Commun. Korean Math. Soc. **37** (2022), No. 1, pp. 105–112
https://doi.org/10.4134/CKMS.c200451
pISSN: 1225-1763 / eISSN: 2234-3024

GENERALIZED ISOMETRY IN NORMED SPACES

Abbas Zivari-Kazempour

Abstract. Let $g : X \longrightarrow Y$ and $f : Y \longrightarrow Z$ be two maps between real normed linear spaces. Then f is called generalized isometry or g-isometry if for each $x, y \in X$,

$$\|f(g(x)) - f(g(y))\| = \|g(x) - g(y)\|.$$

In this paper, under special hypotheses, we prove that each generalized isometry is affine. Some examples of generalized isometry are given as well.

1. Introduction

A map $f : X \longrightarrow Y$ between real normed linear spaces is an *isometry* if for all $x, y \in X$, $\|f(x) - f(y)\| = \|x - y\|$, and f is *affine* if

$$f(tx + (1 - t)y) = tf(x) + (1 - t)f(y)$$

for all $x, y \in X$ and $t \in [0, 1]$. This definition turns out to be equivalent to the requirement that f is linear up to a translation, i.e., $x \longrightarrow f(x) - f(0)$ is a linear map [10].

An isometry need not be affine. For example, define $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $f(x) = (x, \sin x)$, where $\mathbb{R}^2$ equipped with the usual normed linear space structure. Then f is an isometry, but it is not affine (see also Example 2.9 below).

There are two basic results that every isometry is affine. The first result, due to Mazur and Ulam [4], states that every bijective isometry $f : X \longrightarrow Y$ between real normed spaces is affine. For different proofs of the Mazur-Ulam theorem, see [2, 6, 8].

The second result, due to Baker [1], states that every isometry f between real normed spaces is linear up to translation, whenever Y is strictly convex.

Recall that the normed space X is *strictly convex* if $\|tx + (1 - t)y\| < 1$ whenever x and y are different points of S_X and $0 < t < 1$, where S_X is the unit sphere of X.

There are some equivalent version of this definition [5], such as:

(a) The unit sphere S_X contains no line segments;

Received November 28, 2020; Accepted February 16, 2021.

2010 *Mathematics Subject Classification.* Primary 46H40, 47A10.

Key words and phrases. Isometry, Mazur-Ulam theorem, strictly convex, affine map.

(b) If $x, y \in S_X$ and $x \neq y$, then $\|x + y\| < 2$;
(c) If $\|x + y\| = \|x\| + \|y\|$ and $y \neq 0$, then $x = ty$ for some $t \geq 0$.

For example, every inner product space and the spaces l^p for $1 < p < \infty$ are strictly convex, and on the contrary, none of the spaces l^1, l^∞, c_0 and $\mathbb{R}^n$ for $n \geq 2$ are not strictly convex. For more details, we refer the reader to [5].

A map $f : X \longrightarrow Y$ between normed real linear spaces X and Y *preserves equality of distance*, if

$$\|f(x) - f(y)\| = \|f(u) - f(v)\|$$

for every $x, y, u, v \in X$ satisfying $\|x - y\| = \|u - v\|$. Such maps were first studied by Vogt [9], who extended the Mazur-Ulam theorem by proving that every continuous surjective map which preserves equality of distance and takes 0 to 0, is a linear isometry multiplied by a nonzero constant.

A different kind of generalization of the Mazur-Ulam theorem was given by Rassias and Semrl in [7]. They proved, under especial hypotheses that every surjective mapping $f : X \longrightarrow Y$ between real normed linear spaces is affine.

In [3], the authors introduce a new notation of isometry. The mapping $f : X \longrightarrow X$ is called a two-isometry if for all $x, y \in X$,

$$\|f^2(x) - f^2(y)\|^2 - 2\|f(x) - f(y)\|^2 + \|x - y\|^2 = 0.$$

They proved under certain conditions that every continuous two-isometry f is affine. Note that every isometry is a two-isometry, but the converse is false, in general [3].

Recently, in [10], the authors adapted the proof of the Mazur-Ulam theorem for Fréchet algebra [10, Theorem 2.3].

