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YAMABE AND RIEMANN SOLITONS ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

Shruthi Chidananda and Venkatesha Venkatesha

Abstract. In the present paper, we aim to study Yamabe soliton and Riemann soliton on Lorentzian para-Sasakian manifold. First, we proved, if the scalar curvature of an η-Einstein Lorentzian para-Sasakian manifold M is constant, then either $\tau = n(n-1)$ or, $\tau = n-1$. Also we constructed an example to justify this. Next, it is proved that, if a three dimensional Lorentzian para-Sasakian manifold admits a Yamabe soliton for V is an infinitesimal contact transformation and $tr\,\varphi$ is constant, then the soliton is expanding. Also we proved that, suppose a 3-dimensional Lorentzian para-Sasakian manifold admits a Yamabe soliton, if $tr\,\varphi$ is constant and scalar curvature τ is harmonic (i.e., $\Delta\tau = 0$), then the soliton constant λ is always greater than zero with either $\tau = 2$, or $\tau = 6$, or $\lambda = 6$. Finally, we proved that, if an η-Einstein Lorentzian para-Sasakian manifold M represents a Riemann soliton for the potential vector field V has constant divergence then either, M is of constant curvature 1 or, V is a strict infinitesimal contact transformation.

1. Introduction

It is well known that, the notion of Yamabe flow was first introduced by Richard Hamiliton at the same time as Ricci flow [11]. A Yamabe flow is defined as a tool for constructing metrics of constant scalar curvature. On a smooth pseudo Riemannian manifold, Yamabe flow is defined as the evaluation of the metric g_0 in time t to $g = g(t)$ through the equation

$$\frac{\partial}{\partial t} g(t) = -\tau g, \qquad g(0) = g_0, \tag{1.1}$$

where τ is the scalar curvature of the metric $g(t)$. If a pseudo-Riemannian manifold M holds the relation

$$\pounds_V g = 2(\tau - \lambda) g \tag{1.2}$$

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for a vector field V on M and a constant λ, then M is said to have Yamabe soliton. Like the Ricci soliton [17, 18], the Yamabe soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively.

In the past two decades, many authors have studied Yamabe soliton on various types of manifolds [1, 5, 7, 25, 27]. Recently, Venkatesha et al., studied Yamabe soliton on three dimensional contact manifolds [24] and Ghosh studied Yamabe soliton on Kenmotsu manifold [10].

The notion of Ricci flow is generalized to the concept of Riemann flow (see [21], [22]). As an analogous to the Ricci flow, a Riemann flow has been introduced by Hiriča and Udrişte [12] as a natural extension of the Ricci flow to a non-linear PDE and the metric g as a solution of the PDE. A Riemann soliton is defined as a self similar solution to the Riemann flow and is defined as

$$\frac{\partial}{\partial t}G(t) = -2R(g(t)), \qquad t \in [0, I], \tag{1.3}$$

where R denotes the Riemannian curvature tensor associated with metric g, $G = g \owedge g$ and $\owedge$ is Kulkarni-Nomizu product. If C and D are two $(0, 2)$-tensors, then $C \owedge D$ is given by

$$\begin{aligned}(C \owedge D)(W, X, Y, Z) &= C(W, Z)D(X, Y) + C(X, Y)D(W, Z)\\ &\quad - C(W, Y)D(X, Z) - C(X, Z)D(W, Y).\end{aligned} \tag{1.4}$$

A pseudo-Riemannian manifold M is said to admit a Riemann soliton (g, V), if there exist a vector field V and a constant λ on M such that

$$R + \frac{1}{2}\{\lambda g \owedge g + g \owedge \pounds_V g\} = 0, \tag{1.5}$$

where $\pounds_V$ is the Lie-derivative along V. In (1.5), if $V = Df$, where f is some smooth function and D represents the gradient operator of g, then the soliton is called a gradient Riemann soliton and is given by

$$2R + \lambda g \owedge g + 2g \owedge \nabla^2 f = 0. \tag{1.6}$$

By Kulkarni-Nomizu product defined in (1.4) the soliton equation (1.5) becomes

$$\begin{aligned}&2R(W, X, Y, Z) + 2\lambda\{g(X, Y)g(Z, W) - g(Y, W)g(X, Z)\}\\ &+ \{g(W, Z)(\pounds_V g)(X, Y) + g(X, Y)(\pounds_V g)(W, Z)\\ &- g(W, Y)(\pounds_V g)(X, Z) - g(X, Z)(\pounds_V g)(W, Y)\} = 0\end{aligned} \tag{1.7}$$

for all W, X, Y, $Z \in \mathcal{X}(M)$.

Moreover, contraction of the above expression over W, Z gives

$$\begin{aligned}&2S(X, Y) + 2(n-1)\lambda g(X, Y) + (n-2)(\pounds_V g)(X, Y)\\ &+ 2(div\, V)g(X, Y) = 0.\end{aligned} \tag{1.8}$$

Similar to the Yamabe soliton, the Riemann soliton is steady, shrinking or expanding according as $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$, respectively. In [8], [23], Naik et al., studied geometric properties of Riemann soliton in contact manifolds and in almost Kenmotsu manifolds. Further, in [4], we have studied Riemann soliton on non-Sasakian (κ, μ)-contact manifolds. In [6], De et al., studied an almost Riemann soliton in a non-cosymplectic normal almost contact metric manifold. Further, Blaga et al., considered Riemann soliton in (α, β)-contact manifolds and gave some important geometric aspects [2]. This literature survey motivates us to study Yamabe and Riemann soliton on Lorentzian para-Sasakian manifolds.

