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HERMITE-TYPE EXPONENTIALLY FITTED INTERPOLATION FORMULAS USING THREE UNEQUALLY SPACED NODES

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Abstract. Our aim is to construct Hermite-type exponentially fitted interpolation formulas that use not only the pointwise values of an ω-dependent function f but also the values of its first derivative at three unequally spaced nodes. The function f is of the form,

$$f(x) = g_1(x)\cos(\omega x) + g_2(x)\sin(\omega x),\ x \in [a, b],$$

where g_1 and g_2 are smooth enough to be well approximated by polynomials. To achieve such an aim, we first present Hermite-type exponentially fitted interpolation formulas I_N built on the foundation using N unequally spaced nodes. Then the coefficients of I_N are determined by solving a linear system, and some of the properties of these coefficients are obtained. When N is 2 or 3, some results are obtained with respect to the determinant of the coefficient matrix of the linear system which is associated with I_N. For $N = 3$, the errors for I_N are approached theoretically and they are compared numerically with the errors for other interpolation formulas.

1. Introduction

When a table of numerical values is derived from functions or experiments, the data in the table can be interpolated using either Lagrange or Hermite polynomials [1]. However, functions or data with severely vibrating properties may not be properly approximated by these classical polynomials. Fortunately, exponentially fitted interpolation formulas (= EFIFs) can be used to deal with these types of problems that the classical polynomials may not be able to solve. The essence in constructing EFIFs originates from the exponentially fitted techniques, which are explained by Ixaru [6]. In more detail, Ixaru proposed methods to generate exponentially fitted formulas for differentiation, quadrature, and multistep solvers for ordinary differential equations. Since then, many papers have been published, and some of them are as follows. These methods have further been investigated and extended to dealing

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with differential equations [5,9], generating quadrature formulas [7,10,15], and solving integral equations [2]. Some works on EFIFs were done in a situation where equally spaced nodes were selected. More specifically, [12] analyzed EFIFs using not only the values of an ω-dependent function but also the values of its derivatives up to the nth order at the equally spaced nodes, and [13] contributed to formulating EFIFs for a function depending on two frequencies. Error analysis about EFIFs was also considered as follows. [3] presented a theoretical approach to errors resulting from various types of formulas obtained by exponentially fitted techniques. The approach was extended and applied to investigate the errors for EFIFs when two or three equally spaced nodes are selected [11, 14]. In particular, [11] suggested a method for generating EFIFs after expressing three unequally spaced nodes by two parameters. The method followed some procedures different from the ones we deal with in this paper. In addition, the method was not investigated with regard to error analysis as well as the existence of the coefficients of the associated formulas.

EFIFs built on the basis of using equally spaced nodes can not be used in any situation where more than two unequally spaced nodes are given. To overcome this problem, in this paper we first present Hermite-type EFIFs I_N that satisfy the following two conditions at a finite number of unequally spaced nodes:

(i) I_N agrees with f,
(ii) I_N has the same slope as f,

where f is given in the abstract. Then we focus on analyzing I_N for $N = 3$. In general, (i) and (ii) above are conditions imposed when Hermite-type interpolation formulas are generated. However, in the first step of generating I_N in this paper, it is unknown whether I_N satisfies the above two conditions at the unequally spaced nodes. This is because we do not impose such two conditions when generating I_N. Fortunately, it is later found that I_N satisfies the above two conditions. Thus, I_N indeed becomes a Hermite-type interpolation formula to which exponentially fitted techniques are applied. It is demonstrated that the error analysis for EFIFs built on the basis of using equally spaced nodes can also be applied to the error analysis for $I_{N=3}$ using three unequally spaced nodes. The errors for $I_{N=3}$, $I_{N=2}$, and $H_{N=3}$ are compared through some numerical results where $H_{N=3}$ is the classical Hermite polynomial of degree 5. Consequently, it is shown that $I_{N=3}$ is superior to $I_{N=2}$ and $H_{N=3}$ in accuracy. Our new formula I_N can be used even when the nodes are selected with equal spacing. However, if the nodes are equally spaced, the procedures presented by [11] are recommended.

This paper will be organized as follows. In Section 2, our formula I_N, which approximates an ω-dependent function using a finite number of unequally spaced nodes, is presented. The coefficients of I_N are obtained by solving a linear system $AX = Y$ where A is the coefficient matrix, and both X and Y are column vectors. In Section 3, we examine the values that I_N has at N unequally spaced nodes. As a result, I_N becomes a Hermite-type EFIF. In

Section 4, it is investigated how the determinant of the coefficient matrix A with respect to I_N behaves when $N = 2$ or 3. To explain the advantage of I_N by additional use of the first derivative information, a Lagrange-type EFIF, denoted by $\tilde{I}_N$, is introduced that uses only the pointwise function values. In Section 5, the errors for $I_{N=3}$ are compared numerically with the errors for other interpolation formulas such as $I_{N=2}$ and $H_{N=3}$. In Section 6, the errors resulting from $I_{N=3}$ are approached theoretically and illustrated numerically. In Section 7, the errors for $I_{N=3}$, $I_{N=2}$, and $H_{N=3}$ are compared when another example function is selected. Additionally, when $[a, b]$ is partitioned into a finite number of subintervals, the errors for $I_{N=3}$ and $H_{N=3}$ are compared by repeatedly applying the two formulas on each subinterval.

2. Determining the coefficients of I_N

Assume that a function f depending on ω is given by

$$f(x) = g_1(x)\cos(\omega x) + g_2(x)\sin(\omega x),\ x \in [a, b], \tag{1}$$

where g_1 and g_2 are smooth enough to be well approximated by polynomials. The function form given in (1) above has been dealt with for a long time in the field of numerical solutions of ordinary differential equations (see Chapter 1 of [8]). On the other hand, if the data showing strong vibration is related to the function given in (1), exponentially fitted techniques can be applied as a method of connecting the data. From now on, based on the techniques presented in [6], we will look at how to connect data provided at unequally spaced nodes.

To approximate f, we consider formula I_N with coefficients c_j and d_j. This formula uses not only the values of the function f but also the values of its first derivative at $x = x_j$ on $[a, b]$ in the form:

$$f(\bar{x} + ht) \approx I_N(t) = \sum_{j=1}^{N} c_j f(x_j) + h \sum_{j=1}^{N} d_j f^{(1)}(x_j), \quad x_j = \bar{x} + ht_j. \tag{2}$$

In (2), it is assumed that

(i) $\bar{x} = (a + b)/2$, $h = (b - a)/2$, and $-1 \le t \le 1$,
(ii) N is an integer equal to or greater than 2,
(iii) $a = x_1 < x_2 < x_3 < \cdots < x_N = b$,
(iv) once a set of nodes $\{x_j\}_{j=1}^N$ is randomly selected, set t_j by $x_j = \bar{x} + ht_j$,
(v) $f(x_j)$ and $f^{(1)}(x_j)$ are known where $j = 1, 2, \ldots, N$.

At this point we may ask why the first derivative information is used in the formula I_N. There may be cases where not only function values at some nodes but also its first derivative values at the same nodes can be known. For example, a second-order ordinary differential equation with boundary conditions may be solved numerically by a method known as "shooting". The ODE solvers used at this time obtain not only function values but also its first derivative values at the given nodes. Using the full information available in this way, it is expected that more accurate approximation formulas can be produced than formulas using only function values. For this reason, the first derivative information is used in I_N.