In this paper, we study the notation of generalized isometry or g-isometry and we prove the classical Mazur-Ulam theorem and Baker's result for g-isometry.

2. Main result

We first introduce the concept of generalized isometry (g-isometry) between real normed linear spaces.

Definition 2.1. Let $g : X \longrightarrow Y$ and $f : g(X) \subseteq Y \longrightarrow Z$ be two maps between real normed linear spaces. We say that f is a generalized isometry or g-isometry if

$$\|f(g(x)) - f(g(y))\| = \|g(x) - g(y)\|, \qquad x, y \in X. \tag{1}$$

Clearly, every isometry $f : Y \longrightarrow Z$ is a g-isometry for arbitrary mapping $g : X \longrightarrow Y$, but the converse is fails, in general. The following example illustrates this fact.

Example 2.2. (i) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(t) = |t|$, $g(s) = -s^2$. Then f is a g-isometry, but neither f nor g is isometry.

(ii) Let X be a normed space. Consider $g : X \longrightarrow \mathbb{R}^2$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$g(x) = (\|x\|, -\|x\|) \quad \text{and} \quad f(s,t) = (s, |s|)$$

for all $x \in X$ and $s, t \in \mathbb{R}$, where $\mathbb{R}^2$ equipped with the norm

$$\|(s,t)\| = \max\{|s|, |t|\}.$$

Then for all $x, y \in X$,

$$\|f(g(x)) - f(g(y))\| = \|g(x) - g(y)\|.$$

Thus, f is a g-isometry, while f and g are not isometry.

However, if $X = Y$ and g is the identity map, then it follows from (1) that $f : X \longrightarrow Z$ is an isometry. Also, if the mapping $g : X \longrightarrow Y$ is surjective, then $g(X) = Y$ and hence $f : Y \longrightarrow Z$ turns into isometry.

Lemma 2.3 ([1, Lemma 2]). *Let X be a real normed linear space which is strictly convex and $x, y \in X$. Then $u = \frac{1}{2}(x + y)$ is the unique element of X such that*

$$2\|x - u\| = 2\|y - u\| = \|x - y\|.$$

Theorem 2.4. *Let $g : X \longrightarrow Y$ and $f : g(X) \subseteq Y \longrightarrow Z$ be two maps such that*

(i) *g is linear and continuous,*
(ii) *f is a g-isometry,*
(iii) *Z is strictly convex.*

Then f is linear on $g(X)$.

Proof. If $f(0) \neq 0$, then the mapping $h : g(X) \longrightarrow Z$ defined by $h(g(x)) := f(g(x)) - f(0)$ is a g-isometry and $h(0) = 0$. So, without loss of generality we may assume that $f(0) = 0$. Since f is a g-isometry we get

$$2\|f(g(\frac{x+y}{2})) - f(g(x))\| = 2\|g(\frac{x+y}{2}) - g(x)\| = \|g(x) - g(y)\|.$$

Similarly,

$$2\|f(g(\frac{x+y}{2})) - f(g(y))\| = \|g(x) - g(y)\|$$

for all $x, y \in X$. Now it follows from Lemma 2.3 that

$$f(g(\frac{x+y}{2})) = \frac{1}{2}(f(g(x)) + f(g(y))).$$

Let $T : X \longrightarrow Z$ be defined by $T(x) = f(g(x))$. Since g is continuous and f is a g-isometry, f and hence T is continuous. As $f(0) = g(0) = 0$, it follows from Lemma 2.2 of [10] that T is linear. Consequently, f is linear on $g(X)$. □

In Theorem 2.4, if $X = Y$ and $g : X \longrightarrow X$ is the identity map, then we deduce the next result.

Corollary 2.5 ([1]). *Suppose that $f : X \longrightarrow Z$ is an isometry between real normed linear spaces. If Z is strictly convex, then f is linear.*

The following example was constructed by Baker [1]. Hear we adopt it for g-isometry with minor changes.

It shows that a g-isometry can be not only nonlinear but also homogeneous of degree one. Moreover, it proves that the strict convexity of Z in Theorem 2.4 is essential.