The structure of this paper is as follows: After the accumulation of some basic results and formulas in Section 2, we show some non-existence curvature conditions on Lorentzian para-Sasakian manifold M. Also, we show that, if M is an η-Einstein and τ is constant on M, then either $\tau = n(n-1)$, or $\tau = n-1$. Example has been constructed to justify this. In Section 3, we consider studying the Yamabe soliton and we establish a result that, if a three dimensional Lorentzian para-Sasakian metric g represents a Yamabe soliton for an infinitesimal contact transformation V with constant $tr\,\varphi$, then $\lambda > 0$. Further, we prove that, if a three dimensional Lorentzian para-Sasakian manifold with constant $tr\,\varphi$ and $\Delta\tau = 0$ admits a Yamabe soliton, then the soliton is expanding. Section 4, is devoted to study Riemann soliton on M under certain conditions, such as, (1) M is an η-Einstein and $divV$ is constant, (2) for $V = \xi$, (3) $V = Df$ and $divV$ is constant.

2. Preliminaries

The Lorentzian para-Sasakian structure on a differentiable manifold M was first introduced by K. Matsumoto in 1989 and is defined as follows [13]:

An n-dimensional smooth manifold M together with 1-form η , a $(1,1)$ tensor φ, a unit vector field ξ and a Lorentzian metric g is said to have a Lorentzian para-Sasakian structure if it holds the following conditions:

$$\varphi\xi = 0, \quad \eta(\xi) = -1, \quad \varphi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$(\nabla_X \varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \varphi X. \tag{2.4}$$

From the definition, it is known that

$$g(X,\xi) = \eta(X)$$

for all X belongs to $\mathcal{X}(M)$. And so the vector field ξ is time like, i.e.,

$$g(\xi,\xi) = -1$$

and φ is symmetric with respect to the metric g. Moreover, the geometric aspects of the Reeb vector field ξ have been exclusively studied by Wang in [26]. A smooth connected manifold M together with a Lorentzian para-Sasakian structure is said to be a Lorentzian para-Sasakian manifold. In recent years, the Lorentzian para-Sasakian manifold has been studied by many authors, [14–

16, 19, 20]. So we have the following expressions

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.5}$$

$$R(\xi,Y)Z = g(Y,Z)\xi + \eta(Z)Y + 2\eta(Y)\eta(Z)\xi, \tag{2.6}$$

$$Q\xi = (n-1)\xi. \tag{2.7}$$

Moreover, the Reeb vector field ξ is never a Killing, i.e.,

$$(\pounds_\xi g)(Y,Z) = 2g(Z,\varphi Y) \tag{2.8}$$

as φ is linear and the rank of φ is $n-1$, so $\pounds_\xi g \neq 0$ for all vector fields on $\mathcal{X}(M)$. Since, φ is symmetric. Therefore, we have

$$div\,\xi = tr\,\varphi,$$

where *div* and *tr* stand for divergence and trace, respectively.

Definition 2.1. A pseudo-Riemannian manifold M is said to be an η-Einstein if the Ricci operator Q satisfies

$$g(QX,Y) = \alpha g(X,Y) + \beta(\eta\otimes\eta)(X,Y), \tag{2.9}$$

where α, β are the smooth functions on M.

Moreover, from [3], the expression of Q for an η-Einstein Lorentzian para-Sasakian manifold is given by

$$QX = \left\{\frac{\tau}{n-1} - 1\right\}X + \left\{\frac{\tau}{n-1} - n\right\}\eta(X)\xi. \tag{2.10}$$

If M is a three-dimensional Lorentzian para-Sasakian manifold, then the expression of Q is given as

$$QX = \left\{\frac{\tau}{2} - 1\right\}X + \left\{\frac{\tau}{2} - 3\right\}\eta(X)\xi. \tag{2.11}$$

Definition 2.2. On a pseudo-Riemannian manifold M, any vector field V is said to be an infinitesimal contact transformation if it satisfies

$$\pounds_V\eta = \sigma\eta, \tag{2.12}$$

where σ is the smooth function on M. If $\sigma = 0$, then V is called to be strict.

From [9], we have:

Lemma 2.3. *On an n-dimensional pseudo-Riemannian manifold M, if there exists a vector field V such that $\pounds_V g = 2\rho g$, where ρ is a smooth function, then the following equations hold true on M*

$$(\pounds_V S)(X,Y) = g(X,Y)(\Delta\rho) - (n-2)g(\nabla_X D\rho, Y), \tag{2.13}$$

$$\pounds_V\tau = -2\rho\tau + 2(n-1)\Delta\rho, \tag{2.14}$$

where $\Delta\rho = -divD\rho$. If $\rho = \tau - \lambda$, then $\Delta\rho = \Delta\tau = -divD\tau$.

From Yano [28], we deduce the following computational formulas

$$2g((\pounds_V\nabla)(X,Y),Z) = (\nabla_X\pounds_V g)(Y,Z) + (\nabla_Y\pounds_V g)(X,Z) - (\nabla_Z\pounds_V g)(X,Y) \tag{2.15}$$

and

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z). \tag{2.16}$$

Proposition 2.4. *A Lorentzian para-Sasakian manifold M for $\dim M > 1$, never has the following curvature conditions*:

- *η-recurrent Ricci tensor.*
- *cyclic η-recurrent Ricci tensor.*

Proof. Let M be an n-dimensional Lorentzian para-Sasakian manifold and the dimension $n > 1$.

- If suppose the Ricci curvature tensor S on M satisfies $(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z)$ (*i.e.,* Ricci tensor is η-recurrent) for all X, Y, $Z \in \mathcal{X}(M)$.

By taking $X = Y = \xi$ in this expression and from (2.7), we obtain

$$(n-1)\eta(Z) = 0, \tag{2.17}$$

this shows that $n = 1$. Which is a contradiction.