To construct I_N, a functional L is defined as

$$L(f(x), h, \mathcal{K}) = f(x+ht) - \sum_{j=1}^{N} c_j f(x+ht_j) - h \sum_{j=1}^{N} d_j f^{(1)}(x+ht_j), \tag{3}$$

where $\mathcal{K} = (c_1, c_2, \ldots, c_N, d_1, d_2, \ldots, d_N)$. Then, the equations we are interested in are

$$L(x^m \exp(\pm i\omega x), h, \mathcal{K}) = 0, \quad m = 0, 1, 2, \ldots, N-1. \tag{4}$$

Now let's see how the values of c_j and d_j can be obtained from (4). When $\mu = i\omega$ and $f(x) = \exp(\mu x)$, (3) gives

$$L(\exp(\mu x), h, \mathcal{K}) = \exp(\mu x)\phi(\mu h, \mathcal{K}),$$

where

$$\phi(u, \mathcal{K}) = \exp(ut) - \sum_{j=1}^{N} c_j \exp(ut_j) - u \sum_{j=1}^{N} d_j \exp(ut_j). \tag{5}$$

In a similar way, we obtain

$$L(\exp(-\mu x), h, \mathcal{K}) = \exp(-\mu x)\phi(-\mu h, \mathcal{K}).$$

By introducing $Z = u^2 = (\mu h)^2 = -\omega^2 h^2$ and using (5) we define

$$\Phi^+(Z, \mathcal{K}) = \frac{1}{2}(\phi(u, \mathcal{K}) + \phi(-u, \mathcal{K})) \text{ and } \Phi^-(Z, \mathcal{K}) = \frac{1}{2u}(\phi(u, \mathcal{K}) - \phi(-u, \mathcal{K})).$$

To get the expressions corresponding to Φ^+ and Φ^-, Ixaru functions ([6] or Section 3.2 of [8]) are used and given as follows.

Definition 1.

(i)

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0, \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0, \end{cases}$$

(ii)

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0, \\ 1 & \text{if } Z = 0, \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0, \end{cases}$$

(iii) for $s = 1, 2, 3, \ldots,$

$$\eta_s(Z) = \begin{cases} (\eta_{s-2}(Z) - (2s-1)\eta_{s-1}(Z))/Z & \text{if } Z \neq 0, \\ 2^s s!/(2s+1)! & \text{if } Z = 0. \end{cases} \tag{6}$$

The above functions also have the following properties about the power series and differentiation:

(i)

$$\eta_s(Z) = 2^s \sum_{q=0}^{\infty} g_{sq} Z^q/(2q+2s+1)!$$

with

$$g_{sq} = \begin{cases} 1 & \text{if } s=0, \\ (q+1)(q+2)\dots(q+s) & \text{if } s=1,2,3,\dots, \end{cases}$$

(ii)

$$\frac{d}{dZ}\eta_s(Z) = \frac{1}{2}\eta_{s+1}(Z), \quad s = \ -1,\ 0,\ 1,\dots. \tag{7}$$

Then we have

$$\Phi^+(Z,\mathcal{K}) = \eta_{-1}(Zt^2) - \sum_{j=1}^{N} c_j \eta_{-1}(Zt_j^2) - \sum_{j=1}^{N} d_j t_j Z \eta_0(Zt_j^2) \tag{8}$$

and

$$\Phi^-(Z,\mathcal{K}) = t\eta_0(Zt^2) - \sum_{j=1}^{N} c_j t_j \eta_0(Zt_j^2) - \sum_{j=1}^{N} d_j \eta_{-1}(Zt_j^2). \tag{9}$$

If one of (i) or (ii) given in (10) below is established, the other is also established:

$$\begin{array}{ll} \text{(i)} & L(\exp(\pm\mu x), h, \mathcal{K}) = 0, \\ \text{(ii)} & \Phi^{\pm}(Z,\mathcal{K}) = 0. \end{array} \tag{10}$$

Using the above result and chain rule, it can be derived that (i) and (ii) given in (11) below are equivalent to each other:

$$\begin{array}{ll} \text{(i)} & L(x^m \exp(\pm\mu x), h, \mathcal{K}) = 0,\ m = 0,1,\dots,N-1, \\ \text{(ii)} & \frac{d^m}{dZ^m}\Phi^{\pm}(Z,\mathcal{K}) = 0,\ m = 0,1,\dots,N-1. \end{array} \tag{11}$$

The mth derivative of Φ^+ and Φ^- are provided as follows.

Lemma 2. *For* $m = 1,2,3,\dots,$

$$\begin{aligned} \frac{d^m}{dZ^m}\Phi^+(Z,\mathcal{K}) = \frac{1}{2^m} \Bigg[& t^{2m}\eta_{m-1}(Zt^2) - \sum_{j=1}^{N} c_j t_j^{2m} \eta_{m-1}(Zt_j^2) \\ & - \sum_{j=1}^{N} d_j t_j^{2m-1} \left(\eta_{m-2}(Zt_j^2) + \eta_{m-1}(Zt_j^2)\right) \Bigg] \end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{d^m}{dZ^m}\Phi^-(Z,\mathcal{K}) = \frac{1}{2^m}\Bigg[& t^{2m+1}\eta_m(Zt^2) - \sum_{j=1}^{N} c_j t_j^{2m+1}\eta_m(Zt_j^2) \\ & - \sum_{j=1}^{N} d_j t_j^{2m}\eta_{m-1}(Zt_j^2)\Bigg].\end{aligned} \tag{13}$$

Proof. Using (8), (7), and (6), we have

$$\begin{aligned}\frac{d}{dZ}\Phi^+ &= \frac{d}{dZ}\left(\eta_{-1}(Zt^2) - \sum_{j=1}^{N} c_j\eta_{-1}(Zt_j^2) - \sum_{j=1}^{N} d_j t_j Z\eta_0(Zt_j^2)\right) \\ &= \frac{1}{2}t^2\eta_0(Zt^2) - \frac{1}{2}\sum_{j=1}^{N} c_j t_j^2\eta_0(Zt_j^2) - \sum_{j=1}^{N} d_j t_j\left(\eta_0(Zt_j^2) + \frac{1}{2}Zt_j^2\eta_1(Zt_j^2)\right) \\ &= \frac{1}{2}\left[t^2\eta_0(Zt^2) - \sum_{j=1}^{N} c_j t_j^2\eta_0(Zt_j^2) - \sum_{j=1}^{N} d_j t_j\left(\eta_{-1}(Zt_j^2) + \eta_0(Zt_j^2)\right)\right], \\ \frac{d^2}{dZ^2}\Phi^+ &= \frac{1}{2^2}\left[t^4\eta_1(Zt^2) - \sum_{j=1}^{N} c_j t_j^4\eta_1(Zt_j^2) - \sum_{j=1}^{N} d_j t_j^3\left(\eta_0(Zt_j^2) + \eta_1(Zt_j^2)\right)\right], \\ &\vdots \\ \frac{d^m}{dZ^m}\Phi^+ &= \frac{1}{2^m}\Bigg[t^{2m}\eta_{m-1}(Zt^2) - \sum_{j=1}^{N} c_j t_j^{2m}\eta_{m-1}(Zt_j^2) \\ &\qquad - \sum_{j=1}^{N} d_j t_j^{2m-1}\left(\eta_{m-2}(Zt_j^2) + \eta_{m-1}(Zt_j^2)\right)\Bigg].\end{aligned}$$

Similarly, the result for $\frac{d^m}{dZ^m}\Phi^-$ is obtained by differentiating Φ^- given in (9) m times. $\square$

From the results of (12) and (13), it can be found that the system given in (ii) of (11) is linear in c_j and d_j. Therefore, this linear system can be expressed as

$$AX = Y, \tag{14}$$

where A is the coefficient matrix of size $2N\times 2N$, and both X and Y are column vectors with $2N$ components. The detailed forms of X, Y, and A are given as follows.

