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# NEW GENERALIZATION OF THE WRIGHT SERIES IN TWO VARIABLES AND ITS PROPERTIES

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Abstract. The main aim of this paper is to introduce a new generalization of the Wright series in two variables, which is expressed in terms of Hermite polynomials. The properties of the freshly defined function involving its auxiliary functions and the integral representations are established. Furthermore, a Gauss-Hermite quadrature and Gaussian quadrature formulas have been established to evaluate some integral representations of our main results and compare them with our theoretical evaluations using graphical simulations.

## 1. Introduction

Over the past decades, the special functions have received a particular attention from the mathematical physics researchers for its significance in many applications like engineering [2,3], optics communications [17,26], creation of new laser beams [6, 12, 36], among many others. Some extensions of these functions as Gamma, beta, poly-Bernoulli numbers, hypergeometric, Wright, Wright-Bessel and Fox-Wright functions have been developed [1, 4, 7–11, 13, 18, 23, 27, 32, 34].

In the thirties, Wright investigated, in the partitions theory, a convergent series representation named the Wright function [40–43]. By working on the time-fractional diffusion-wave equation, Mainardi introduced in his analysis two auxiliary functions of the Wright type interrelated through $F_\nu(z) = \nu z M_\nu(z)$ to study fractional calculus and probability theory [24, 25]. At the beginning of the last century, Mittag-Leffler studied an entire function referred as $E_\alpha(z)$ and defined a series representation which gives a simple generalization of the exponential function [14, 28–31]. In 1905, Wiman introduced a Mittag-Leffler function $E_{\alpha,\beta}(z)$ with two parameters [39] which is examined in the fifties by Humbert and Agarwal [16]. In 2011, Özergin et al. presented some generalizations of Gamma, beta and hypergeometric functions and their transformation formulas and properties [34]. In an interesting paper that was published in

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2015, El-Shahed and Salem introduced an extension of the classical Wright function $W^{\gamma,\delta}_{\alpha,\beta}(z)$, Kummer confluent hypergeometric function and two auxiliary functions $M_\alpha(z)$ and $F_\alpha(z)$ [13].

A few years ago, a new type of integral expressions associated with the generalized (Wright) hypergeometric function are established by Khan et al. [20]. Based on the extended beta function, the extension of Wright-Bessel function and its properties are introduced by Arshad et al. [4], while Khan and Nizar developed an integral formula involving Wright generalized Bessel function as well as some new integral expressions as particular cases [18].

On the other hand, based on the work of Özergin et al. [34], Khan et al. have recently developed a new extension of the generalized Wright function [21] by using generalized beta function. Also, they analyzed some properties of this new series. Lately, Khan et al. derived some properties of certain integral formulas involving the generalized Wright function [22] and the Redheffer-type of the inequalities including generalized Fox-Wright function are defined by Naheed et al. [32].

This study provides some information about one of the special functions known as Wright function that is denoted by $W_{\alpha,\beta}(\cdot)$. This function, introduced by Wright in 1933, is defined by the following series (see [40–42])

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n+\beta)} \tag{1.1}$$

with $\alpha > -1$ and $\beta \in \mathbb{C}$.

The function $W_{\alpha,\beta}(z)$, defined in the whole complex plane, is an entire function and its order is $\frac{1}{(1+\alpha)}$.

In 2015, El-Shahed and Salem [13] generalized the Wright function by introducing the following entire function also of order $\frac{1}{(1+\alpha)}$

$$W^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{1.2}$$

where $\alpha$ is a real ($\alpha > -1$), $\beta$, $\gamma$, $\delta$, $z \in \mathbb{C}$, $\delta \neq 0, -1, -2, \dots, |z| < 1$, $\Gamma(\cdot)$ is the gamma function and $(\chi)_n = \frac{\Gamma(\chi+n))}{\Gamma(\chi)}$ is the usual Pochhammer symbol.

In a continuation of this investigation, Khan et al. [21] introduced a new generalization of the Wright function by using the generalization of gamma and Euler's beta functions proposed by Özergin et al. [34], defined by

$$W^{\gamma,\delta,\lambda,\sigma;c,d}_{\alpha,\beta}(z;p) = \sum_{n=0}^{\infty} \frac{B^{c,d}_p(\gamma+n,\lambda-\gamma)}{B(\gamma,\lambda-\gamma)} \frac{(\sigma)_n}{(\delta)_n} \frac{z^n}{n!\Gamma(\alpha n+\beta)}, \tag{1.3}$$

where $\alpha > -1$, $\delta \neq 0, -1, -2, \dots; \gamma$, $\delta$, $\lambda$, $\sigma$, $\beta$, $\alpha \in \mathbb{C}$, with $z \in \mathbb{C}$ and $|z| < 1$.

The current paper introduces a new generalization of the Wright series in two variables, which is expressed in terms of the Hermite polynomials and also its auxiliary functions.

For more information, on the Hermite polynomials, we suggest to refer to [19, 33, 37].

**Definition.** A new generalization of the Wright series in two variables is defined as

$$B^{\gamma,\delta}_{\alpha,\beta}(u,v) = 2iu\sqrt{\pi}e^{-v^2}\sum_{n=0}^{\infty}\frac{1}{a^{\gamma,\delta}_{\alpha,\beta}(n)}\frac{u^n}{n!}H_n(v), \tag{1.4}$$

where $H_n$ is the Hermite polynomial of order $n$ and

$$a^{\gamma,\delta}_{\alpha,\beta}(n) = \frac{(\delta)_n}{(\gamma)_n}\Gamma(\alpha n+\beta), \tag{1.5}$$

with $\delta \neq 0, -1, -2, \ldots$.