Similarly,

- If S is cyclic η-recurrent on M, then

$$(\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) = \eta(Y)S(X,Z) + \eta(Z)S(X,Y) + \eta(X)S(Y,Z). \tag{2.18}$$

In this, by taking $Y = Z = \xi$, we get

$$-3(n-1)\eta(X) = 0, \tag{2.19}$$

which leads to the contradiction as $n > 1$. Hence the result is proved. □

Lemma 2.5. *On a Lorentzian para-Sasakian manifold, the following condition holds true*:

$$(\nabla_\xi Q)Y = 2(tr\,\varphi)\varphi^2 Y - 2\varphi QY. \tag{2.20}$$

Proof. Taking covariant derivative of (2.8) along the direction of X and from (2.3) we deduce

$$(\nabla_X\pounds_\xi g)(Y,Z) = 2\{g(X,Y)\eta(Z) + \eta(Y)g(X,Z) + 2\eta(X)\eta(Y)\eta(Z)\}. \tag{2.21}$$

In view of (2.15) and (2.21), we find

$$(\pounds_\xi\nabla)(Y,Z) = 2g(\varphi Y,\varphi Z)\xi. \tag{2.22}$$

Now, in (2.22), with the help of (2.3) and (2.4), we infer

$$(\nabla_X\pounds_\xi\nabla)(Y,Z) = 2g(\varphi Y,\varphi Z)\varphi X + 2\eta(Y)g(X,\varphi Z)\xi + 2\eta(Z)g(X,\varphi Y)\xi. \tag{2.23}$$

By virtue of this, we obtain

$$(\nabla_Y \pounds_\xi \nabla)(X,Z) = 2g(\varphi X, \varphi Z)\varphi Y + 2\eta(X)g(Y,\varphi Z)\xi + 2\eta(Z)g(X,\varphi Y)\xi. \tag{2.24}$$

On substituting the foregoing relations in (2.16) and then contracting (2.16) over X with respect to an orthonormal basis, gives

$$(\pounds_\xi S)(Y,Z) = 2g(\varphi Y, \varphi Z)(tr\,\varphi). \tag{2.25}$$

On the other hand, computing the left hand side of (2.25) by using (2.4) leads to

$$(\pounds_\xi S)(Y,Z) = g((\nabla_\xi Q)Y,Z) + 2g(\varphi QY, Z). \tag{2.26}$$

Hence, by equating (2.25) with (2.26) we obtain (2.20). This finishes the proof. □

Lemma 2.6. *On an η-Einstein Lorentzian para-Sasakian manifold M we have*

$$\xi\tau = -2\left(\frac{\tau}{n-1} - n\right)(tr\,\varphi). \tag{2.27}$$

Proof. Since M is η-Einstein, covariant derivative of equation (2.10) leads to obtain

$$(\nabla_X Q)Y = \left(\frac{X\tau}{n-1}\right)Y + \left(\frac{X\tau}{n-1}\right)\eta(Y)\xi + \left(\frac{\tau}{n-1} - n\right)\{g(X,\varphi Y)\xi + \eta(Y)\varphi X\}. \tag{2.28}$$

Hence, fetching $Y = \xi$ in the above relation and then taking contraction over X gives the condition (2.27). □

Theorem 2.7. *Let τ be the scalar curvature of an n-dimensional η-Einstein Lorentzian para-Sasakian manifold M. If τ is constant, then either $\tau = n(n-1)$ with $(tr\,\varphi) = \pm(n-1)$, or $\tau = (n-1)$ with $(tr\,\varphi) = 0$.*

Proof. Suppose τ is constant on M, then $\xi\tau = 0$ and from (2.27), we get

$$(\tau - n(n-1))\,(tr\,\varphi) = 0. \tag{2.29}$$

From (2.28) we get $(\nabla_\xi Q)X = 0$, which in (2.20) for $Y = \varphi Y$ implies

$$(tr\,\varphi)\varphi Y - Q\varphi^2 Y = 0. \tag{2.30}$$

Contracting this over Y and with the help of (2.10), we find

$$(tr\,\varphi)^2 - \tau + (n-1) = 0. \tag{2.31}$$

On solving (2.31) by using (2.29) we obtain, either $\tau = (n-1)$ with $(tr\,\varphi) = 0$, or $\tau = n(n-1)$ with $(tr\,\varphi) = \pm(n-1)$. Hence the result is proved. □

From the above theorem, we can also state that:

Theorem 2.8. *Let M be an n-dimensional Lorentzian para-Sasakian manifold and the scalar curvature τ is constant on M. If τ is neither $n(n-1)$ nor $(n-1)$, then M never be an η-Einstein manifold.*

Example 2.9. Here we construct the 5-dimensional Lorentzian para-Sasakian manifold M. We consider $M = \{(u, v, w, x, y) \in \mathbb{R}^5\}$, where (u, v, w, x, y) are the standard coordinates in $\mathbb{R}^5$.

Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the basis for M and the Lorentzian metric g is defined as the

$$g(v_i, v_j) = \begin{cases} 0 & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j \quad \text{and} \quad i \neq 3, \\ -1 & \text{for} \quad i = j = 3. \end{cases} \tag{2.32}$$

Let ∇ be the Levi-Civita connection corresponding to g and we have

$$[v_1, v_2] = 0, \quad [v_1, v_3] = -v_1, \quad [v_1, v_4] = 0,$$

$$[v_1, v_5] = v_1, \quad [v_2, v_3] = -v_2, \quad [v_2, v_4] = v_2,$$

$$[v_2, v_5] = v_2, \quad [v_3, v_4] = v_4, \quad [v_3, v_5] = v_5, \quad [v_4, v_5] = -v_5.$$

Let the $(1, 1)$ tensor field φ is defined by

$$\varphi v_1 = -v_1, \quad \varphi v_2 = -v_2, \quad \varphi v_3 = 0, \quad \varphi v_4 = -v_4, \quad \varphi v_5 = -v_5. \tag{2.33}$$

Let η be the 1-form defined by $\eta(X) = g(X, v_3)$ for any vector field X on $\mathcal{X}(M)$. Then, by the linearity of φ and g, we find

$$\eta(v_3) = -1, \tag{2.34}$$

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.35}$$

$$g(\varphi\cdot, \varphi\cdot) = (g + \eta \otimes \eta)(\cdot, \cdot). \tag{2.36}$$

