*

$$(Y, \mu_2, f(\mathcal{H}))$$

is $\mu_2 f(\mathcal{H})$-countably compact.

Corollary 3.4. *Let $f : (X, \mu_1, \mathcal{H}) \longrightarrow (Y, \mu_2)$ be a (μ_1, μ_2)-continuous surjection function. If Y is a finite space and $(X, \mu_1, \mathcal{H})$ is a $\mu_1\mathcal{H}$-countably compact space, then (Y, μ_2) is a μ_2-countably compact space.*

Proof. By the assumption and from Corollary 3.3, $(Y, \mu_2, f(\mathcal{H}))$ is a $\mu_2 f(\mathcal{H})$-countably compact space. Since Y is finite, the class $f(\mathcal{H})$ is of finite subsets and apply Theorem 2.2 to get (Y, μ_2) is a μ_2-countably compact space. □

As a reverse work of Corollary 1.28, we get this result via a (μ_1, μ_2)-open bijection.

Corollary 3.5. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2, \mathcal{H})$ be a (μ_1, μ_2)-open bijection. If $(Y, \mu_2, \mathcal{H})$ is a $\mu_2\mathcal{H}$-countably compact space, then*

$$(X, \mu_1, f^{-1}(\mathcal{H}))$$

is a $\mu_1 f^{-1}(\mathcal{H})$-countably compact space.

Proof. From the assumption, we get $f^{-1} : (Y, \mu_2, \mathcal{H}) \longrightarrow (X, \mu_1)$ is a (μ_2, μ_1)-continuous surjection. Since $(Y, \mu_2, \mathcal{H})$ is a $\mu_2\mathcal{H}$-countably compact space, by applying Corollary 3.4, we get the space $(X, \mu_1, f^{-1}(\mathcal{H}))$ is a $\mu_1 f^{-1}(\mathcal{H})$-countably compact space. □

4. Conclusions

We have introduced the notion of μ-countably compact spaces and $\mu\mathcal{H}$-countably compact spaces in the sense of generalized topology given in [4] and the notion of hereditary class $\mathcal{H}$. An example of a $\mu\mathcal{H}$-countably compact space which is not μ-countably compact is presented. Also, it is proved that these notions preserve hereditary property under μ-closedness. And interesting characterizations of μ-countably compact space and $\mu\mathcal{H}$-countably compact space are given in Theorems 2.5 and 2.6. Moreover, some other interesting results are given by using μ_θ-accumulation points [13]. In Section 3 of the paper, some preservations of function properties are studied and investigated.

As a future work, it is expected to extend our work to introduce more generalizations of countably compact spaces as in [1, 2] in the sense of generalized topology and hereditary classes, or by replacing generalized topology by weaker framework as weaker structures WS [7]. Although WS spaces are not closed under arbitrary union, some modifications can be made to get interesting and even analogous results obtained in this paper. In addition, the construction arising from a generalized topology and a hereditary class $\mathcal{H}$ remains valid, if the generalized topology μ is replaced by weaker structures WS as it is presented in [14]. In conclusion, this paper can be used to furnish more research in different paths of generalizations of μ-countably compactness via hereditary classes in future.

Acknowledgement. The authors would like to thank the referees for their valuable comments and suggestions to improve this paper.

References

[1] Z. Altawallbeh, *More on almost countably compact spaces*, Italian J. Pure Appl. Math. **43** (2020), 177–184.

[2] Z. Altawallbeh and A. Al-Momany, *Nearly countably compact spaces*, Int. Electron. J. Pure Appl. Math. **8** (2014), no. 4, 59–66. https://doi.org/10.12732/iejpam.v8i4.7

[3] C. Carpintero, E. Rosas, M. Salas-Brown, and J. Sanabria, *μ-compactness with respect to a hereditary class*, Bol. Soc. Parana. Mat. (3) **34** (2016), no. 2, 231–236. https://doi.org/10.5269/bspm.v34i2.27177

[4] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar. **96** (2002), no. 4, 351–357. https://doi.org/10.1023/A:1019713018007

[5] Á. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. **106** (2005), no. 1-2, 53–66. https://doi.org/10.1007/s10474-005-0005-5

[6] Á. Császár, *Modification of generalized topologies via hereditary classes*, Acta Math. Hungar. **115** (2007), no. 1-2, 29–36. https://doi.org/10.1007/s10474-006-0531-9

[7] Á. Császár, *Weak structures*, Acta Math. Hungar. **131** (2011), no. 1-2, 193–195. https://doi.org/10.1007/s10474-010-0020-z

[8] K. Kuratowski, *Topologie*, Warszawa. **I** (1933).

[9] A. Pavlović, *Local function versus local closure function in ideal topological spaces*, Filomat **30** (2016), no. 14, 3725–3731. https://doi.org/10.2298/FIL1614725P

[10] A. Qahis, H. H. AlJarrah, and T. Noiri, *μ-Lindelöfness in terms of a hereditary class*, Missouri J. Math. Sci. **28** (2016), no. 1, 15–24. http://projecteuclid.org/euclid.mjms/1474295352

[11] L. E. de Arruda Saraiva, *Generalized quotient topologies*, Acta Math. Hungar. **132** (2011), no. 1-2, 168–173. https://doi.org/10.1007/s10474-010-0047-1

[12] M. S. Sarsak, *Weak separation axioms in generalized topological spaces*, Acta Math. Hungar. **131** (2011), no. 1-2, 110–121. https://doi.org/10.1007/s10474-010-0017-7

[13] M. S. Sarsak, *Weakly μ-compact spaces*, Demonstratio Math. **45** (2012), no. 4, 929–938.

[14] A. M. Zahran, K. El-Saady, and A. Ghareeb, *Modification of weak structures via hereditary classes*, Appl. Math. Lett. **25** (2012), no. 5, 869–872. https://doi.org/10.1016/j.aml.2011.10.034

Zuhier Altawallbeh
Department of Mathematics
Tafila Technical University
Tafila 66110, Jordan
Email address: Zuhier1980@gmail.com

Ibrahim Jawarneh
Department of Mathematics
Al-Hussein Bin Talal University
Ma'an 71111, Jordan
Email address: ibrahim.a.jawarneh@ahu.edu.jo