It is known that there are four auxiliary functions of Wright function for $0<\alpha<1$ which are defined as

$$M_\alpha(z) = W_{-\alpha,1-\alpha}(-z); \quad M^{\gamma,\delta}_\alpha(z) = W^{\gamma,\delta}_{-\alpha,1-\alpha}(-z); \tag{1.6}$$

$$F_\alpha(z) = W_{-\alpha,0}(-z); \quad F^{\gamma,\delta}_\alpha(z) = W^{\gamma,\delta}_{-\alpha,0}(-z). \tag{1.7}$$

We define two new auxiliary functions of any order $\alpha \in (0,1)$ and for each $z \neq 0$ as follows

$$\mathcal{M}^{\gamma,\delta}_\alpha(u,v) = B^{\gamma,\delta}_{-\alpha,1-\alpha}(-u,v) = \sum_{n=0}^{\infty}\frac{(-u)^n}{n!}\frac{H_n(v)}{a_{-\alpha,1-\alpha}(n)} \tag{1.8}$$

and

$$\mathcal{F}^{\gamma,\delta}_\alpha(u,v) = B^{\gamma,\delta}_{-\alpha,0}(-u,v) = \sum_{n=0}^{\infty}\frac{(-u)^n}{n!}\frac{H_n(v)}{a_{-\alpha,0}(n)}, \tag{1.9}$$

where

$$a_{-\alpha,1-\alpha}(n) = \frac{(\delta)_n}{(\gamma)_n}\Gamma(1-\alpha(n+1)), \tag{1.10}$$

and

$$a_{-\alpha,0}(n) = \frac{(\delta)_n}{(\gamma)_n}\Gamma(-\alpha n). \tag{1.11}$$

## 2. Properties of the generalized Wright function

Below, we give some properties of these series by establishing some theorems.

**Theorem 2.1.** *Let $\alpha > -1$, $\Re(p) > 0$ $\delta \neq 0, -1, -2, \ldots$, and $z, \beta, \gamma, \delta \in \mathbb{C}$. Then the function* (1.4) *can be represented by*

$$B^{\gamma,\delta}_{\alpha,\beta}(u,v) = \int_{-\infty}^{+\infty} e^{-pz^2+2qz}W^{\gamma,\delta}_{\alpha,\beta}(z)dz, \tag{2.1}$$

*where*

$$u = \frac{1}{2i\sqrt{p}} \quad \textit{and} \quad v = \frac{iq}{\sqrt{p}}. \tag{2.2}$$

*Proof.* By substituting (1.2) into (2.1) one finds

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} e^{-pz^2+2qz} W_{\alpha,\beta}^{\gamma,\delta}(z)dz \\ &= \sum_{n=0}^{\infty} \frac{1}{a_{\alpha,\beta}^{\gamma,\delta}(n)} \frac{1}{n!} \int_{-\infty}^{+\infty} z^n e^{-pz^2+2qz} dz. \end{aligned} \tag{2.3}$$

By using the identity [5]

$$\int_{-\infty}^{+\infty} z^n e^{-pz^2+2qz} dz = e^{q^2/p} \sqrt{\frac{\pi}{p}} \left(\frac{1}{2i\sqrt{p}}\right)^n H_n\left(\frac{iq}{\sqrt{p}}\right) \tag{2.4}$$

with $\Re(p) > 0$, (2.3) can be written as

$$I = e^{q^2/p} \sqrt{\frac{\pi}{p}} \sum_{n=0}^{\infty} \frac{1}{a_{\alpha,\beta}^{\gamma,\delta}(n)} \frac{(1/2i\sqrt{p})^n}{n!} H_n\left(\frac{iq}{\sqrt{p}}\right). \tag{2.5}$$

By taking $u$ and $v$ given by (2.2), Theorem 2.1 is proved. □

*Remark* 2.2. It is interesting to see that (2.1) can be written as

$$B_{\alpha,\beta}^{\gamma,\delta}(u,v) = \int_{-\infty}^{+\infty} e^{\frac{z^2}{4u^2} - \frac{vz}{u}} W_{\alpha,\beta}^{\gamma,\delta}(z)dz. \tag{2.6}$$

**Theorem 2.3.** *Let $\alpha > -1$, $\beta$, $\gamma$, $\delta \in \mathbb{C}$. Then the auxiliary functions introduced earlier can be expressed as*

$$\mathcal{M}_{\alpha}^{\gamma,\delta}(u,v) = \int_{-\infty}^{+\infty} e^{\frac{z^2}{4u^2} + \frac{vz}{u}} M_{\alpha}^{\gamma,\delta}(-z)dz \tag{2.7}$$

*and*

$$\mathcal{F}_{\alpha}^{\gamma,\delta}(u,v) = \int_{-\infty}^{+\infty} e^{\frac{z^2}{4u^2} + \frac{vz}{u}} F_{\alpha}^{\gamma,\delta}(-z)dz. \tag{2.8}$$

*Proof.* By using (2.1) and (1.8), one finds

$$\begin{aligned} B_{-\alpha,1-\alpha}^{\gamma,\delta}(-u,v) &= \int_{-\infty}^{+\infty} e^{\frac{z^2}{4u^2} + \frac{vz}{u}} W_{-\alpha,1-\alpha}^{\gamma,\delta}(z)dz \\ &= \mathcal{M}_{\alpha}^{\gamma,\delta}(u,v) \\ &= \int_{-\infty}^{+\infty} e^{\frac{z^2}{4u^2} + \frac{vz}{u}} M_{\alpha}^{\gamma,\delta}(-z)dz, \end{aligned} \tag{2.9}$$

which proves (2.7). For (2.8), (1.9) and (2.1) is used for $-\alpha$ and $\beta = 0$. It is easy to deduce the following identities

$$
\begin{aligned}
B^{\gamma,\delta}_{-\alpha,0}(-u,v) &= \mathcal{F}^{\gamma,\delta}_{\alpha}(u,v) \\
&= \int_{-\infty}^{+\infty} e^{\frac{z^2}{4u^2}+\frac{vz}{u}} W^{\gamma,\delta}_{-\alpha,0}(z)dz \\
&= \int_{-\infty}^{+\infty} e^{\frac{z^2}{4u^2}+\frac{vz}{u}} F^{\gamma,\delta}_{\alpha}(-z)dz.
\end{aligned}
\tag{2.10}
$$

This completes the proof of Theorem 2.3. □

**Theorem 2.4.** *Let* $\beta$ *and* $\delta \neq 0, -1, -2, \ldots$. *Then* (1.4) *can be expressed in two variables as*

$$
B^{\gamma,\delta}_{\alpha,\beta}(u,v) = \frac{1}{\Gamma(\beta)} \sum_{n=l=0}^{\infty} \frac{(\gamma)_{n+2l}}{(\beta)_{\alpha(n+2l)}(\delta)_{n+2l}} \frac{(2uv)^n}{n!} \frac{(-u^2)^l}{l!}.
\tag{2.11}
$$

*Proof.* By using the following expansion of Hermite polynomial [15]

$$
H_n(v) = \sum_{l=0}^{[n/2]} \frac{(-1)^l n!}{l!(n-2l)!} (2v)^{n-2l},
\tag{2.12}
$$

(1.4) becomes

$$
B^{\gamma,\delta}_{\alpha,\beta}(u,v) = \sum_{n=0}^{\infty} \sum_{l=0}^{[n/2]} A(l,n),
\tag{2.13}
$$

where

$$
A(l,n) = \frac{(-1)^l}{l!(n-2l)! a^{\gamma,\delta}_{\alpha,\beta}(n)} u^n (2v)^{n-2l}.
\tag{2.14}
$$

With the help of the identity [38] of the double summation

$$
\sum_{n=0}^{\infty} \sum_{l=0}^{[n/2]} A(l,n) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} A(l, n+2l)
\tag{2.15}
$$

with $[x]$ denotes the greatest integer in $x$, we obtain (2.11). □

In the next theorem, we give the Mellin-Barnes contour integral representation of the generalized Wright function $B^{\gamma,\delta}_{\alpha,\beta}$.