Theorem 3. *X, Y, and A are given as*

(i) *for* $m = 1, 2, \ldots, N$,

$X(m) = c_m,\ X(N+m) = d_m,$

$Y(m) = t^{2(m-1)}\eta_{m-2}(Zt^2),\ Y(N+m) = t^{2(m-1)+1}\eta_{m-1}(Zt^2),$

(ii) *for* $m = 1, 2, \ldots, N$ *and* $j = 1, 2, \ldots, N$,

$$A(m, j) = A(N+m, N+j) = t_j^{2(m-1)}\eta_{m-2}(Zt_j^2),$$
$$A(N+m, j) = t_j^{2(m-1)+1}\eta_{m-1}(Zt_j^2),$$

(iii) *for* $m = 1, 2, \ldots, N-1$ *and* $j = 1, 2, \ldots, N$,

$$A(1, N+j) = t_j Z\eta_0(Zt_j^2),$$
$$A(m+1, N+j) = t_j^{2(m-1)+1}\left(\eta_{m-2}(Zt_j^2) + \eta_{m-1}(Zt_j^2)\right).$$

Proof. First apply (8), (9), and the results of Lemma 2 to (ii) given in (11). Then transform the resulting equations into the linear system as in (14). As a result, X, Y, and A are obtained. □

Finally, the coefficients of I_N, c_j and d_j, are determined by solving the linear system obtained. Note that c_j and d_j depend on ω, h, t, and $\{t_j\}_{j=1}^N$. Nevertheless, we will continue to use the notations c_j and d_j for simplicity, instead of $c_j\left(\omega, h, t, \{t_j\}_{j=1}^N\right)$ and $d_j\left(\omega, h, t, \{t_j\}_{j=1}^N\right)$.

3. Properties of c_j and d_j

Based on the results of the previous section, c_j and d_j are determined in a way that satisfies the equations given in (ii) of (11). At this point, we do not know if I_N has the same values as f at the nodes. That is, we can not guarantee

$$[f(x)]_{x=\bar{x}+ht_j} = [I_N(t)]_{x=\bar{x}+ht_j},\ j = 1, 2, \ldots, N. \tag{15}$$

This is because we did not impose the above conditions when c_j and d_j were determined. However, we will see in Theorem 5 that not only the conditions given in (15) but also the other conditions given in (16) below are satisfied:

$$\left[\frac{d}{dx}f(x)\right]_{x=\bar{x}+ht_j} = \left[\frac{d}{dx}I_N(t)\right]_{x=\bar{x}+ht_j},\ j = 1, 2, \ldots, N. \tag{16}$$

Therefore, I_N will truly be a Hermite-type interpolation formula, not just an approximation of f. To achieve such an end, we need some results given in Lemma 4.

Lemma 4. *For* $j, k = 1, 2, \ldots, N$,

(i)

$$c_j = \begin{cases} 1 & \textit{if } t = t_j \\ 0 & \textit{if } t = t_k\ (k \neq j) \end{cases} \quad \textit{and} \quad \frac{d}{dt}c_j = 0 \textit{ if } t = t_k,$$

(ii)

$$d_j = 0 \; if \; t = t_k \quad and \quad \frac{d}{dt} d_j = \begin{cases} 1 & if \; t = t_j, \\ 0 & if \; t = t_k \; (k \neq j) \; . \end{cases}$$

Proof. From A and Y given in Theorem 3, we get

(i) for $m = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$,

$$[Y(m)]_{t=t_j} = A(m, j) \text{ and } [Y(N+m)]_{t=t_j} = A(N+m, j),$$

(ii)

$$\frac{d}{dt} Y(1) = \frac{d}{dt} \eta_{-1}(Zt^2) = tZ\eta_0(Zt^2),$$

(iii) for $m = 1, 2, \ldots, N-1$,

$$\begin{aligned} \frac{d}{dt} Y(m+1) &= \frac{d}{dt} t^{2m} \eta_{m-1}(Zt^2) \\ &= 2mt^{2m-1} \eta_{m-1}(Zt^2) + t^{2m+1} Z\eta_m(Zt^2) \qquad \text{(use (7))} \\ &= t^{2(m-1)+1} \left(\eta_{m-2}(Zt^2) + \eta_{m-1}(Zt^2) \right), \qquad \text{(use (6))} \end{aligned}$$

(iv) using (7) and (6) similarly as in (iii) above,

$$\begin{aligned} \frac{d}{dt} Y(N+m) &= \frac{d}{dt} t^{2(m-1)+1} \eta_{m-1}(Zt^2) \\ &= (2(m-1)+1) t^{2(m-1)} \eta_{m-1}(Zt^2) + t^{2(m-1)+2} Z\eta_m(Zt^2) \\ &= t^{2(m-1)} \eta_{m-2}(Zt^2), \text{ where } m = 1, 2, \ldots, N. \end{aligned}$$

From the above results, we have that, for $j = 1, 2, \ldots, N$,

$$A(\cdot, j) = [Y(\cdot)]_{t=t_j} \tag{17}$$

and

$$A(\cdot, N+j) = \left[\frac{d}{dt} Y(\cdot) \right]_{t=t_j} . \tag{18}$$

Using (17) and (18), not only the values of c_j and d_j, but also their first derivative values are given at $t = t_k$ as follows. Apply Cramer's rule to the linear system, $AX = Y$. Then, for $j = 1, 2, \ldots, N$,

$$c_j = \frac{\det(A_j)}{\det(A)} \text{ and } d_j = \frac{\det(A_{N+j})}{\det(A)},$$

where A_i is a matrix A with its ith column replaced by Y. Finally, Lemma 4 is proved from the following property for the determinant of a matrix: if two columns of a square matrix are equal, its determinant is equal to zero. For example, let's find out the values of c_1 and $\frac{d}{dt}c_1$ at $t = t_k$ for $N = 2$. When N is 2, the coefficient matrix A has size 4×4 and is given by

$$A = \begin{bmatrix} \eta_{-1}(Zt_1^2) & \eta_{-1}(Zt_2^2) & t_1 Z\eta_0(Zt_1^2) & t_2 Z\eta_0(Zt_2^2) \\ t_1^2\eta_0(Zt_1^2) & t_2^2\eta_0(Zt_2^2) & t_1(\eta_{-1}(Zt_1^2)+\eta_0(Zt_1^2)) & t_2(\eta_{-1}(Zt_2^2)+\eta_0(Zt_2^2)) \\ t_1\eta_0(Zt_1^2) & t_2\eta_0(Zt_2^2) & \eta_{-1}(Zt_1^2) & \eta_{-1}(Zt_2^2) \\ t_1^3\eta_1(Zt_1^2) & t_2^3\eta_1(Zt_2^2) & t_1^2\eta_0(Zt_1^2) & t_2^2\eta_0(Zt_2^2) \end{bmatrix}.$$

We also have

$$X = \begin{bmatrix} c_1 & c_2 & d_1 & d_2 \end{bmatrix}^T$$

and

$$Y = \begin{bmatrix} \eta_{-1}(Zt^2) & t^2\eta_0(Zt^2) & t\eta_0(Zt^2) & t^3\eta_1(Zt^2) \end{bmatrix}^T.$$