By the Koszul's formula, we find

$\nabla_{v_1} v_1 = -v_3 - v_5$, $\nabla_{v_1} v_2 = 0$, $\nabla_{v_1} v_3 = -v_1$, $\nabla_{v_1} v_4 = 0$, $\nabla_{v_1} v_5 = v_1$,
$\nabla_{v_2} v_1 = 0$, $\nabla_{v_2} v_2 = -v_3 - v_4 - v_5$, $\nabla_{v_2} v_3 = -v_2$, $\nabla_{v_2} v_4 = v_2$, $\nabla_{v_2} v_5 = v_2$,
$\nabla_{v_3} v_1 = 0$, $\nabla_{v_3} v_2 = 0$, $\nabla_{v_3} v_3 = 0$, $\nabla_{v_3} v_4 = 0$, $\nabla_{v_3} v_5 = 0$,
$\nabla_{v_4} v_1 = 0$, $\nabla_{v_4} v_2 = 0$, $\nabla_{v_4} v_3 = -v_4$, $\nabla_{v_4} v_4 = -v_3$, $\nabla_{v_4} v_5 = 0$,
$\nabla_{v_5} v_1 = 0$, $\nabla_{v_5} v_2 = 0$, $\nabla_{v_5} v_3 = -v_5$, $\nabla_{v_5} v_4 = v_5$, $\nabla_{v_5} v_5 = -v_3 - v_4$.

Hence, we can conclude that (φ, v_3, η, g) defines a Lorentzian para-Sasakian structure on M and so M is a Lorentzian para-Sasakian manifold. Let R be the Riemannian curvature and S is the Ricci tensor and by the above relations, we evaluated the following conditions

$R(v_1, v_2)v_2 = 0$, $R(v_1, v_3)v_3 = -v_1$, $R(v_1, v_4)v_4 = v_1$, $R(v_1, v_5)v_5 = 0$,
$R(v_2, v_3)v_3 = -v_2$, $R(v_2, v_4)v_4 = 0$, $R(v_2, v_5)v_5 = -v_2$, $R(v_3, v_4)v_4 = v_3$,
$R(v_3, v_5)v_5 = v_3 + v_4$, $R(v_4, v_5)v_5 = 0$.

And from the above relations, we obtain

$$S(v_1, v_1) = 2, \quad S(v_2, v_2) = 0, \quad S(v_3, v_3) = -4,$$
$$S(v_4, v_4) = 2, \quad S(v_5, v_5) = 0.$$

Since, M is 5-dimensional and the scalar curvature is 8. Moreover, $S(v_1, v_1) \neq S(v_2, v_2)$ shows that M is never an η-Einstein. Hence this verifies Theorem 2.8.

Example 2.10. Let us consider a manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$ and the orthonormal basis $\{u_1, u_2, u_3\}$ on M, with the Lorentzian metric g satisfying

$$g(u_i, u_j) = 0 \quad \text{for} \quad i \neq j,$$
$$g(u_1, u_1) = g(u_2, u_2) = 1,$$
$$g(u_3, u_3) = -1.$$

Define 1-form η and the vector field ξ by

$$\eta(X) = g(X, u_3), \qquad \xi = u_3.$$

Let ∇ be the Levi-Civita connection corresponding to g and is defined by

$$[u_1, u_2] = 0, \quad [u_1, u_3] = -u_1, \quad [u_2, u_3] = -u_2,$$

and the tensor field φ is defined by

$$\varphi u_1 = -u_1, \quad \varphi u_2 = -u_2, \quad \varphi u_3 = 0.$$

Use of Koszul's formula gives the following relations

$$\nabla_{u_1} u_1 = -u_3, \quad \nabla_{u_1} u_2 = 0, \quad \nabla_{u_1} u_3 = -u_1,$$
$$\nabla_{u_2} u_1 = 0, \quad \nabla_{u_2} u_2 = -u_3, \quad \nabla_{u_2} u_3 = -u_2,$$
$$\nabla_{u_3} u_1 = 0, \quad \nabla_{u_3} u_2 = 0, \quad \nabla_{u_3} u_3 = 0.$$

From the above relations, it is clear that $(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$ and $\nabla_X \xi = \varphi X$ for any vector fields X, Y. Hence, the defined structure $(\varphi, \xi = u_3, \eta, g)$ is a Lorentzian para-Sasakian structure on M. Then the corresponding Riemannian curvature tensor and Ricci tensor have been calculated as follows:

$$R(u_1, u_2)u_2 = u_1, \quad R(u_1, u_3)u_3 = -u_1, \quad R(u_2, u_1)u_1 = u_2,$$
$$R(u_2, u_3)u_3 = -u_2, \quad R(u_3, u_1)u_1 = u_3, \quad R(u_3, u_2)u_2 = u_3,$$

and

$$S(u_1, u_1) = S(u_2, u_2) = 2, \quad S(u_3, u_3) = -2,$$
$$S(u_1, u_2) = S(u_1, u_3) = S(u_2, u_3) = 0.$$

Clearly, the constructed structure (φ, ξ, η, g), for $\xi = u_3$ is an Einstein Lorentzian para-Sasakian structure with $\tau = 6$ and $tr\, \varphi = -2$. This verifies Theorem 2.7.

3. Yamabe soliton

Theorem 3.1. *If a Lorentzian para-Sasakian metric g represents a Yamabe soliton, then the scalar curvature τ is constant if and only if V is Killing.*

Proof. Suppose M has a constant scalar curvature and g is a Yamabe soliton. Then by equation (1.2) we can deduce that, $\nabla_X \pounds_V g = 0$. And by using this in the computational formula (2.15), we obtain

$$(\pounds_V \nabla)(Y, Z) = 0, \tag{3.1}$$

this implies getting

$$(\nabla_X \pounds_V \nabla)(Y, Z) = 0. \tag{3.2}$$

As a result, the preceding condition in (2.16) produces

$$(\pounds_V R)(X, Y)Z = 0. \tag{3.3}$$

Substituting $Y = Z = \xi$ in the previous relation and then tracing the resulting equation with the aid of (1.2), we find

$$\eta(\pounds_V \xi) = \tau - \lambda = 0. \tag{3.4}$$

Therefore, use of this in (1.2) proves that V is Killing.