**Theorem 2.5.** *Let* $\alpha > -1$, $\beta$, $\gamma$, $\delta$ *and* $z \in \mathbb{C}$, $\delta \neq 0, -1, -2, \ldots$. *Then the Mellin-Barnes contour integral representation of* $B^{\gamma,\delta}_{\alpha,\beta}$ *is given by*

$$
{}^{\gamma,\delta}_{\alpha,\beta}(u,v) = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \frac{ue^{-v^2/2}}{\sqrt{\pi}} \int_L \frac{(\sqrt{2})^t \Gamma(\gamma+t)}{\Gamma(\delta+t)\Gamma(\beta+\alpha t)} \Gamma(-t)(-u)^t D_t\left(\sqrt{2}v\right) dt,
\tag{2.16}
$$

*where* $L$ *is the Mellin-Barnes contour and* $D_t$ *is the parabolic function.*

*Proof.* By the use of Theorem 1 of [13], our (2.1) can be written as

$$B_{\alpha,\beta}^{\gamma,\delta}(u,v)=\int_{-\infty}^{+\infty}e^{-pz^2+2qz}W_{\alpha,\beta}^{\gamma,\delta}(z)dz$$

$$=\frac{\Gamma(\delta)}{2\pi i\Gamma(\gamma)}\int_{-\infty}^{+\infty}e^{-pz^2+2qz}dz\int_L\frac{\Gamma(\gamma+t)}{\Gamma(\delta+t)\Gamma(\beta+\alpha t)}\Gamma(-t)(-z)^t dt, \tag{2.17}$$

where $u=\frac{1}{2i\sqrt{p}}$, and $v=\frac{iq}{\sqrt{p}}$.

On interchanging the order of integration in the last equation and using (2.1), we obtain

$$B_{\alpha,\beta}^{\gamma,\delta}(u,v)=\frac{\Gamma(\delta)}{2\pi i\Gamma(\gamma)}\int_L\frac{(-1)^t\Gamma(\gamma+t)}{\Gamma(\delta+t)\Gamma(\beta+\alpha t)}\Gamma(-t)dt\int_{-\infty}^{+\infty}e^{-pz^2+2qz}z^t dz. \tag{2.18}$$

With the help of the following identity [35]

$$\int_{-\infty}^{+\infty}x^{\nu}e^{-\beta^2x^2-irx}dx=\frac{\sqrt{\pi}}{\left(i\sqrt{2}\right)^{\nu}}\frac{1}{\beta^{\nu+1}}e^{-r^2/8\beta^2}\,D_{\nu}\left(\frac{r}{\beta\sqrt{2}}\right) \tag{2.19}$$

with $\Re(\beta)>0$ and $\Re(\nu)>-1$, (2.4) can be written as

$$\int_{-\infty}^{+\infty}z^t e^{-pz^2+2qz}dz=\sqrt{\frac{\pi}{p}}e^{q^2/2p}\left(\frac{1}{i\sqrt{2p}}\right)^t D_t\left(\sqrt{2}\frac{iq}{\sqrt{p}}\right). \tag{2.20}$$

By using the definitions of $u$ and $v$ given by (2.2), we obtain (2.16). This completes the proof. □

**Theorem 2.6.** *Let $\alpha>-1$, $\beta$, $\gamma$, $\delta$ and $z\in\mathbb{C}$, $\delta\neq 0,-1,-2,\ldots$. Then $B_{\alpha,\beta}^{\gamma,\delta}$ can be represented as*

$$\begin{aligned}B_{\alpha,\beta}^{\gamma,\delta}(u,v)&=2iu\sqrt{\pi}e^{-v^2}\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\sum_{n=0}^{\infty}\frac{u^n}{n!\Gamma(\alpha n+\beta)}H_n(v)\\&\quad\times\int_0^1 x^{n+\gamma-1}(1-x)^{\delta-\gamma-1}dx.\end{aligned} \tag{2.21}$$

*Proof.* By the use of Theorem 2 of [13], (2.1) becomes

$$B_{\alpha,\beta}^{\gamma,\delta}(u,v)=\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_{-\infty}^{+\infty}e^{-pz^2+2qz}dz\int_0^1x^{\gamma-1}(1-x)^{\delta-\gamma-1}W_{\alpha,\beta}(zx)\,dx$$

$$=\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1x^{\gamma-1}(1-x)^{\delta-\gamma-1}dx\int_{-\infty}^{+\infty}e^{-pz^2+2qz}W_{\alpha,\beta}(xz)\,dz, \tag{2.22}$$

where

$$W_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\alpha n+\beta)} \tag{2.23}$$

with $\alpha>-1$ and $\beta\in\mathbb{C}$.

Hence, by using (2.4), (2.22) can be rearranged to write
(2.24)
$$B_{\alpha,\beta}^{\gamma,\delta}(u,v)=\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\sum_{n=0}^{\infty}\frac{I_n}{n!\Gamma(\alpha n+\beta)}\int_0^1 x^{n+\gamma-1}(1-x)^{\delta-\gamma-1}dx,$$

where
$$I_n=e^{q^2/p}\sqrt{\frac{\pi}{p}}\left(\frac{1}{2i\sqrt{p}}\right)^n H_n\left(\frac{iq}{\sqrt{p}}\right).$$

By using the definitions of $u$ and $v$, (2.21) is proved. This completes the proof of Theorem 2.6. □

**Corollary 2.7.** *We know that* (*see* [35])
$$\int_0^1 x^{\mu-1}\left(1-x^{\lambda}\right)^{\nu-1}dx=\frac{1}{\lambda}B\left(\frac{\mu}{\lambda},\nu\right)\ \textit{with}\ \Re(\mu)>0,\ \Re(\nu)>0\ \textit{and}\ \lambda>0.$$