Applying Cramer's rule to $AX = Y$, we have $c_1 = \frac{\det(A_1)}{\det(A)}$ where

$$A_1 = \begin{bmatrix} \eta_{-1}(Zt^2) & \eta_{-1}(Zt_2^2) & t_1 Z\eta_0(Zt_1^2) & t_2 Z\eta_0(Zt_2^2) \\ t^2\eta_0(Zt^2) & t_2^2\eta_0(Zt_2^2) & t_1(\eta_{-1}(Zt_1^2)+\eta_0(Zt_1^2)) & t_2(\eta_{-1}(Zt_2^2)+\eta_0(Zt_2^2)) \\ t\eta_0(Zt^2) & t_2\eta_0(Zt_2^2) & \eta_{-1}(Zt_1^2) & \eta_{-1}(Zt_2^2) \\ t^3\eta_1(Zt^2) & t_2^3\eta_1(Zt_2^2) & t_1^2\eta_0(Zt_1^2) & t_2^2\eta_0(Zt_2^2) \end{bmatrix}.$$

Then we get

(i) $[c_1]_{t=t_1} = 1$, because $[\det(A_1)]_{t=t_1} = \det(A)$,
(ii) $[c_1]_{t=t_2} = 0$, because $[A_1(\cdot,1)]_{t=t_2} = A_1(\cdot,2)$,
(iii) $\left[\frac{d}{dt}c_1\right]_{t=t_1} = 0$, because $\left[\frac{d}{dt}A_1(\cdot,1)\right]_{t=t_1} = A_1(\cdot,3)$,
(iv) $\left[\frac{d}{dt}c_1\right]_{t=t_2} = 0$, because $\left[\frac{d}{dt}A_1(\cdot,1)\right]_{t=t_2} = A_1(\cdot,4)$. □

Now we conclude this section as follows.

Theorem 5. *For $j = 1, 2, 3, \dots, N$, we have*

(i)
$$[f(x)]_{x=\bar{x}+ht_j} = [I_N(t)]_{x=\bar{x}+ht_j},$$

(ii)
$$\left[\frac{d}{dx}f(x)\right]_{x=\bar{x}+ht_j} = \left[\frac{d}{dx}I_N(t)\right]_{x=\bar{x}+ht_j}.$$

Proof. For $j = 1, 2, 3, \dots, N$, (i) and (ii) are proved as follows:
(i)

$$\begin{aligned} & [I_N(t)]_{x=\bar{x}+ht_j} \\ = \; & [I_N(t)]_{t=t_j} \qquad \text{(use } x = \bar{x}+ht) \\ = \; & \sum_{k=1}^{N} [c_k]_{t=t_j} f(\bar{x}+ht_k) + h\sum_{k=1}^{N} [d_k]_{t=t_j} f^{(1)}(\bar{x}+ht_k) \qquad \text{(see (2))} \\ = \; & [c_j]_{t=t_j} f(\bar{x}+ht_j) = 1 \cdot f(\bar{x}+ht_j) \qquad \text{(see Lemma 4)} \\ = \; & [f(x)]_{x=\bar{x}+ht_j}. \end{aligned}$$

(ii)

$$
\begin{aligned}
&\left[\frac{d}{dx} I_N(t)\right]_{x=\bar{x}+ht_j} \\
&= \left[\frac{d}{dt} I_N(t)\right]_{t=t_j} \cdot \frac{1}{h} \qquad \text{(use chain rule)} \\
&= \left(\sum_{k=1}^{N} \left[\frac{d}{dt} c_k\right]_{t=t_j} f(\bar{x}+ht_k) + h\sum_{k=1}^{N} \left[\frac{d}{dt} d_k\right]_{t=t_j} f^{(1)}(\bar{x}+ht_k)\right) \cdot \frac{1}{h} \\
&= \left(h \left[\frac{d}{dt} d_j\right]_{t=t_j} f^{(1)}(\bar{x}+ht_j)\right) \cdot \frac{1}{h} \\
&= 1 \cdot f^{(1)}(\bar{x}+ht_j) \qquad \text{(see Lemma 4)} \\
&= \left[\frac{d}{dx} f(x)\right]_{x=\bar{x}+ht_j}.
\end{aligned}
$$

□

Thus, Theorem 5 says that I_N agrees with f at $x = \bar{x} + ht_j$, $j = 1, 2, \ldots, N$, and it has the same slope as f at the nodes. It is therefore expected that I_N will provide an approximation to interpolate f at the nodes with considerable accuracy over the interval of interest.

4. About the determinant of A where N is 2 or 3

To obtain the values of c_j and d_j, the determinant of A must not be zero in the linear system $AX = Y$. Thus, we examine the determinant of A for the cases where N is 2 or 3. To eliminate ambiguity, if $N = 2, 3$, then A in the linear system is denoted by $A_{N=2}$, $A_{N=3}$, respectively. First, for $N = 2$, the determinant of A is given as follows.

Lemma 6. *When $t_1 = -1$ and $t_2 = 1$, we have*

$$\det(A_{N=2}) = \frac{\cos 4v + 8v^2 - 1}{2v^4},$$

where $v = \omega h$.

Proof. From Theorem 3, $A_{N=2}$ has size 4×4. First substitute $t_1 = -1$ and $t_2 = 1$ into $A_{N=2}$ and then evaluate the determinant of $A_{N=2}$. Next, using the definition of η_s and $v = \omega h$, we have

$$
\begin{aligned}
\det(A_{N=2}) &= 4\left(2\eta_{-1}^2(Z)\eta_0^2(Z) - \eta_{-1}^3(Z)\eta_1(Z) - \eta_{-1}^2(Z)\eta_0(Z)\eta_1(Z) - Z\eta_0^4(Z)\right) \\
&= \frac{\cos 4v + 8v^2 - 1}{2v^4}.
\end{aligned}
$$

□

Now, for $v > 0$, it can be seen that the determinant of $A_{N=2}$ is positive.

Theorem 7. *If the same conditions as in Lemma* 6 *are assumed, then*

$$\det(A_{N=2}) = \frac{\cos 4v + 8v^2 - 1}{2v^4} > 0, \;\; v > 0. \tag{19}$$

Proof. Define a function p for $v \geq 0$ by

$$p(v) = \cos 4v + 8v^2 - 1.$$

Notice that p corresponds to the numerator in the determinant of $A_{N=2}$. It is clear that $p(0) = 0$ and $p'(v) > 0$ for $v > 0$. This result gives

$$p(v) > 0 \;\text{ for } v > 0.$$

Therefore, we can conclude that $\det(A_{N=2}) > 0$ for $v > 0$. □

Let's take a moment to consider another interpolation formula $\tilde{I}_N$ to only use the values of f at the nodes, not using its first derivative information:

$$f(\bar{x} + ht) \approx \tilde{I}_N(t) = \sum_{j=1}^{N} \tilde{c}_j f(\bar{x} + ht_j). \tag{20}$$

The linear system for $\tilde{I}_N$,

$$\tilde{A}\tilde{X} = \tilde{Y}, \tag{21}$$

can be obtained by similarly performing the procedure in Section 2 as the linear system for I_N was obtained by (14). When $N = 2$, we denote $\tilde{A}$ given in (21) as $\tilde{A}_{N=2}$. Then, if the same conditions as in Lemma 6 are assumed, it can be seen that

$$\det(\tilde{A}_{N=2}) = \frac{\sin(2v)}{v}, \;\; v > 0. \tag{22}$$

Therefore, as v increases, the determinant of $\tilde{A}_{N=2}$ repeatedly becomes zero. Next, let's examine the cases for $N = 3$.