Conversely, if the soliton vector field V is Killing, then from the expression (1.2), it is obvious that $\tau = \lambda$. Since λ is constant, which means τ is also constant. This completes the proof. □

Corollary 3.2. *If g is a Lorentzian para-Sasakian metric, then g never satisfies Yamabe equation for $V = \xi$.*

Proof. If suppose a Lorentzian para-Sasakian metric g is a Yamabe soliton for $V = \xi$, then the equation (1.2), on (ξ, ξ) gives $\tau - \lambda = 0$. Later, this in (1.2) shows ξ is Killing. But, as we know, if ξ is Killing then by the condition (2.8) $\varphi = 0$, which is a contradiction. Therefore, V is never a Reeb vector field ξ. □

Here we justify the above theorem by the following example:

Example 3.3. In Example 2.9, if manifold M holds Yamabe soliton for $V = \xi = v_3$, then, by computing (1.2) on (v_3, v_3), we acquire

$$(\pounds_{v_3} g)(v_3, v_3) = 2(\lambda - \tau) = 0, \tag{3.5}$$

this implies $\tau = \lambda$, at one more time, evaluating (1.2) on (v_2, v_2) gives

$$2g(\nabla_{v_2} v_3, v_2) = -2 = 0,$$

which is a contradiction. Therefore it verifies Corollary 3.2.

Theorem 3.4. *Let g be a Lorentzian para-Sasakian metric and it admits Yamabe soliton for V is an infinitesimal contact transformation, if τ is constant in the direction of ξ then V is Killing.*

Proof. From Definition 2.2 and from the equation (1.2) we can easily find that

$$\sigma = (\tau - \lambda), \tag{3.6}$$

and as we know η is closed on M, i.e., $d\eta = 0$, therefore applying d on both sides of relation (2.12) provides

$$(d\sigma \wedge \eta)(X, Y) = 0. \tag{3.7}$$

In the above equation for $X = \xi$ we get $Y\sigma = -(\xi\sigma)\eta(Y)$. So σ is constant if $\xi\sigma$ is zero. Since $\xi\tau = 0$, then by (3.6), we have $\xi\sigma = 0$, which shows σ is constant on M and consequently τ is also constant on M. Therefore, from Theorem 3.1 the proof is completed. □

Theorem 3.5. *Let M be a three-dimensional Lorentzian para-Sasakian manifold and admits a Yamabe soliton for the potential vector field V, where V is an infinitesimal contact transformation. If the trace of φ is constant, then the soliton is expanding.*

Proof. For a 3-dimensional Lorentzian para-Sasakian manifold the expression of Ricci tensor is given by

$$S = \left\{\frac{\tau}{2} - 1\right\} g + \left\{\frac{\tau}{2} - 3\right\} \eta \otimes \eta. \tag{3.8}$$

Taking the Lie-derivative of the above condition in the direction of V results in the following

$$\begin{aligned}(\pounds_V S)(Y, Z) = &\left(\frac{\pounds_V \tau}{2}\right) g(Y, Z) + \left\{\frac{\tau}{2} - 1\right\} (\pounds_V g)(Y, Z) + \left(\frac{\pounds_V \tau}{2}\right) \eta(Y)\eta(Z)\\ &+ \left\{\frac{\tau}{2} - 3\right\} (\pounds_V \eta \otimes \eta)(Y, Z).\end{aligned} \tag{3.9}$$

We can also have

$$\begin{aligned}g((\pounds_V Q)Y, Z) = &\left(\frac{\pounds_V \tau}{2}\right) g(Y, Z) + \left(\frac{\pounds_V \tau}{2}\right) \eta(Y)\eta(Z) + \left\{\frac{\tau}{2} - 3\right\} \{\eta(Z)(\pounds_V \eta)Y\\ &+ g(\pounds_V \xi, Z)\eta(Y)\}.\end{aligned} \tag{3.10}$$

From equation (1.2), we derive

$$(\pounds_V S)(Y, Z) - g((\pounds_V Q)Y, Z) = 2(\tau - \lambda)S(Y, Z). \tag{3.11}$$

As from (1.2), we have $\eta(\pounds_V \xi) = (\tau - \lambda)$. Next, by putting $Y = Z = \xi$ in equation (3.11) and with the help of (3.9) and (3.10) we find that

$$(\pounds_V S)(\xi, \xi) = -4(\tau - \lambda). \tag{3.12}$$

Since, from (2.13) we have

$$(\pounds_V S)(\xi, \xi) = -\Delta\tau - g(\nabla_\xi D\tau, \xi). \tag{3.13}$$

On equating (3.12) with (3.13), we obtain

$$4(\tau - \lambda) = \Delta\tau + \xi(\xi\tau). \tag{3.14}$$

Since V is an infinitesimal contact transformation, thus, from the conditions (2.12) and (1.2), we have that $X\sigma = X\tau = 0$ for all X orthogonal to ξ. Later, this implies getting

$$D\tau = -(\xi\tau)\xi. \tag{3.15}$$

Now differentiating this along Y provides

$$\nabla_Y D\tau = -\{Y(\xi\tau)\}\xi - (\xi\tau)\nabla_Y\xi. \tag{3.16}$$

Further, we proceed with the condition $tr\,\varphi = constant$. If the trace of φ is constant, then from (2.27) we obtain

$$\xi(\xi\tau) = -(\xi\tau)(tr\,\varphi) = (\tau - 6)(tr\,\varphi)^2. \tag{3.17}$$

In equation (2.27), the fact that $g(X, D\tau) = 0$ for any X orthogonal to ξ enables us to find

$$X(\xi\tau) = -(X\tau)(tr\,\varphi) = 0, \tag{3.18}$$

for all X perpendicular to ξ.