*Taking* $\mu=n+\gamma$, $\lambda=1$ *and* $\nu=\delta-\gamma$, *we find the following result*

$$\int_0^1 x^{n+\gamma-1}(1-x)^{\delta-\gamma-1}dx=B\left(n+\gamma,\delta-\gamma\right). \tag{2.25}$$

*With the use of this last equation, one can write* (2.21) *as*

$$\begin{aligned}B_{\alpha,\beta}^{\gamma,\delta}(u,v)&=2iu\sqrt{\pi}e^{-v^2}\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\\&\quad\times\sum_{n=0}^{\infty}\frac{u^n}{n!\Gamma(\alpha n+\beta)}H_n(v)\,B(n+\gamma,\delta-\gamma).\end{aligned} \tag{2.26}$$

**Theorem 2.8.** *The Mellin transform of* $B_{\alpha,\beta}^{\gamma,\delta}$ *is given by*

$$\begin{aligned}\mathfrak{M}\left\{B_{\alpha,\beta}^{\gamma,\delta}(u,v);s\right\}&=\frac{\Gamma(s)}{2}u^s(-4u^2)^{\left(\frac{1-s}{2}\right)}\\&\quad\times\sum_{n=0}^{\infty}\frac{\left[1+(-1)^{n-s}\right]}{a_{\alpha,\beta}^{\gamma,\delta}(n)}\frac{(-4u^2)^{\frac{n}{2}}}{n!}\Gamma\left(\frac{n+1-s}{2}\right).\end{aligned} \tag{2.27}$$

*Proof.* The definition of the Mellin transform of $B_{\alpha,\beta}^{\gamma,\delta}$ yields with the help of (2.6)

$$\begin{aligned}\mathfrak{M}\left\{B_{\alpha,\beta}^{\gamma,\delta}(u,v);s\right\}&=\int_0^{+\infty}v^{s-1}B_{\alpha,\beta}^{\gamma,\delta}(u;v)dv\\&=\int_0^{+\infty}v^{s-1}dv\int_{-\infty}^{+\infty}e^{\frac{z^2}{4u^2}-\frac{vz}{u}}W_{\alpha,\beta}^{\gamma,\delta}(z)dz.\end{aligned} \tag{2.28}$$

On interchanging the order of integration in (2.28) and by using the identity

$$\mathfrak{M}\left\{e^{-at};s\right\}=\int_0^{+\infty}v^{s-1}e^{-av}dv=a^{-s}\Gamma(s), \tag{2.29}$$

we obtain

$$\mathfrak{M}\left\{B_{\alpha,\beta}^{\gamma,\delta}(u,v);s\right\}=\Gamma(s)u^{s}I_{s}, \tag{2.30}$$

where

$$I_{s}=\int_{-\infty}^{+\infty}z^{-s}e^{\frac{z^{2}}{4u^{2}}}W_{\alpha,\beta}^{\gamma,\delta}(z)dz. \tag{2.31}$$

By the use of the expansion of $W_{\alpha,\beta}^{\gamma,\delta}$ given by (1.2) and the identity (2.29), the last integral in (2.31) can be written as

$$\begin{aligned} I_{s}&=\sum_{n=0}^{\infty}\frac{1}{n!a_{\alpha,\beta}^{\gamma,\delta}(n)}\left[1+(-1)^{n-s}\right]\int_{0}^{+\infty}t^{n-s}e^{t^{2}/4u^{2}}dt\\ &=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!a_{\alpha,\beta}^{\gamma,\delta}(n)}\left[1+(-1)^{n-s}\right]\left(\frac{-1}{4u^{2}}\right)^{-\left(\frac{n+1-s}{2}\right)}\Gamma\left(\frac{n+1-s}{2}\right), \end{aligned} \tag{2.32}$$

and finally, we find (2.27). This completes the proof of Theorem 2.8. □

**Theorem 2.9.** *Let $\beta\in\mathbb{C}$ and $m\in\mathbb{N}$. Then*

$$\begin{aligned} B_{0,\beta}^{1,m}(u,v)=&\frac{\Gamma(m)}{\Gamma(\beta)}2i\sqrt{\pi}e^{-v^{2}}u^{2-m}\\ &\times\left\{\left[e^{(2v-u)u}-1\right]H_{1-m}(v-u)-\sum_{k=1}^{m-2}u^{k}H_{1-m+k}(v)\right\}. \end{aligned} \tag{2.33}$$

*Proof.* Starting from the following relation of $W_{0,\beta}^{1,\delta}$ and the Mittag-Leffer function [13, 14]

$$W_{0,\beta}^{1,\delta}(z)=\frac{\Gamma(\delta)}{\Gamma(\beta)}E_{1,\delta}(z) \tag{2.34}$$

with

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}\quad\text{with }\alpha>0. \tag{2.35}$$

For $\delta=m$,

$$W_{0,\beta}^{1,m}(z)=\frac{\Gamma(m)}{\Gamma(\beta)}E_{1,m}(z), \tag{2.36}$$

where

$$E_{1,m}(z)=z^{1-m}\left(e^{z}-\sum_{k=0}^{m-2}\frac{z^{k}}{k!}\right). \tag{2.37}$$

By using (2.1), $B_{0,\beta}^{1,m}$ can be written as

$$\begin{aligned} B_{0,\beta}^{1,m}(u,v) &= \frac{\Gamma(m)}{\Gamma(\beta)} \int_{-\infty}^{+\infty} e^{-pz^2+2qz} E_{1,m}(z)dz \\ &= \frac{\Gamma(m)}{\Gamma(\beta)} \left( I_1 - \sum_{k=0}^{m-2} \frac{I_{2k}}{k!} \right), \end{aligned} \tag{2.38}$$

where

$$I_1 = \int_{-\infty}^{+\infty} z^{1-m} e^{-pz^2+(2q+1)z} dz \tag{2.39}$$

and

$$I_{2k} = \int_{-\infty}^{+\infty} z^{1-m+k} e^{-pz^2+2qz} dz. \tag{2.40}$$

By the use of the identity (2.4), (2.38) becomes
(2.41)

$$\begin{aligned} &B_{0,\beta}^{1,m}(u,v) = \frac{\Gamma(m)}{\Gamma(\beta)} \sqrt{\frac{\pi}{p}} e^{q^2/p} \left( \frac{1}{2i\sqrt{p}} \right)^{1-m} \\ &\times \left\{ \left[ e^{\left(q+\frac{1}{4}\right)/p} - 1 \right] H_{1-m} \left[ \frac{i}{\sqrt{p}} \left( q + \frac{1}{2} \right) \right] - \sum_{k=1}^{m-2} \left( \frac{1}{2i\sqrt{p}} \right)^k H_{1-m+k} \left( \frac{iq}{\sqrt{p}} \right) \right\}. \end{aligned}$$