Example 2.6. Let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$\phi(x,y) = \begin{cases} y & y \in [0,x], \quad \text{or} \quad y \in [x,0], \\ x & x \in [0,y], \quad \text{or} \quad x \in [y,0], \\ 0 & \text{otherwise.} \end{cases}$$

Then

(i) ϕ is homogeneous, i.e., $\phi(\lambda x, \lambda y) = \lambda \phi(x,y)$ for all $x, y, \lambda \in \mathbb{R}$,

(ii) For every $(x,y), (a,b) \in \mathbb{R}^2$,

$$|\phi(x,y) - \phi(a,b)| \leq \sqrt{(x-a)^2 + (y-b)^2}.$$

(iii) ϕ is not linear.

Let $X = Y = \mathbb{R}^2$ with the usual normed linear space structure, and let $Z = \mathbb{R}^3$ with the usual vector space structure. Then, Z with norm

$$\|(x,y,z)\| = \max\{\sqrt{x^2+y^2}, |z|\},$$

is a normed linear space. Define $g : X \longrightarrow Y$ by $g(x,y) = (y,x)$ and $f : Y \longrightarrow Z$ via

$$f(x,y) = (x, y, \phi(x,y)).$$

Then g is linear and it follows from (i), (ii) and (iii) that f is a homogeneous g-isometry which is not linear.

Following [6], let

$$\operatorname{def}(\phi) = \|\phi(\frac{x+y}{2}) - \frac{1}{2}(\phi(x) + \phi(y))\|,$$

denote the possible "affine defect" of $\phi : X \longrightarrow Y$.

Next we prove the Mazur-Ulam theorem for g-isometry.

Theorem 2.7. *Let $g : X \longrightarrow Y$ and $f : g(X) \subseteq Y \longrightarrow Z$ be two maps such that g is affine and f is a surjective g-isometry. Then f is affine on $g(X)$.*

Proof. Let $x, y \in X$ arbitrary and fixed. For $T := f \circ g$, we have

$$\begin{aligned} \operatorname{def}(T) &\leq \frac{1}{2}\|T(\frac{x+y}{2}) - T(x)\| + \frac{1}{2}\|T(\frac{x+y}{2}) - T(y)\| \\ &= \frac{1}{2}\|g(\frac{x+y}{2}) - g(x)\| + \frac{1}{2}\|g(\frac{x+y}{2}) - g(y)\| = \frac{1}{2}\|g(x) - g(y)\|. \end{aligned}$$

Therefore $\frac{1}{2}\|g(x) - g(y)\|$ is uniform bound on the defect. Define $h : Z \longrightarrow Z$ by

$$h(z) = T(x) + T(y) - z,$$

and consider $f_1 : g(X) \longrightarrow g(X)$ with $f_1 := f^{-1} \circ h \circ f$. Then

$$f_1(g(x)) = f^{-1} \circ h \circ f(g(x)) = f^{-1} \circ f(g(y)) = g(y),$$

and similarly, $f_1(g(y)) = g(x)$. Since f is surjective, for $z_1, z_2 \in Z$ there exist $x, y \in X$ such that $f(g(x)) = z_1$ and $f(g(y)) = z_2$. Then

$$\|z_1 - z_2\| = \|f(g(x)) - f(g(y))\| = \|g(x) - g(y)\| = \|f^{-1}(z_1) - f^{-1}(z_2)\|.$$

Thus, $f^{-1} : Z \longrightarrow g(X)$ is an isometry and hence

$$\begin{aligned}
\mathrm{def}(f_1 \circ g) &= \|f_1 \circ g(\frac{x+y}{2}) - \frac{1}{2}(f_1 \circ g(y) + f_1 \circ g(x))\| \\
&= \|f^{-1} \circ h \circ T(\frac{x+y}{2}) - \frac{1}{2}(g(x) + g(y))\| \\
&= \|f^{-1}\big(T(x) + T(y) - T(\frac{x+y}{2})\big) - f^{-1}\big(T(\frac{x+y}{2})\big)\| \\
&= \|T(x) + T(y) - 2T(\frac{x+y}{2})\| \\
&= 2\mathrm{def}(T).
\end{aligned}$$