Next, tracing (3.16) over Y and then using above relation yields

$$-\Delta\tau = -\{\xi(\xi\tau)\} - (\xi\tau)(tr\,\varphi). \tag{3.19}$$

On substituting (3.17) and (3.19) in (3.14) we get

$$-4(\tau - \lambda) = -2(\tau - 6)(tr\,\varphi)^2 + (\tau - 6)(tr\,\varphi)^2, \tag{3.20}$$

differentiating (3.20) along ξ and using (2.27), we have

$$(\tau - 6)\{4(tr\,\varphi) - (tr\,\varphi)^3\} = 0. \tag{3.21}$$

Note that the trace of φ is constant. Therefore, from the above equation, there are three cases that arise: either $\tau = 6$, or $(tr\,\varphi) = 0$, or $(tr\,\varphi)^2 = 4$. *First case* itself proves the result. Next, let us deal with *second case*, i.e., $(tr\,\varphi) = 0$, which in (2.27) finds $\xi\tau = 0$ and from (2.28) for $n = 3$ gives $(\nabla_\xi Q)Y = 0$, use of this in (2.20) enables us to find $\tau = 2$. Finally, if $(tr\,\varphi)^2 = 4$, which in (3.20) finds $\lambda = 6$. Hence, by Theorem 3.1 the proof is completed. □

Theorem 3.6. *Let M be a Lorentzian para-Sasakian manifold of dimension three and admits a Yamabe soliton (g, V, λ). If $tr\,\varphi$ is constant and the scalar curvature τ is harmonic, i.e., $\Delta\tau = 0$, then the soliton is expanding with either V is Killing, or $\lambda = 6$.*

Proof. Suppose a three-dimensional Lorentzian para-Sasakian manifold M admits a Yamabe soliton. If $tr\,\varphi$ is constant and $\Delta\tau = 0$, then from (2.27) we have

$$\xi(\xi\tau) = (\tau - 6)(tr\,\varphi)^2. \tag{3.22}$$

Use of foregoing condition in (3.14) and the harmonic scalar curvature condition provides

$$4(\tau - \lambda) - (\tau - 6)(tr\,\varphi)^2 = 0. \tag{3.23}$$

Taking covariant derivative of preceding relation along ξ and from (2.27), we yields

$$(\tau - 6)(tr\,\varphi)\{4 - (tr\,\varphi)^2\} = 0. \tag{3.24}$$

Hence, from the above equation we conclude that either $\tau = 6$, or $\tau = 2$, or $\lambda = 6$. This finishes the proof. $\square$

4. Riemann soliton

Theorem 4.1. *Let M $(\dim M = n > 2)$ be an η-Einstein Lorentzian para-Sasakian manifold and represents a Riemann soliton for V has a constant divergence. Then either V is strict infinitesimal contact transformation or M is of constant curvature* 1.

Proof. By the hypothesis, $divV$ is constant. Therefore, the contraction of equation (1.8) gives an expression for τ and shows τ is constant on M. Taking the covariant derivative of equation (2.10) leads to obtaining

$$g((\nabla_X Q)Y, Z) = \left(\frac{\tau}{n-1} - n\right)\{\eta(Z)g(\varphi X, Y) + \eta(Y)g(\varphi X, Z)\}. \tag{4.1}$$

In view of the above condition and from (1.8), we derive

$$(\nabla_X \pounds_V g)(Y, Z) = \frac{-2}{n-2}\left(\frac{\tau}{n-1} - n\right)\{\eta(Z)g(\varphi X, Y) + \eta(Y)g(\varphi X, Z)\}.$$

Use of foregoing relation in the computational formula (2.15) yields

$$(\pounds_V \nabla)(X, Y) = \frac{-2}{n-2}\left(\frac{\tau}{n-1} - n\right)g(X, \varphi Y)\xi.$$

By the help of above condition and equation (2.3), we obtain

$$\begin{aligned}(\nabla_X \pounds_V \nabla)(Y, Z) = \frac{-2}{n-2}\left(\frac{\tau}{n-1} - n\right)&\{g(\varphi X, \varphi Y)\eta(Z)\xi \\ &+ g(\varphi X, \varphi Z)\eta(Y)\xi + g(Y, \varphi Z)\varphi X\}.\end{aligned}$$

With the help of previous equation, the right side of the relation (2.16) is computed as

$$\begin{aligned}(\pounds_V R)(X, Y)Z = \frac{-2}{n-2}\left(\frac{\tau}{n-1} - n\right)&\{g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &+ g(Y, \varphi Z)\varphi X - g(X, \varphi Z)\varphi Y\}.\end{aligned}$$

Tracing this over X implies

$$(\pounds_V S)(Y, Z) = \frac{-2}{n-2}\left(\frac{\tau}{n-1} - n\right)\{(tr\,\varphi)g(Y, \varphi Z)\}. \tag{4.2}$$

In equation (4.2), by placing $Z = \xi$ and from (2.7), we obtain

$$(n-1)(\pounds_V \eta)Y = g(QY, \pounds_V \xi). \tag{4.3}$$

In order to find $g(QY, \pounds_V\xi)$, we go through an η-Einstein condition. By taking an inner product of (2.10) with $\pounds_V\xi$ we find the following:

$$g(QX, \pounds_V\xi) = \left(\frac{\tau}{n-1} - 1\right) g(X, \pounds_V\xi) + \left(\frac{\tau}{n-1} - n\right) \eta(X)\eta(\pounds_V\xi). \tag{4.4}$$