Finally, if the expressions of $u$ and $v$ are used, it is easy to find (2.33). This completes the proof of Theorem 2.9. □

**Theorem 2.10.** *Let $\beta$, $\gamma$ and $\delta \in \mathbb{C}$, and $\mathrm{Re}\,(\delta) > 0$. Then*

$$B_{1,\beta}^{\gamma,\delta}(u,v) = \frac{2i\sqrt{\pi}}{\Gamma(\beta)} u e^{-v^2} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n (\beta)_n} \frac{u^n}{n!} H_n(v). \tag{2.42}$$

*Proof.* By the help of the following identity [13]

$$W_{1,\beta}^{\gamma,\delta}(z) = \frac{1}{\Gamma(\beta)} {}_1F_2(\gamma;\delta,\beta;z) = \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n (\beta)_n} \frac{z^n}{n!}, \tag{2.43}$$

$B_{1,\beta}^{\gamma,\delta}$ is written as

$$B_{1,\beta}^{\gamma,\delta}(u,v) = \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n (\beta)_n n!} \int_{-\infty}^{+\infty} z^n e^{-pz^2+2qz} dz. \tag{2.44}$$

By the use of (2.4), one finds easily (2.42). This completes the proof of Theorem 2.10. □

**Theorem 2.11.** *Let $\beta \in \mathbb{C}$, $u$, $v \in \mathbb{C}$. Then*

$$\begin{aligned} B_{-1,\beta}^{1-\beta,1-\beta}(u,v) &= B_{-1,\beta}(u,v) \\ &= \frac{i\sqrt{2\pi}}{\Gamma(\beta)}\left(\sqrt{2}u\right)^{\beta} e^{-\frac{1}{8u^2}\left(4u^2v^2-4uv-1\right)} D_{\beta-1}\left(\frac{2uv+1}{u\sqrt{2}}\right), \end{aligned} \tag{2.45}$$

*where $D$ is the parabolic function.*

*Proof.* Taking $\alpha = -1$, $\gamma = \delta = 1-\beta$ and using (2.1), we obtain

$$\begin{aligned} B_{-1,\beta}^{1-\beta,1-\beta}(u,v) &= \frac{1}{\Gamma(\beta)} \int_{-\infty}^{+\infty} (1+z)^{\beta-1} e^{-pz^2+2qz} dz \\ &= \frac{1}{\Gamma(\beta)} e^{-(p+2q)} I, \end{aligned} \tag{2.46}$$

where

$$I = \int_{-\infty}^{+\infty} t^{\beta-1} e^{-pt^2+2(p+q)t} dt. \tag{2.47}$$

With the help of (2.19) and taking $\nu = \beta - 1$ and $r = 2i(p+q)$, the integral of (2.47) can be expressed as

$$I = \frac{\sqrt{\pi}}{\left(\sqrt{2}i\right)^{\beta-1} p^{\beta/2}} e^{\frac{4(p+q)^2}{8p}} D_{\beta-1}\left(\frac{2i(p+q)}{\sqrt{2p}}\right). \tag{2.48}$$

The expressions of $u$ and $v$ used in the proof of Theorem 2.7 yield (2.45). This completes the proof of Theorem 2.11. □

**Theorem 2.12.** *Let $\gamma$, $\delta \in \mathbb{C}$. Then*

$$\begin{aligned} &B_{-1/2,1}^{\gamma,\delta}(u,v) \\ &= 2i\sqrt{\pi}ue^{-v^2}\left\{1 - \frac{\gamma}{\delta\sqrt{\pi}} u \sum_{n=0}^{\infty} \frac{(1/2)_n \left(\frac{1+\gamma}{2}\right)_n \left(\frac{2+\gamma}{2}\right)_n}{(3/2)_n \left(\frac{1+\delta}{2}\right)_n \left(\frac{2+\delta}{2}\right)_n} \frac{\left(-u^2/4\right)^n}{n!} H_{2n+1}(v)\right\}. \end{aligned} \tag{2.49}$$

*Proof.* To evaluate $B_{-1/2,1}^{\gamma,\delta}(u,v)$, we use the following expression [13]

$$W_{-1/2,1}^{\gamma,\delta}(-z) = 1 - \frac{\gamma z}{\delta\sqrt{\pi}} {}_3F_3\left(\frac{1}{2}, \frac{1+\gamma}{2}, \frac{2+\gamma}{2}; \frac{3}{2}, \frac{1+\gamma}{2}, \frac{2+\delta}{2}; -\frac{z^2}{4}\right). \tag{2.50}$$

Therefore, $B_{-1/2,1}^{\gamma,\delta}$ can be expressed as

$$B_{-1/2,1}^{\gamma,\delta}(u,v) = A_1 - \frac{\gamma}{\delta\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(1/2)_n \left(\frac{1+\gamma}{2}\right)_n \left(\frac{2+\gamma}{2}\right)_n}{(3/2)_n \left(\frac{1+\delta}{2}\right)_n \left(\frac{2+\delta}{2}\right)_n} \frac{\left(-1/4\right)^n}{n!} A_{2n}, \tag{2.51}$$

where

$$A_1 = \int_{-\infty}^{+\infty} e^{-pz^2+2qz} dz \tag{2.52}$$

and

$$A_{2n}=\int_{-\infty}^{+\infty} z^{2n+1}e^{-pz^2+2qz}dz. \tag{2.53}$$

Using (2.4) yields the expression of these last integrals and one finds easily (2.49). ☐

**Theorem 2.13.** *Let $\gamma$ and $\delta\in\mathbb{C}$. Then*

$$\mathcal{M}_{1/2}^{\gamma,\delta}(u,v)=2iue^{-v^2}\sum_{n=0}^{\infty}\frac{(\gamma/2)_n\left(\frac{1+\gamma}{2}\right)_n}{(\delta/2)_n\left(\frac{1+\delta}{2}\right)_n}\frac{\left(-{}^{u^2}/_4\right)^n}{n!}H_{2n}(v). \tag{2.54}$$