Lemma 8. *When $t_1 = -1, t_2 = x,$ and $t_3 = 1,$ we have*

$$\det(A_{N=3}) = 4G(v, x)/v^9,$$

where $-1 < x < 1$, $v = \omega h$, and

$$\begin{aligned}
&G(v,x)\\
&= v^3x^4 - 2v^3x^2 - 3v\cos^2(v) + v\cos^4(v) - 2\cos^3(v)\sin(v) + v^3\\
&\quad - 2v\cos^2(vx) + 4v\cos^2(vx)\cos^2(v) + 2\cos^2(vx)\cos(v)\sin(v)\\
&\quad + 2x^2\cos^3(v)\sin(v) - 2vx^2\cos^2(vx) - 2vx^2\cos^4(v) - vx^4\cos^2(v)\\
&\quad + vx^4\cos^4(v) - 2x^2\cos^2(vx)\cos(v)\sin(v) + 4vx^2\cos^2(vx)\cos^2(v)\\
&\quad + 8vx\cos(vx)\sin(vx)\cos(v)\sin(v).
\end{aligned} \tag{23}$$

Proof. From Theorem 3, $A_{N=3}$ has size 6×6. First substitute $t_1 = -1, t_2 = x$, and $t_3 = 1$ into $A_{N=3}$ and then express η_s in $A_{N=3}$ by v. Thus the determinant of $A_{N=3}$ becomes a function of two variables v and x. Matlab [16] is used to obtain $G(v, x)$ given in (23). □

Note that $G(v, -1) = G(v, 1) = 0$. Therefore, when $x = -1$ or 1, $\det(A_{N=3}) = 0$ for $v > 0$. Using $\eta_s(Z)$ at $Z = 0$ given in Definition 1, we get

$$\det(A_{N=3}) = \frac{16(x^2-1)^4}{135} \quad \text{at } v = 0. \tag{24}$$

We also have

$$\lim_{v\to 0} \det(A_{N=3}) = \lim_{v\to 0} 4G(v, x)/v^9 = \frac{16(x^2-1)^4}{135}, \tag{25}$$

which is exactly the same result as (24). To obtain the final result in (25), apply l'Hospital's rule nine times. When $0 \le v \le 3$ and $-1 \le x \le 1$, the graph of $z = \det(A_{N=3})$ is given by Fig. 1, where the values of the determinant of $A_{N=3}$ are less than 0.12 and are positive for $-1 < x < 1$. As $v \ge 3$ increases, the values of the determinant of $A_{N=3}$ decrease consistently while the sign of the determinant never changes and remains positive for $-1 < x < 1$. For example, when $13 \le v \le 16$ and $-1 < x < 1$, the values of the determinant of $A_{N=3}$ are positive and are less than 10^{-6} as shown in Fig. 2. Eventually, when $27 \le v \le 30$ and $-1 < x < 1$, the values of the resulting determinant are still positive and are less than 1.2×10^{-8} as shown in Fig. 3. Even if $v \ge 30$, the determinant of $A_{N=3}$ is expected to show a consistently decreasing behavior in a similar manner as above.

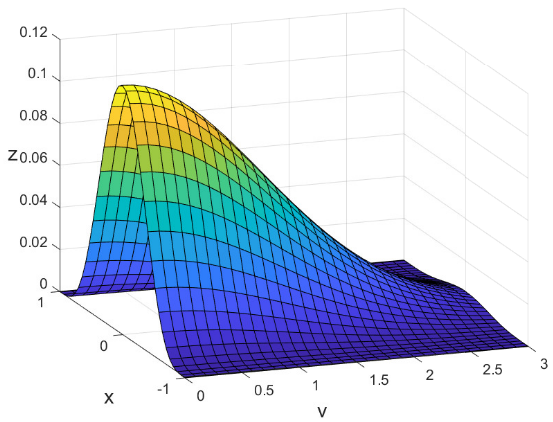

FIGURE 1. $z = \det(A_{N=3}), 0 \le v \le 3, -1 \le x \le 1$

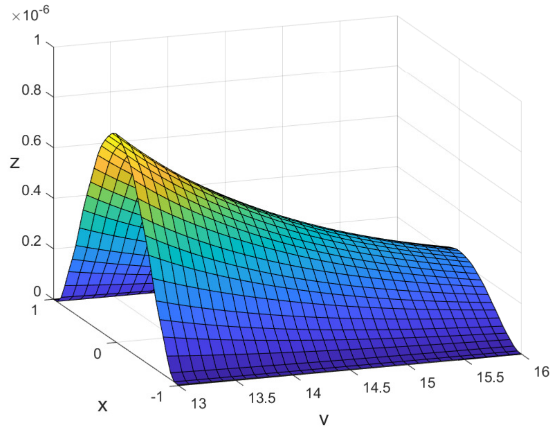

FIGURE 2. $z = \det(A_{N=3}), 13 \leq v \leq 16, -1 \leq x \leq 1$

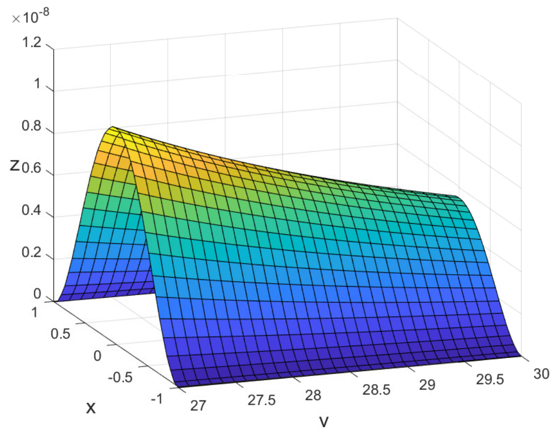

FIGURE 3. $z = \det(A_{N=3}), 27 \leq v \leq 30, -1 \leq x \leq 1$

Based on these observations, the sign of the determinant of $A_{N=3}$ is given as follows.

Proposition 9. *If the same conditions as in Lemma* 8 *are assumed, then*

$$\det(A_{N=3}) > 0, \tag{26}$$

where $0 \leq v \leq 30$ *and* $-1 < x < 1$.

In the above proposition, the range of v is given as $0 \le v \le 30$. However, the values of ω and h can be selected more diversely because $v = \omega h$. Next, when $N = 3$, we denote $\tilde{A}$ given in (21) as $\tilde{A}_{N=3}$. Then $\tilde{A}_{N=3}$ is obtained from the following three equations,

$$
\begin{aligned}
&\tilde{L}(1, h, \tilde{\mathcal{K}}) = 1 - \sum_{j=1}^{3} \tilde{c}_j = 0,\\
(27) \qquad &\tilde{\Phi}^{+}(Z, \tilde{\mathcal{K}}) = \eta_{-1}(Zt^2) - \sum_{j=1}^{3} \tilde{c}_j \eta_{-1}(Zt_j^2) = 0,\\
&\tilde{\Phi}^{-}(Z, \tilde{\mathcal{K}}) = t\eta_0(Zt^2) - \sum_{j=1}^{3} \tilde{c}_j t_j \eta_0(Zt_j^2) = 0,
\end{aligned}
$$

where $\tilde{\mathcal{K}} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$. To construct $\tilde{I}_N$ for $N = 3$, three unknowns $\tilde{c}_1$, $\tilde{c}_2$, and $\tilde{c}_3$ must be determined. So it is necessary to add one more equation to $\tilde{\Phi}^{\pm}(Z, \tilde{\mathcal{K}}) = 0$. This is why the first equation in (27) is added newly to the last two equations in (27). For further details, refer to [6]. If the same conditions as in Lemma 8 are assumed, then

$$
(28) \qquad \det(\tilde{A}_{N=3}) = \frac{2\sin(v)(\cos(vx) - \cos(v))}{v}, \ v > 0.
$$

Therefore, as v increases, the determinant of $\tilde{A}_{N=3}$ repeatedly becomes zero. As seen in (22) and (28), there are some critical values in v where the determinant of $\tilde{A}$ becomes zero. However, if the first derivative information is additionally used as in I_N, such critical values disappear as shown in (19) and (26). Therefore, I_N can be used more flexibly than $\tilde{I}_N$ when N is 2 or 3.