In (1.8), for $Y = \xi$ and the expansion of $\pounds_V g$ provides

$$(n-2)g(X, \pounds_V\xi) = \{2(n-1)(1+\lambda) + 2(divV)\}\eta(X) + (n-2)(\pounds_V\eta)X. \tag{4.5}$$

For $n > 2$, by taking $Y = \xi$ in (4.3) and by the fact that $Q\xi = (n-1)\xi$ we obtain the value $\eta(\pounds_V\xi) = 0$. Finally, substituting (4.5) in (4.4) (minding that $n > 2$) and then the use of the resulting equation in (4.3) gives

$$\left(n - \frac{\tau}{n-1}\right)(\pounds_V\eta)X = \left(\frac{\tau}{n-1} - 1\right)\left(\frac{2(n-1)(1+\lambda) + 2(divV)}{n-2}\right)\eta(X). \tag{4.6}$$

For an η-Einstein Lorentzian para-Sasakian manifold with constant τ, we have from Theorem 2.7 that either $\tau = n-1$ or $\tau = n(n-1)$. Therefore, if $\tau = n-1$, then the preceding equation shows that V is a strictly infinitesimal contact transformation. This completes the either part of the theorem. Next, if suppose $\tau = n(n-1)$, then from (4.6) we infer

$$(n-1)(1+\lambda) + divV = 0. \tag{4.7}$$

Moreover, contraction of (1.8) leads to achieve

$$n + n\lambda + 2(divV) = 0. \tag{4.8}$$

On solving (4.7) and (4.8), we obtain $\lambda = -1$ and $divV = 0$. Making use of the resulting equations and $QX = (n-1)X$ in (1.8) provides $\pounds_V g = 0$, i.e., V is Killing. Thus, from (1.7), we conclude that, manifold M is of constant curvature 1. □

Theorem 4.2. *If (φ, ξ, η, g) is a Lorentzian para Sasakian structure on an n-dimensional manifold M, then for $n > 2$, g never a Riemann soliton (g, ξ).*

Proof. If suppose a Lorentzian para-Sasakian metric g is a Riemann soliton for $V = \xi$, then from (1.8) we have

$$2S(X,Y) + \{2(n-1)\lambda + 2(tr\,\varphi)\}g(X,Y) + 2(n-2)g(\varphi X, Y) = 0. \tag{4.9}$$

Choosing $X = Y = \xi$ in the foregoing relation we get

$$tr\,\varphi = -(n-1)(1+\lambda). \tag{4.10}$$

Contracting (4.9) over X, Y, and from the above condition we find

$$\tau = -\lambda n(n-1) + 2(n-1)(n-1)(1+\lambda). \tag{4.11}$$

Since λ is constant, which implies τ is constant on M and from (4.9), we deduce

$$(\nabla_X Q)Y = -(n-2)(\nabla_X\varphi)Y. \tag{4.12}$$

In the above relation putting $Y = \xi$ and then contracting over X finds $(n-2)(n-1) = 0$. But this is a contradiction to our assumption that $n > 2$. This completes the proof. $\square$

Example 4.3. In Example 2.10, if g represents a Riemann soliton (g, ξ), then in equation (1.7) for $W = Z = u_1$ and $X = Y = u_2$, we have

$$2 + 2\lambda + (\pounds_{u_3} g)(u_2, u_2) + (\pounds_{u_3} g)(u_1, u_1) = 0, \tag{4.13}$$

which finds $\lambda = -1$. Again, in (1.7) for $W = Z = u_2$ and $X = Y = u_3$ we get

$$-2 + 2 + 2g(\nabla_{u_2} u_3, u_2) = 0. \tag{4.14}$$

Since, $g(\nabla_{u_2} u_3, u_2) = -1$, use of this in the preceding relation leads to a contradiction. Hence, g never admits a Riemann soliton for V being a Reeb vector field ξ.

Theorem 4.4. *If a Lorentzian para-Sasakian metric g supports a Riemann soliton for $V = Df$ with divergence of V (i.e., $divDf = -\Delta f$) constant, then M is of constant curvature 1 and the scalar curvature $\tau = n(n-1)$.*

Proof. If the vector V in (1.7) is a gradient of a smooth function f, then the relation (1.8) reduces to

$$QW + \lambda(n-1)W - (\Delta f)W + (n-2)\nabla_W Df = 0. \tag{4.15}$$

If Δf is constant, then the contraction of (4.15) shows that the scalar curvature τ constant. Further, from equation (4.15), we derive the following relation

$$(\nabla_X Q)W = -(n-2)\{\nabla_X \nabla_W Df + \nabla_{\nabla_X W} Df\}. \tag{4.16}$$

So, from this and equation (4.15), we find

$$(n-2)R(X, W)Df = -(\nabla_X Q)W + (\nabla_W Q)X. \tag{4.17}$$

For $n \geq 3$, in the above expression setting $X = \xi$ and then taking the scalar product of the resulting condition with ξ gives $g(R(\xi, W)Df, \xi) = 0$. Next, contraction of (4.17) over X with respect to an orthonormal basis provides $(n-2)QDf = 0$. This implies f is constant along ξ. Further, the use of equation (2.5) in $g(R(\xi, W)\xi, Df) = 0$ shows $Wf = 0$, i.e., f is constant. Hence, the equation (1.7) turns to

$$R(X, Y)Z = -\lambda\{g(Y, Z)X - g(X, Z)Y\}. \tag{4.18}$$

Replacing Y and Z by ξ and X by φX in (4.18) and by the virtue of (2.5), we get the value of λ as -1. Hence the theorem is proved. $\square$

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References

[1] A. M. Blaga, *Some geometrical aspects of Einstein, Ricci and Yamabe solitons*, J. Geom. Symmetry Phys. **52** (2019), 17–26. https://doi.org/10.7546/jgsp-52-2019-17-26

[2] A. M. Blaga and D. R. Laţcu, *Remarks on Riemann and Ricci solitons in* (α, β)*-contact metric manifolds*, J. Geom. Symmetry Phys. **58** (2020), 1–12. https://doi.org/10.7546/jgsp-58-2020-1-12

[3] S. K. Chaubey, *Some properties of LP-Sasakian manifolds equipped with* m*-projective curvature tensor*, Bull. Math. Anal. Appl. **3** (2011), no. 4, 50–58.