*Proof.* By taking $\alpha=1/2$ in (1.8) and using the following expression [13]

$$M_{1/2}^{\gamma,\delta}(-z)=\frac{1}{\sqrt{\pi}}{}_2F_2\left(\frac{\gamma}{2},\frac{1+\gamma}{2};\frac{\delta}{2},\frac{1+\delta}{2};-\frac{z^2}{4}\right), \tag{2.55}$$

the result is:

$$\mathcal{M}_{1/2}^{\gamma,\delta}(u,v)=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(\gamma/2)_n\left(\frac{1+\gamma}{2}\right)_n}{(\delta/2)_n\left(\frac{1+\delta}{2}\right)_n}\frac{\left(-{}^{1}/_4\right)^n}{n!}\int_{-\infty}^{+\infty}z^{2n}e^{-pz^2+2qz}dz, \tag{2.56}$$

which yields easily (2.54) by the use of (2.4). This completes the proof of Theorem 2.13. ☐

**Theorem 2.14.** *Let $\gamma\in\mathbb{C}$ and $u$, $v\in\mathbb{C}$. Then*

$$\begin{aligned}\mathcal{M}_{1/3}^{\gamma,\gamma+1}(u,v)&=2i\sqrt{\pi}ue^{-v^2}\\&\times\left\{\frac{1}{\Gamma(2/3)}\sum_{n=0}^{\infty}\frac{(\gamma/2)_n}{(2/3)_n({}^{3+\gamma}/_3)_n}\frac{({}^{u}/_3)^{3n}}{n!}H_{3n}(v)\right.\\&\left.-\frac{\gamma u}{(\gamma+1)\Gamma(1/3)}\sum_{n=0}^{\infty}\frac{\left(\frac{1+\gamma}{3}\right)_n}{(4/3)_n\left(\frac{4+\gamma}{3}\right)_n}\frac{({}^{u}/_3)^{3n}}{n!}H_{3n+1}(v)\right\}.\end{aligned} \tag{2.57}$$

*Proof.* Applying the following relation [13]

$$\begin{aligned}M_{1/3}^{\gamma,\gamma+1}(z)&=\frac{1}{\Gamma(2/3)}{}_1F_2\left(\frac{\gamma}{2};\frac{2}{3},\frac{3+\delta}{3};\frac{z^3}{27}\right)\\&-\frac{\gamma z}{(\gamma+1)\Gamma(1/3)}{}_1F_2\left(\frac{1+\gamma}{3};\frac{4}{3},\frac{4+\delta}{3};\frac{z^3}{27}\right)\end{aligned} \tag{2.58}$$

on (2.1) and using (1.6), we obtain for $\alpha=\frac{1}{3}$ and $\delta=\gamma+1$

(2.59)

$$\begin{aligned}\mathcal{M}_{1/3}^{\gamma,\gamma+1}(u,v)&=\frac{1}{\Gamma({}^2/_3)}\sum_{n=0}^{\infty}\frac{(\gamma/2)_n}{(2/3)_n\left(\frac{3+\gamma}{3}\right)_n}\frac{\left({}^1/_{27}\right)^n}{n!}\int_{-\infty}^{+\infty}z^{3n}e^{-pz^2+2qz}dz\\&-\frac{\gamma}{(\gamma+1)\Gamma({}^1/_3)}\sum_{n=0}^{\infty}\frac{\left(\frac{1+\gamma}{3}\right)_n}{(4/3)_n\left(\frac{4+\gamma}{3}\right)_n}\frac{\left({}^1/_{27}\right)^n}{n!}\int_{-\infty}^{+\infty}z^{3n+1}e^{-pz^2+2qz}dz.\end{aligned}$$

With the help of the identity (2.4), this last equation becomes

(2.60)

$$\mathcal{M}_{1/3}^{\gamma,\gamma+1}(u,v) = \frac{e^{q^2/p}\sqrt{\pi/p}}{\Gamma\left(2/3\right)} \sum_{n=0}^{\infty} \frac{(\gamma/2)_n}{(2/3)_n\left(\frac{3+\gamma}{3}\right)_n} \frac{\left(i/216p^{3/2}\right)^n}{n!} H_{3n}\left(\frac{iq}{\sqrt{p}}\right)$$
$$- \frac{\gamma e^{q^2/p}\sqrt{\pi/p}}{(\gamma+1)\Gamma\left(1/3\right)} \frac{1}{2i\sqrt{p}} \sum_{n=0}^{\infty} \frac{\left(\frac{1+\gamma}{3}\right)_n}{(4/3)_n\left(\frac{4+\gamma}{3}\right)_n} \frac{\left(i/216p^{3/2}\right)^n}{n!} H_{3n+1}\left(\frac{iq}{\sqrt{p}}\right).$$

After some simplifications (2.57) is obtained. Thus, the proof of Theorem 2.14 is complete. □

## 3. Graphical simulations of $B_{\alpha,\beta}^{\gamma,\delta}(u,v)$

In this section, some numerical simulations of Theorems 2.1 and 2.6 are performed with respect to the variable $u$ in order to show an agreement between the series and integral representations as well as the effect of some parameters in the evolution of $B_{\alpha,\beta}^{\gamma,\delta}(u,v)$. Below, the used numerical methods are: Gauss-Hermite quadrature and Gaussian quadrature.

**Evaluation of Theorem 2.1**

Fig. 1 illustrates a graphical simulation of (1.4) and (2.1) vs. the variable $u$. The parameters are chosen as $\gamma = 5$, $\delta = 2$ and $v = 2$. Fig. 1 shows that the obtained results are similar. Note that (2.1) is evaluated in this case by Gauss-Hermite quadrature method.

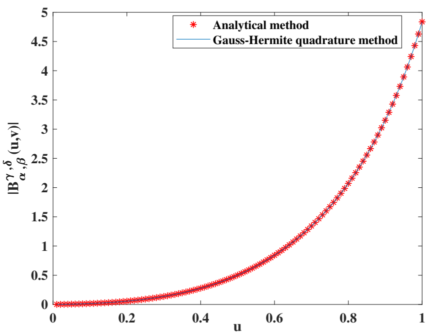
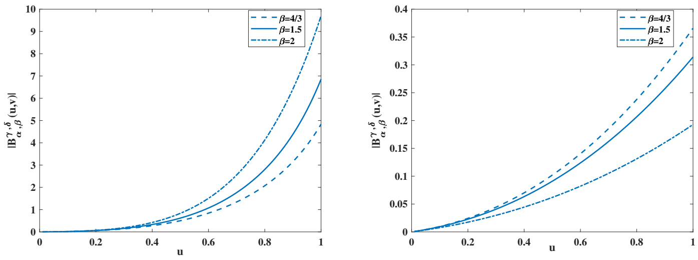



FIGURE 1. Representation of $B_{\alpha,\beta}^{\gamma,\delta}(u,v)$ in terms of $u$ evaluated from (1.4) and (2.1) with $\beta = \frac{4}{3}$ and $\alpha = -\frac{1}{3}$.