In general, when $N > 3$, the determinant of A in the linear system $AX = Y$ involves three or more variables. However, once the values of v (or Z) and $\{t_j\}_{j=1}^{N}$ are set, the determinant of A, if it exists, can be evaluated by referring to Theorem 3 and used to generate I_N.

5. Error comparison

To investigate the error generated by I_N, we will proceed as follows.

(a) Present an example function f for the investigation.
(b) When $N = 2$ or 3, denote I_N given in (2) as $I_{N=2}$ or $I_{N=3}$, respectively.
(c) Provide numerical results regarding $f - I_N$.

First, the example function f is presented by

$$
(29) \qquad f(x) = \cos(x)\cos(\omega x) - \sin(x)\sin(\omega x), \ \omega = 100,
$$

on the domain $[0, 0.1]$. Note that the function f is of the form (1) when $g_1(x) = \cos(x)$ and $g_2(x) = -\sin(x)$. Assume three unequally spaced nodes in $[0, 0.1]$ are chosen as

$$
(30) \qquad x = 0, \ 0.0375, \ 0.1.
$$

Then our new formula $I_{N=3}$ using three unequally spaced nodes given in (30) is provided by

$$I_{N=3}(t) = \sum_{j=1}^{3} c_j f(0.05 + 0.05t_j) + 0.05 \sum_{j=1}^{3} d_j f^{(1)}(0.05 + 0.05t_j), \tag{31}$$

where $t_1 = -1$, $t_2 = -1/4$, and $t_3 = 1$. Meanwhile, in order for our current situation of using those three unequally spaced nodes to be approached under the framework of EFIFs using equally spaced nodes, EFIFs using only two nodes, say $\tilde{I}_{N=2}$ or $I_{N=2}$, have to be used to approximate f. In other words, in the above situation where three unequally spaced nodes are selected, EFIFs using three equally spaced nodes cannot be used. Therefore, if the above situation proceeds by EFIFs using equally spaced nodes, $I_{N=2}$ can be applied to approximate f on $[0, 0.1]$ as follows. First, divide the interval $[0, 0.1]$ into two subintervals,

$$[0, 0.1] = [0, 0.0375] \cup [0.0375, 0.1],$$

and then apply $I_{N=2}$ on each subinterval. Therefore the values of $\bar{x}$, h, and $\{t_j\}_{j=1}^{2}$ in $I_{N=2}$,

$$I_{N=2}(t) = \sum_{j=1}^{2} c_j f(\bar{x} + ht_j) + h \sum_{j=1}^{2} d_j f^{(1)}(\bar{x} + ht_j), \tag{32}$$

are set as follows:

(i) $\bar{x} = 0.01875$ and $h = 0.01875$ in the first subinterval $[0, 0.0375]$,
(ii) $\bar{x} = 0.06875$ and $h = 0.03125$ in the second subinterval $[0.0375, 0.1]$,
(iii) $t_1 = -1$ and $t_2 = 1$ in both subintervals.

As for the errors with respect to $I_{N=2}$ and $I_{N=3}$ described in (32) and (31), respectively, numerical results are illustrated and compared in Fig. 4. As seen in Fig. 4, $I_{N=3}$ approximates f more accurately than $I_{N=2}$ even if $I_{N=2}$ is applied on each of the two subintervals and the error for $I_{N=2}$ has values between -2×10^{-4} and 6×10^{-4}. The error for $I_{N=3}$ looks like the x-axis because the error for $I_{N=3}$ is relatively less than that for $I_{N=2}$. In fact, the error for $I_{N=3}$ in Fig. 4 is given by

$$|f - I_{N=3}| < 1.4 \times 10^{-5} \quad \text{on} \quad [0, 0.1]. \tag{33}$$

Let us consider a classical Hermite polynomial (see Chapter 3 of [1]), denoted by $H_{N=3}$, of degree 5 agreeing with f and $f^{(1)}$ at three nodes x_1, x_2, and x_3. Then $H_{N=3}$ is given by

$$H_{N=3}(x) = \sum_{j=1}^{3} f(x_j) H_{3,j}(x) + \sum_{j=1}^{3} f^{(1)}(x_j) \bar{H}_{3,j}(x), \tag{34}$$

where

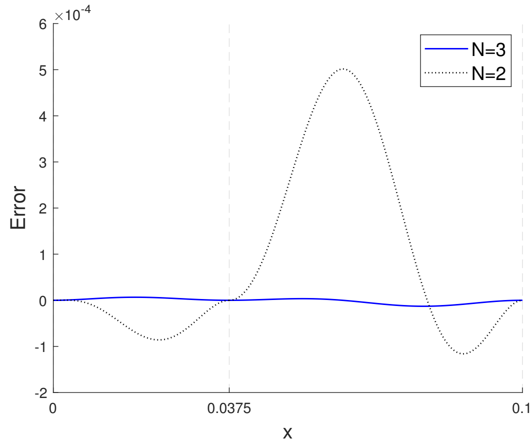

FIGURE 4. Error $= f - I_N$

(i)

$$H_{3,j}(x) = \left(1 - 2(x - x_j)L_{3,j}^{(1)}(x_j)\right) L_{3,j}^2(x),\ \ \bar{H}_{3,j}(x) = (x - x_j)L_{3,j}^2(x),$$

(ii)

$$L_{3,1}(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)},$$
$$L_{3,2}(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)},$$
$$L_{3,3}(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}.$$

When x_1, x_2, and x_3 in (34) are selected as three unequally spaced nodes given in (30), the error for $H_{N=3}$ is illustrated in Fig. 5 and it is given by

$$|f - H_{N=3}| < 0.95 \quad \text{on} \quad [0, 0.1]. \tag{35}$$

Thus, (33) and (35) give the following result:

$$\max_{x\in[0,0.1]} |f(x) - I_{N=3}(t)| \approx 1.47 \times 10^{-5} \max_{x\in[0,0.1]} |f(x) - H_{N=3}(x)|. \tag{36}$$

As can be seen from (36), the error for $I_{N=3}$ is much less than the error for $H_{N=3}$.

6. Error analysis

The errors generated by I_N can be theoretically approached. This approach is achieved by extending the studies of Coleman and Ixaru [3] and Ghizzetti

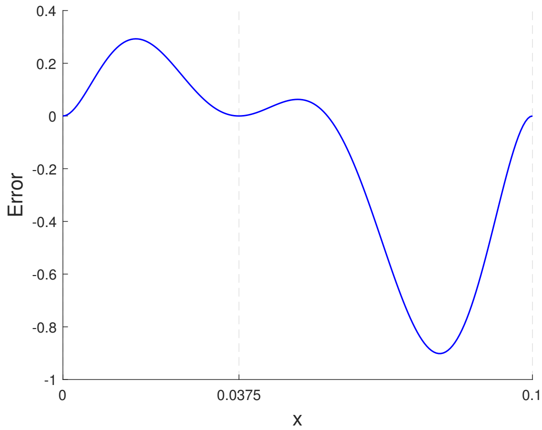

FIGURE 5. Error $= f - H_{N=3}$

and Ossicini [4]. In fact, the errors for exponentially fitted interpolation formulas using either two nodes or equally spaced nodes were investigated in [11,14]. These investigations also make it possible to analyze the errors for I_N using unequally spaced nodes. Therefore, the procedure for finding a formula regarding the errors for I_N begins as follows. Ghizzetti and Ossicini considered quadrature formulas $Q_{N,m}$ with coefficients $C_{k,j}$ of the form,