Now by the same method as in the proof of [6], we get $\mathrm{def}(T) = 0$. Hence

$$T(\frac{x+y}{2}) = \frac{1}{2}(T(x) + T(y))$$

for all $x, y \in X$. Therefore, T is affine by Lemma 2.2 of [10]. As g is affine, we conclude that f is affine on $g(X)$. □

Corollary 2.8 ([4])**.** *Every bijective isometry $f : X \longrightarrow Z$ between real normed linear spaces is affine.*

Example 2.9. Let $X = c_0$, the Banach space of all sequences of scalars that converge to 0, with the norm $\|x_j\|_\infty = \sup\{|x_j| : j \in \mathbb{N}\}$ and let $f : X \longrightarrow X$ be defined by

$$f(x) = f(x_1, x_2, x_3, \ldots) = (x_1, 1 - |x_1|, x_2, x_3, \ldots)$$

for all $x \in X$. Then for $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$ in X we have

$$\begin{aligned}
\|f(x) - f(y)\|_\infty &= \|f(x_1, x_2, x_3, \ldots) - f(y_1, y_2, y_3, \ldots)\|_\infty \\
&= \|(x_1 - y_1, |y_1| - |x_1|, x_2 - y_2, \ldots)\|_\infty \\
&= \|(x_1 - y_1, x_2 - y_2, \ldots)\|_\infty \\
&= \|x - y\|_\infty.
\end{aligned}$$

Thus, f is an isometry but it is not affine. Therefore the surjectivity of f in preceding corollary is essential. Moreover, this example shows that the assumption Z of being strictly convex in Corollary 2.5 can not be removed.

A mapping $f : X \longrightarrow Y$ between two real normed linear spaces satisfies the distance one preserving property ($DOPP$) if for all $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$.

Theorem 2.10. *Let $g : X \longrightarrow Y$ and $f : g(X) \subseteq Y \longrightarrow Z$ be two maps such that*

(i) *g is linear and* $\dim X \geq 1$.
(ii) *for all $x, y \in X$,*

$$\|f(g(x)) - f(g(y))\| \leq \|g(x) - g(y)\|.$$

(iii) *f satisfies the $(DOPP)$ on $g(X)$.*

Then f is a g-isometry.

Proof. Let $x, y \in X$ with $\|g(y) - g(x)\| < 1$. This is possible, because let $a, b \in X$ with $a \neq b$. Take $\alpha = \|g(a)\|$ and $\beta = \|g(b)\|$. Since g is linear, there exist $x, y \in X$ such that $g(x) = \frac{1}{4\alpha} g(a)$ and $g(y) = \frac{1}{4\beta} g(b)$. So

$$\|g(y) - g(x)\| \leq \|g(y)\| + \|g(x)\| \leq \frac{1}{4} + \frac{1}{4} < 1.$$

Suppose that

$$\|f(g(x)) - f(g(y))\| < \|g(x) - g(y)\|. \tag{2}$$

Since g is linear, we have

$$g(x) + \frac{1}{\|g(x) - g(y)\|}(g(y) - g(x)) \in g(X).$$

Thus, there exists $z \in X$ such that

$$g(z) = g(x) + \frac{1}{\|g(x) - g(y)\|}(g(y) - g(x)).$$

Hence

$$\|g(z) - g(x)\| = 1, \qquad \|g(z) - g(y)\| = 1 - \|g(y) - g(x)\|.$$

From (iii) we get

$$\begin{aligned} 1 = \|f(g(z)) - f(g(x))\| &\leq \|f(g(z)) - f(g(y))\| + \|f(g(y)) - f(g(x))\| \\ &< \|g(z) - g(y)\| + \|g(y) - g(x)\| \\ &= 1 - \|g(y) - g(x)\| + \|g(y) - g(x)\| = 1, \end{aligned}$$

which is not possible. Therefore the equality in (2) holds, i.e.,

$$\|f(g(x)) - f(g(y))\| = \|g(x) - g(y)\|, \qquad x, y \in X,$$

and hence f is a g-isometry. □

Example 2.11. Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $g(x, y) = (y, x)$ and $f(x, y) = (x, |x|)$. Then for all $a, b, x, y \in \mathbb{R}$,

$$\begin{aligned} \|f(g(x, y)) - f(g(a, b))\| &= \|(y, |y|) - (b, |b|)\| \\ &= \max\{|y - b|, |y| - |b|\} \\ &= |y - b| \\ &\leq \|g(x, y) - g(a, b)\|. \end{aligned}$$

Consequently, the conditions (i) and (ii) of above theorem are fulfilled. However, f is not g-isometry, because the condition (iii) is false, in general.