The figures below display (1.4) for various values of $\beta$ and $\alpha$. The other parameters are similar of these in Fig. 1. For Fig. 2(A-B), one can note that, when $\alpha = -\frac{1}{3}$, the quantity $B_{\alpha,\beta}^{\gamma,\delta}(u,v)$ takes a large value with the increases of the parameters $\beta$, and when the parameter $\alpha$ takes a positive value it changes its behaviour.




(A) (B)

FIGURE 2. Illustration of $B^{\gamma,\delta}_{\alpha,\beta}(u,v)$ in terms of $u$ evaluated from (1.4) with three values of $\beta$ for (A) $\alpha = -\frac{1}{3}$ and (B) $\alpha = 2$.

**Evaluation of Theorem 2.5**

Fig. 3 presents a numerical representation of (1.4) and (2.17). The expression (2.17) is solved by using Gaussian quadrature method. The parameters are taken as $\gamma = 6$, $\beta = \frac{4}{3}$, $\alpha = -\frac{1}{3}$ and $v = 2$. The results show a good agreement between the numerical and analytical formulas.

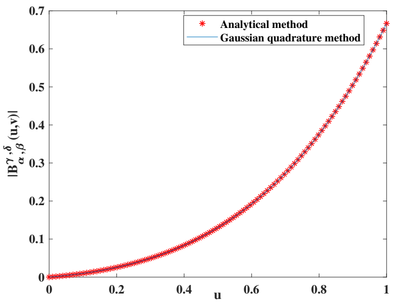
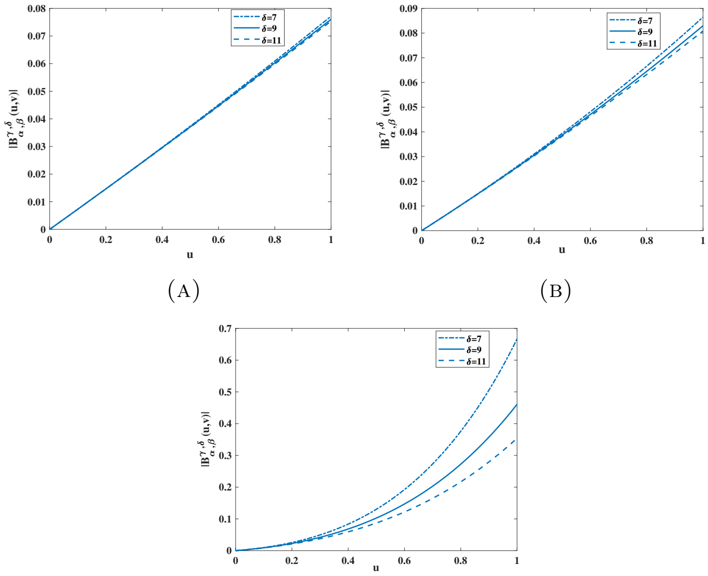



FIGURE 3. Representation of $B^{\gamma,\delta}_{\alpha,\beta}(u,v)$ in terms of $u$ evaluated from (1.4) and (2.17) with $\delta = 7$.

Fig. 4 shows the evolution of $B^{\gamma,\delta}_{\alpha,\beta}(u,v)$ established in (1.4) with three values of $\delta$ and $\gamma$.




(A) (B) (C)

FIGURE 4. Illustration of $B^{\gamma,\delta}_{\alpha,\beta}(u,v)$ in terms of $u$ evaluated from (1.4) for $\delta = 7, 9$ and 11, with (A) $\gamma = 0.1$, (B) $\gamma = 0.3$ and (C) $\gamma = 6$.

From these plots, the conclusion is: when $\Re(\gamma) \to 0$, there is no effect of the parameter $\beta$ on the evolution of $B^{\gamma,\delta}_{\alpha,\beta}(u,v)$, but when $\gamma$ is equal to six, the quantity $B^{\gamma,\delta}_{\alpha,\beta}(u,v)$ increases with increasing $\delta$.

## 4. Conclusion

This study has investigated a new generalization of the Wright series in two variables. The properties of these functions are derived and some integral transforms, with known special functions, are provided. We have illustrated some graphical representations of some results by using numerical and analytical methods to show the agreement between the series and integral representations.

## References

[1] F. Al-Musallam and S. L. Kalla, *Further results on a generalized gamma function occurring in diffraction theory*, Integral Transform. Spec. Funct. **7** (1998), no. 3-4, 175–190. https://doi.org/10.1080/10652469808819198

[2] L. C. Andrews, *Special functions of mathematics for engineers*, reprint of the 1992 second edition, SPIE Optical Engineering Press, Bellingham, WA, 1998.

[3] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge University Press, Cambridge, 1999. `https://doi.org/10.1017/CBO9781107325937`

[4] M. Arshad, S. Mubeen, K. S. Nisar, and G. Rahman, *Extended Wright-Bessel function and its properties*, Commun. Korean Math. Soc. **33** (2018), no. 1, 143–155. `https://doi.org/10.4134/CKMS.c170039`

[5] A. Belafhal, Z. Hricha, L. Dalil Essakali and T. Usman, *A note on some integrals involving Hermite polynomials and their applications*, Advanced Mathematical Models & Applications. **5** (2020), 313–319.

[6] A. Belafhal and F. Saad, *Conversion of circular beams by a spiral phase plate*: *generation of generalized Humbert beams*, Optik. **138** (2017), 516–528.

[7] M. A. Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, *Extension of Euler's beta function*, J. Comput. Appl. Math. **78** (1997), no. 1, 19–32. `https://doi.org/10.1016/S0377-0427(96)00102-1`

[8] M. A. Chaudhry, N. M. Temme, and E. J. M. Veling, *Asymptotics and closed form of a generalized incomplete gamma function*, J. Comput. Appl. Math. **67** (1996), no. 2, 371–379. `https://doi.org/10.1016/0377-0427(95)00018-6`

[9] M. A. Chaudhry and S. M. Zubair, *Generalized incomplete gamma functions with applications*, J. Comput. Appl. Math. **55** (1994), no. 1, 99–124. `https://doi.org/10.1016/0377-0427(94)90187-2`

[10] M. A. Chaudhry and S. M. Zubair, *On the decomposition of generalized incomplete gamma functions with applications of Fourier transforms*, J. Comput. Appl. Math. **59** (1995), no. 3, 253–284. `https://doi.org/10.1016/0377-0427(94)00026-W`

[11] M. A. Chaudhry and S. M. Zubair, *Extended incomplete gamma functions with applications*, J. Math. Anal. Appl. **274** (2002), no. 2, 725–745. `https://doi.org/10.1016/S0022-247X(02)00354-2`

[12] A. A. A. Ebrahim, F. Saad, L. Ez-zariy, and A. Belafhal, *Theoretical conversion of the hypergeometric-Gaussian beams family into a high-order spiraling Bessel beams by a curved fork-shaped hologram*, Opt. Quant. Electron. **49** (2017), 169–186.