$$\begin{aligned} \int_a^b f(r)g(r)dr &= Q_{N,m} + E[f] \\ &= \sum_{j=1}^{N} \sum_{k=0}^{m-1} C_{k,j} \frac{d^k f(r_j)}{dr^k} + E[f], \end{aligned} \tag{37}$$

where $a \le r_1 < r_2 < \cdots < r_N \le b$. In the above, $E[f]$ corresponds to the error term and it is equal to zero if f is a solution of a linear differential equation $L_r[f] = 0$ of order m. The differential operator L_r is given by

$$L_r = \sum_{k=0}^{m} \epsilon_k(r) \frac{d^{m-k}}{dr^{m-k}}$$

with $\epsilon_0(r) = 1$. Here it is assumed that the conditions required for f, g, and ϵ_k are satisfied. Refer to Chapters 1 and 2 of [4] for detailed conditions for f, g, and ϵ_k. If the end points of the integration interval are not quadrature nodes, set $r_0 = a$ and $r_{N+1} = b$. Theorem 2.4.1 of [4] suggests that the error term is given by

$$E[f] = \int_a^b \Psi(r) L_r[f(r)]dr. \tag{38}$$

In (38), Ψ satisfies that, for $i = 0, 1, \ldots, N$,

(i)
$$\Psi(r) = \psi_i(r), \quad r_i \le r \le r_{i+1},$$

(ii)
$$\psi_i(r) = -\int_a^r S(z,r)g(z)dz + \sum_{k=0}^{m-1}\sum_{j=1}^{i} C_{k,j}\left[\frac{\partial^k}{\partial z^k}S(z,r)\right]_{z=r_j}, \tag{39}$$

where the sum does not appear for $i = 0$. In (39), $S(z,r)$ is the solution of $L_z[S] = 0$ such that

$$\left[\frac{\partial^k}{\partial z^k}S(z,r)\right]_{z=r} = \delta_{k,m-1},\ k = 0, 1, \ldots, m-1, \tag{40}$$

where

$$\delta_{p,q} = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases}$$

At this point, our concern is to construct a formula that corresponds to the error term generated by I_N. Therefore, consider g in (37) when g is given by

$$g = g(x,r) = \begin{cases} \infty, & \text{if } x = r, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$f(x) = \sum_{j=1}^{N}\sum_{k=0}^{m-1} C_{k,j}(x)\frac{d^k f(r_j)}{dr^k} + E[f](x),\ a \le x \le b.$$

The error term is expressed by

$$E[f](x) = \int_a^b \Psi(x,r)L_r[f(r)]dr, \tag{41}$$

where

(i)
$$\Psi(x,r) = \psi_i(x,r) \ \text{ for } \ r_i \le r \le r_{i+1},\ i = 0, 1, \ldots, N,$$

(ii)
$$\psi_i(x,r) = -S(x,r)U(r-x) + \sum_{k=0}^{m-1}\sum_{j=1}^{i} C_{k,j}(x)\left[\frac{\partial^k}{\partial z^k}S(z,r)\right]_{z=r_j},$$

(iii) S is the solution of $L_z(S) = 0$ and it satisfies the conditions given in (40),

(iv) U is the unit step function defined by

$$U(\tau) = \begin{cases} 1, & \tau \ge 0, \\ 0, & \tau < 0. \end{cases}$$

Thus, if $C_{k,j}(x)$ are known for any value x in $[a, b]$, the corresponding error term can be evaluated by (41).

When f is presented as (29), let's find the error term $E[f](x)$ for our new formula $I_{N=3}$ given in (31). Since $I_{N=3}$ is exact for $f(x) = x^m \exp(\pm i\omega x)$, $m = 0, 1, 2$, we have

$$L_r = \left(\frac{d^2}{dr^2} + \omega^2\right)^3.$$

Based on (iii) of (41), we obtain

$$\begin{aligned} S(z,r) = {} & \frac{3}{8\omega^5}\sin(\omega(z-r)) - \frac{3}{8\omega^4}(z-r)\cos(\omega(z-r)) \\ & - \frac{1}{8\omega^3}(z-r)^2\sin(\omega(z-r)). \end{aligned}$$

When $k = 2, 3, 4, 5$ and $j = 1, 2, 3$, set

$$C_{0,j}(x) = c_j, \quad C_{1,j}(x) = hd_j, \quad \text{and} \quad C_{k,j}(x) = 0.$$

Then the error term becomes

$$\begin{aligned} E[f](x) &= \int_0^{0.1} \Psi(x,r) L_r[f(r)]dr \\ &= \int_{r_1}^{r_2} \psi_1(x,r) L_r[f(r)]dr + \int_{r_2}^{r_3} \psi_2(x,r) L_r[f(r)]dr, \end{aligned} \tag{42}$$

where $r_1 = 0, r_2 = 0.0375$, and $r_3 = 0.1$. In (42) we have

(i)

$$\psi_1(x,r) = -S(x,r)U(r-x) + c_1 S(r_1,r) + hd_1 \left[\frac{\partial}{\partial z} S(z,r)\right]_{z=r_1},$$

(ii)

$$\psi_2(x,r) = \psi_1(x,r) + c_2 S(r_2,r) + hd_2 \left[\frac{\partial}{\partial z} S(z,r)\right]_{z=r_2}.$$

Finally, the error term $E[f]$ derived in (42) is calculated at $x = 0.0005k$, $k = 0, 1, 2, \ldots, 200$, and illustrated in Figure 6, where f is given in (29). When the graph given in Figure 6 was obtained, Matlab's symbolic operations were used. As can be expected, magnifying the error (= solid line) corresponding to $N = 3$ in Figure 4 shows the same result as the error given in Figure 6. Therefore, it is verified that the error term $E[f](x)$ given in (42) is derived correctly.

7. Another example and conclusion

Another example function $\tilde{f}$ is presented by

$$\tilde{f}(x) = \exp(x)\cos(\omega x) + \exp(-x)\sin(\omega x), \ \omega = 100, \tag{43}$$

on the domain $[0, 0.1]$. In the situation of Section 5 where the errors ($= f - I_N, f - H_{N=3}$) regarding f given in (29) are obtained, if we apply $\tilde{f}$ given in

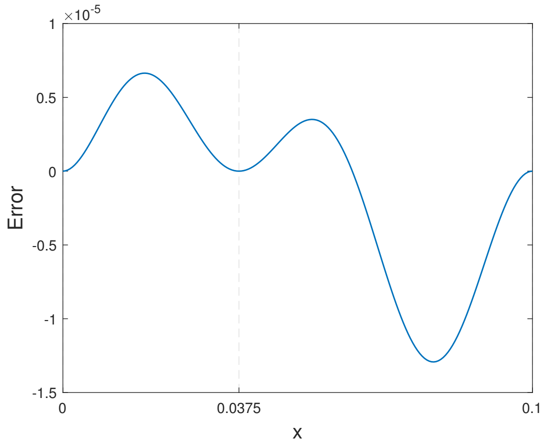

FIGURE 6. Error $= E[f](x)$

(43) instead of f, we get the results in Figures 7 and 8 where the same three nodes as (30) are used. In details, the errors for $I_{N=3}$, $I_{N=2}$, and $H_{N=3}$ on $[0, 0.1]$ are given by

(i)

$$\left|\tilde{f} - I_{N=3}\right| < 1.4 \times 10^{-5}, \tag{44}$$

(ii)

$$\left|\tilde{f} - I_{N=2}\right| < 7.5 \times 10^{-4}, \tag{45}$$

(iii)

$$\left|\tilde{f} - H_{N=3}\right| < 1.85. \tag{46}$$

From (44) and (46), we have

$$\max_{x\in[0,0.1]} \left|\tilde{f}(x) - I_{N=3}(t)\right| \approx 0.76 \times 10^{-5} \max_{x\in[0,0.1]} \left|\tilde{f}(x) - H_{N=3}(x)\right|. \tag{47}$$

From (44), (45), and (47), it is clearly seen that the error for our formula $I_{N=3}$ is relatively less than the errors for $I_{N=2}$ and the classical Hermite polynomial $H_{N=3}$.