Let $f : X \longrightarrow X$ be an f-isometry, i.e., for all $x, y \in X$,

$$\|f^2(x) - f^2(y)\| = \|f(x) - f(y)\|.$$

Then, f need not be isometry or affine. Of course, f is an isometry whenever it is surjective and hence in this case f is affine by Corollary 2.8.

Proposition 2.12. *Suppose that $f : X \longrightarrow X$ is an f-isometry. If f is continuous with dense range, then f is an isometry.*

Proof. For $x, y \in X$, there exist sequences (x_n), (y_n) in X such that $f(x_n) \longrightarrow x$ and $f(y_n) \longrightarrow y$. Now it follows from the continuity of norm that

$$\|f(x_n) - f(y_n)\| \longrightarrow \|x - y\|.$$

On the other hand, by the continuity of f, $f^2(x_n)$ and $f^2(y_n)$ tends to $f(x)$ and $f(y)$, respectively. Hence

$$\|f^2(x_n) - f^2(y_n)\| \longrightarrow \|f(x) - f(y)\|.$$

Since f is an f-isometry, we get $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in X$. □

The continuity and the condition that f has a dense range in above result are essential as is shown the following example.

Example 2.13. (i) *Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(t) = |t|$. Then f is an f-isometry and it is continuous, but the range of f is not dense in $\mathbb{R}$. However, f is not isometry.*

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Then f is an f-isometry and its range is dense in $\mathbb{R}$, but it is false to be continuous. However, f is not isometry.

Is $f : X \longrightarrow X$ affine with the same hypotheses of Proposition 2.12? More generally, the following question can be raised.

Question 2.14. Is every dense range isometry $f : X \longrightarrow Y$ between real normed linear spaces affine?

Acknowledgments. The author gratefully acknowledges the helpful comments of the anonymous referees.

References

[1] J. A. Baker, *Isometries in normed spaces*, Amer. Math. Monthly **78** (1971), 655–658. https://doi.org/10.2307/2316577

[2] T. Figiel, P. Šemrl, and J. Väisälä, *Isometries of normed spaces*, Colloq. Math. **92** (2002), no. 1, 153–154. https://doi.org/10.4064/cm92-1-13

[3] H. Khodaei and A. Mohammadi, *Generalizations of Alesandrov problem and Mazur-Ulam theorem for two-isometries and two-expansive mappings*, Commun. Korean Math. Soc. **34** (2019), no. 3, 771–782. https://doi.org/10.4134/CKMS.c180200

[4] S. Mazur and S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C. R. Acad. Sci. Paris. **194** (1932), 946–948.

[5] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, 183, Springer-Verlag, New York, 1998. https://doi.org/10.1007/978-1-4612-0603-3

[6] B. Nica, *The Mazur-Ulam theorem*, Expo. Math. **30** (2012), no. 4, 397–398. https://doi.org/10.1016/j.exmath.2012.08.010

[7] T. M. Rassias and P. Šemrl, *On the Mazur-Ulam theorem and the Aleksandrov problem for unit distance preserving mappings*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 919–925. https://doi.org/10.2307/2160142

[8] J. Väisälä, *A proof of the Mazur-Ulam theorem*, Amer. Math. Monthly **110** (2003), no. 7, 633–635. https://doi.org/10.2307/3647749

[9] A. Vogt, *Maps which preserve equality of distance*, Studia Math. **45** (1973), 43–48. https://doi.org/10.4064/sm-45-1-43-48

[10] A. Zivari-Kazempour and M. R. Omidi, *On the Mazur-Ulam theorem for Fréchet algebras*, Proyecciones **39** (2020), no. 6, 1647–1654.

Abbas Zivari-Kazempour
Department of Mathematics
Ayatollah Borujerdi University
Borujerd, Iran
Email address: zivari@abru.ac.ir, zivari6526@gmail.com