[13] M. El-Shahed and A. Salem, *An extension of Wright function and its properties*, J. Math. **2015** (2015), Art. ID 950728, 11 pp. `https://doi.org/10.1155/2015/950728`

[14] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, second edition, Springer Monographs in Mathematics, Springer, Berlin, t 2020. `https://doi.org/10.1007/978-3-662-61550-8`

[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, fifth edition, Academic Press, Inc., San Diego, CA, 1996.

[16] P. Humbert and R. P. Agarwal, *Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations*, Bull. Sci. Math. (2) **77** (1953), 180–185.

[17] D. F. V. James and S. S. Agarwal, *The generalized Fresnel transform and its application to optics*, Optics Commun. **126** (1996), 207–212.

[18] W. A. Khan and K. S. Nisar, *Beta type integral formula associated with Wright generalized Bessel function*, Acta Math. Univ. Comenian. (N.S.) **87** (2018), no. 1, 117–125.

[19] W. A. Khan, K. S. Nisar, M. Acikgoz, and U. Duran, *A novel kind of Hermite based Frobenius type Eulerian polynomials*, Proc. Jangjeon Math. Soc. **22** (2019), no. 4, 551–563.

[20] W. A. Khan, K. S. Nisar, and J. Choi, *An integral formula of the Mellin transform type involvilng the extended Wright-Bessel function*, FJMS. **102** (2017), 2903–2912.

[21] N. U. Khan, T. Usman, and M. Aman, *Some properties concerning the analysis of generalized Wright function*, J. Comput. Appl. Math. **376** (2020), 112840, 8 pp. `https://doi.org/10.1016/j.cam.2020.112840`

[22] N. U. Khan, T. Usman, and M. Aman, *Computation of certain integral formulas involving generalized Wright function*, Adv. Difference Equ. **2020** (2020), Paper No. 491, 10 pp. https://doi.org/10.1186/s13662-020-02948-8

[23] T. Kim, D. S. Kim, and J.-J. Seo, *Fully degenerate poly-Bernoulli numbers and polynomials*, Open Math. **14** (2016), no. 1, 545–556. https://doi.org/10.1515/math-2016-0048

[24] F. Mainardi, *On the initial value problem for the fractional diffusion-wave equation*, in Waves and stability in continuous media (Bologna, 1993), 246–251, Ser. Adv. Math. Appl. Sci., 23, World Sci. Publ., River Edge, NJ, 1994.

[25] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010. https://doi.org/10.1142/9781848163300

[26] K. S. Mathur, *Fundamentals of Fiber Optics Communications*, Zorba books, India, 2018.

[27] A. R. Miller, *Reductions of a generalized incomplete gamma function, related Kampé de Fériet functions, and incomplete Weber integrals*, Rocky Mountain J. Math. **30** (2000), no. 2, 703–714. https://doi.org/10.1216/rmjm/1022009290

[28] G. M. Mittag-Leffler, *Une generalisation de l'integrale de Laplace-Abel*, C. R. Acad. Sci. Paris (Ser. II) **137** (1903), 537–539.

[29] G. M. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$, C.R. Acad. Sci. Paris (Ser. II). **137** (1903), 554–558.

[30] G. M. Mittag-Leffler, *Sopra la funzione* $E_\alpha(x)$, Rendiconti R. Accademia Lincei (Ser. V). **13** (1904), 3–5.

[31] G. M. Mittag-Leffler, *Sur la représentation analytique d'une branche uniforme d'une fonction monogène*, Acta Math. **29** (1905), no. 1, 101–181. https://doi.org/10.1007/BF02403200

[32] S. Naheed, S. Mubeen, G. Rahman, M. R. Alharthi, and K. S. Nisar, *Some new inequalities for the generalized Fox-Wright functions*, AIMS Math. **6** (2021), no. 6, 5452–5464. https://doi.org/10.3934/math.2021322

[33] K. S. Nisar and W.A. Khan, *Notes on q-Hermite based unified Apostol type polynomials*, J. Interdiscip. Math. **22** (2019), 1185–1203.

[34] E. Özergin, M. A. Özarslan, and A. Altın, *Extension of gamma, beta and hypergeometric functions*, J. Comput. Appl. Math. **235** (2011), no. 16, 4601–4610. https://doi.org/10.1016/j.cam.2010.04.019

[35] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Table of Integrals, Series, and Products*, Academic Press, New York, 2007.

[36] F. Saad, E. M. El Halba, and A. Belafhal, *Generation of generalized spiraling Bessel beams of arbitrary order by curved fork-shaped holograms*, Opt. Quant. Electron. **48** (2016), 454–465.

[37] V. Singh, M. A. Khan, A. H. Khan, and K. S. Nisar, *A note on modified Hermite matrix polynomials*, J . Math. Computer Sci. **22** (2021), 333–346.

[38] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1984.

[39] A. Wiman, *Über den Fundamentalsatz in der Teorie der Funktionen* $E^a(x)$, Acta Math. **29** (1905), no. 1, 191–201. https://doi.org/10.1007/BF02403202

[40] E. M. Wright, *On the Coefficients of Power Series Having Exponential Singularities*, J. London Math. Soc. **8** (1933), no. 1, 71–79. https://doi.org/10.1112/jlms/s1-8.1.71

[41] E. M. Wright, *The Asymptotic Expansion of the Generalized Bessel Function*, Proc. London Math. Soc. (2) **38** (1935), 257–270. https://doi.org/10.1112/plms/s2-38.1.257

[42] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Journal London Math. Soc. **10** (1935), 287–293.

[43] E. M. Wright, *The generalized Bessel function of order greater than one*, Quart. J. Math. Oxford Ser. **11** (1940), 36–48. https://doi.org/10.1093/qmath/os-11.1.36

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