[4] S. Chidananda and V. Venkatesha, *Riemann soliton on non-Sasakian* (κ, μ)*-contact manifolds*, Differ. Geom. Dyn. Syst. **23** (2021), 40–51.

[5] B.-Y. Chen and S. Deshmukh, *Yamabe and quasi-Yamabe solitons on Euclidean submanifolds*, Mediterr. J. Math. **15** (2018), no. 5, Paper No. 194, 9 pp. https://doi.org/10.1007/s00009-018-1237-2

[6] K. De and U. C. De, *A note on almost Riemann soliton and gradient almost Riemann soliton*, https://arxiv.org/abs/2008.10190.

[7] S. Deshmukh and B. Y. Chen, *A note on Yamabe solitons*, Balkan J. Geom. Appl. **23** (2018), no. 1, 37–43.

[8] M. N. Devaraja, H. Aruna Kumara, and V. Venkatesha, *Riemann soliton within the framework of contact geometry*, Quaest. Math. **44** (2021), no. 5, 637–651. https://doi.org/10.2989/16073606.2020.1732495

[9] I. K. Erken, *Yamabe solitons on three-dimensional normal almost paracontact metric manifolds*, Period. Math. Hungar. **80** (2020), no. 2, 172–184. https://doi.org/10.1007/s10998-019-00303-3

[10] A. Ghosh, *Yamabe soliton and quasi Yamabe soliton on Kenmotsu manifold*, Math. Slovaca **70** (2020), no. 1, 151–160. https://doi.org/10.1515/ms-2017-0340

[11] R. S. Hamilton, *The Ricci flow on surfaces*, in Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988. https://doi.org/10.1090/conm/071/954419

[12] I. E. Hirică and C. Udrişte, *Ricci and Riemann solitons*, Balkan J. Geom. Appl. **21** (2016), no. 2, 35–44.

[13] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. **12** (1989), no. 2, 151–156.

[14] K. Matsumoto and I. Mihai, *On a certain transformation in a Lorentzian para-Sasakian manifold*, Tensor (N.S.) **47** (1988), no. 2, 189–197.

[15] I. Mihai and R. Roşca, *On Lorentzian* P*-Sasakian manifolds*, in Classical analysis (Kazimierz Dolny, 1991), 155–169, World Sci. Publ., River Edge, NJ, 1992.

[16] I. Mihai, A. A. Shaikh, and U. C. De, *On Lorentzian para-Sasakian manifolds*, Rendiconti del Seminario Matematico di Messina, Serie II. (1999) **3**.

[17] D. M. Naik, *Ricci solitons on Riemannian manifolds admitting certain vector field*, Ricerche di Matematica (2021). https://doi.org/10.1007/s11587-021-00622-z

[18] D. M. Naik, V. Venkatesha, and H. A. Kumara, *Ricci solitons and certain related metrics on almost co-Kaehler manifolds*, Zh. Mat. Fiz. Anal. Geom. **16** (2020), no. 4, 402–417.

[19] A. A. Shaikh and K. K. Baishya, *Some results on LP-Sasakian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **49(97)** (2006), no. 2, 193–205.

[20] M. Tarafdar and A. Bhattacharyya, *On Lorentzian para-Sasakian manifolds*, in Steps in differential geometry (Debrecen, 2000), 343–348, Inst. Math. Inform., Debrecen, 2001.

[21] C. Udrişte, *Riemann flow and Riemann wave*, An. Univ. Vest Timiş. Ser. Mat.-Inform. **48** (2010), no. 1-2, 265–274.

[22] C. Udrişte, *Riemann flow and Riemann wave via bialternate product Riemannian metric*, preprint (2012). arXiv.org/math.DG/1112.4279v4

[23] V. Venkatesha, H. A. Kumara, and D. M. Naik, *Riemann solitons and almost Riemann solitons on almost Kenmotsu manifolds*, Int. J. Geom. Methods Mod. Phys. **17** (2020), no. 7, 2050105, 22 pp. https://doi.org/10.1142/S0219887820501054

[24] V. Venkatesha and D. M. Naik, *Yamabe solitons on 3-dimensional contact metric manifolds with $Q_\varphi = \varphi Q$*, Int. J. Geom. Methods Mod. Phys. **16** (2019), no. 3, 1950039, 9 pp. https://doi.org/10.1142/S0219887819500397

[25] Y. Wang, *Yamabe solitons on three-dimensional Kenmotsu manifolds*, Bull. Belg. Math. Soc. Simon Stevin **23** (2016), no. 3, 345–355. http://projecteuclid.org/euclid.bbms/1473186509

[26] Y. Wang, *Minimal and harmonic Reeb vector fields on trans-Sasakian 3-manifolds*, J. Korean Math. Soc. **55** (2018), no. 6, 1321–1336. https://doi.org/10.4134/JKMS.j170689

[27] Y. Wang, *Almost Kenmotsu $(k,\mu)'$-manifolds with Yamabe solitons*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 1, Paper No. 14, 8 pp. https://doi.org/10.1007/s13398-020-00951-y

[28] K. Yano, *Integral Formulas in Riemannian Geometry*, Pure and Applied Mathematics, No. 1, Marcel Dekker, Inc., New York, 1970.

Shruthi Chidananda
Department of Mathematics
Kuvempu University
Shankaraghatta-577 451
Karnataka, India
Email address: c.shruthi28@gmail.com

Venkatesha Venkatesha
Department of Mathematics
Kuvempu University
Shankaraghatta-577 451
Karnataka, India
Email address: vensmath@gmail.com