If more than three nodes are randomly selected on $[a, b]$, the functions can be interpolated by first dividing the interval $[a, b]$ into a finite number of subintervals and then applying $I_{N=3}$ on each subinterval. For example, suppose that

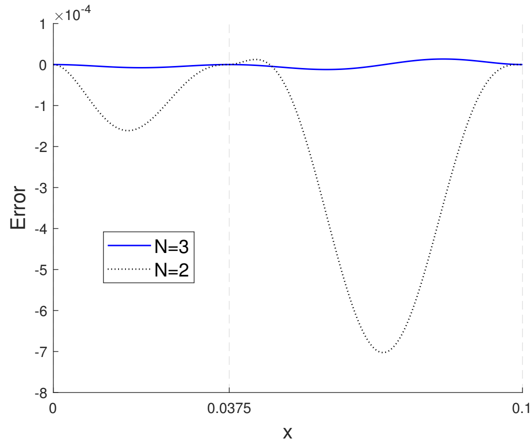

FIGURE 7. Error $= \tilde{f} - I_N$

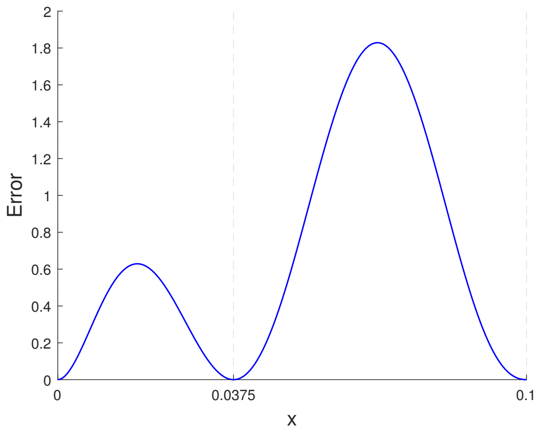

FIGURE 8. Error $= \tilde{f} - H_{N=3}$

21 nodes including both endpoints are randomly selected on $[0, 1]$ as follows:

$$
\begin{array}{llll}
p_1 = 0, & p_2 = 0.0377, & p_3 = 0.0987, & p_4 = 0.1366, \\
p_5 = 0.1978, & p_6 = 0.2919, & p_7 = 0.3154, & p_8 = 0.3655, \\
p_9 = 0.4574, & p_{10} = 0.5342, & p_{11} = 0.5721, & p_{12} = 0.6038, \\
p_{13} = 0.6797, & p_{14} = 0.7150, & p_{15} = 0.7791, & p_{16} = 0.7962, \\
p_{17} = 0.8537, & p_{18} = 0.8852, & p_{19} = 0.9133, & p_{20} = 0.9680, \\
p_{21} = 1. & & &
\end{array}
$$

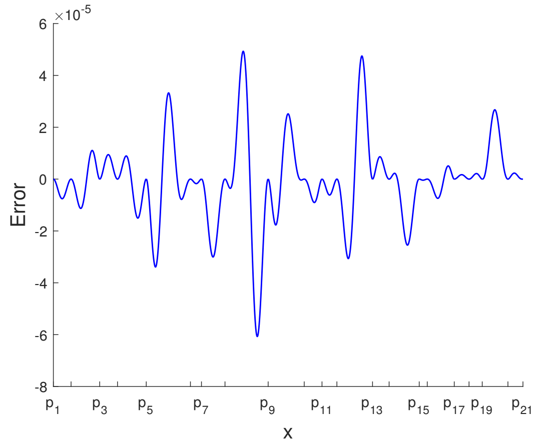

FIGURE 9. Error $= \tilde{f} - I_{N=3}$

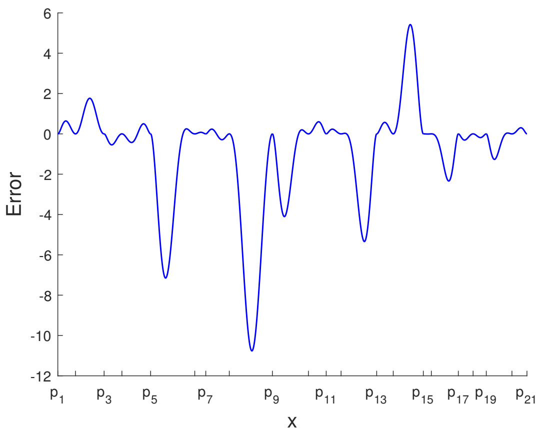

FIGURE 10. Error $= \tilde{f} - H_{N=3}$

First, divide $[0, 1]$ into 10 subintervals,

$$[0, 1] = [p_1, p_3] \cup [p_3, p_5] \cup [p_5, p_7] \cup \cdots \cup [p_{19}, p_{21}].$$

Next, apply $I_{N=3}$ given in (2) on each subinterval $[p_{2(j-1)+1}, p_{2(j-1)+3}]$ for $j = 1, 2, 3, \ldots, 10$, where

(i) x_1, x_2, and x_3 on each subinterval are given by

$$x_1 = p_{2(j-1)+1}, \quad x_2 = p_{2(j-1)+2}, \quad \text{and} \quad x_3 = p_{2(j-1)+3}, \tag{48}$$

(ii) f is given by $\tilde{f}$ in (43).

As a result, the error $(= \tilde{f} - I_{N=3})$ on $[0,1]$ is obtained as shown in Figure 9. Therefore we have

$$\left|\tilde{f} - I_{N=3}\right| < 6.5 \times 10^{-5} \quad \text{on } [0,1]. \tag{49}$$

In the above procedure, instead of applying $I_{N=3}$ on each subinterval $[p_{2(j-1)+1}, p_{2(j-1)+3}]$ if the classical Hermite polynomial $H_{N=3}$ given in (34) is applied on $[p_{2(j-1)+1}, p_{2(j-1)+3}]$, the error regarding $H_{N=3}$ is obtained as shown in Figure 10. Thus we have

$$\left|\tilde{f} - H_{N=3}\right| < 11 \quad \text{on} \quad [0,1]. \tag{50}$$

Now (49) and (50) give

$$\max_{x\in[0,1]} \left|\tilde{f}(x) - I_{N=3}(t)\right| \approx 0.59 \times 10^{-5} \max_{x\in[0,1]} \left|\tilde{f}(x) - H_{N=3}(x)\right|. \tag{51}$$

As can be seen from (51), our new formula $I_{N=3}$ with respect to the function $\tilde{f}$ given in (43) provides much better accuracy than the classical Hermite polynomial $H_{N=3}$. If six nodes, $p_1, p_2, \ldots, p_6$, are randomly selected, divide the whole interval $[p_1, p_6]$ into three subintervals,

$$[p_1, p_6] = [p_1, p_3] \cup [p_3, p_5] \cup [p_5, p_6].$$

Then it would be a good choice to apply $I_{N=3}$ on the first and second subintervals, and $I_{N=2}$ on the last subinterval, respectively.

This paper shows that even if the ω-dependent function f is changed, our new formula $I_{N=3}$ consistently provides a relatively more accurate approximation than $I_{N=2}$ or $H_{N=3}$. All computational results are obtained from Matlab [16]. Our research may provide a cornerstone for constructing more general formulas involving derivatives with orders greater than one.

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