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DECOMPOSITIONS OF GRADED MAXIMAL SUBMODULES

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Abstract. In this paper, we present different decompositions of graded maximal submodules of a graded module. From these decompositions, we derive decompositions of the graded Jacobson radical of a graded module. Using these decompositions, we prove new theorems about graded maximal submodules, improve old theorems, and give other proofs for old theorems.

1. Introduction

Graded maximal submodules and ideals played important role in the study of Graded Ring and Module Theory. Up to the author's knowledge, the definition of graded maximal submodules was almost involved in any study of such submodules. Not much attention was paid to the possibility that graded maximal submodules possess a special decomposition. Because knowing a special decomposition for graded maximal submodules will lead us to a deeper study of such modules and related concepts, the task of this paper is to introduce such a decomposition. The decompositions permit us to prove new theorems, improve theorems, reprove old theorems, and give a simple method to construct graded maximal submodules and ideals.

While the second section gives a quick review for the basics of graded rings and modules, the third section presents two decompositions for graded maximal submodules along with different applications of the decompositions. For example, we show that the decomposition of graded maximal submodules of graded modules over first strongly graded rings is different from the decomposition of graded maximal submodules of some other modules. Also, in contrast to maximal submodules, we show that graded maximal submodules of certain decomposition cannot be graded direct summands unless the graded module submits a strong restriction. In the fourth section, we use the decompositions of the graded maximal submodules to construct decompositions for the graded

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Jacobson radical of a graded module and for the units of a graded ring, followed by some applications.

2. Preliminaries

This section presents a quick review of graded rings and graded modules. More details can be found in the references (for example [1, 4, 6]) and the literature.

Let G be a group with identity e. Let R be a ring with nonzero unity 1. We say R is graded by G if $R = \bigoplus_{g\in G} R_g$, where R_g is an abelian subgroup of R, and $R_gR_h \subseteq R_{gh}$ for every $g, h \in G$. The set $\operatorname{supp}(R, G) = \{g \in G : R_g \neq 0\}$ is called the support of R. The set $h(R) = \bigcup_{g\in G} R_g$ is the set of homogeneous elements of R. The elements of R_g are called homogeneous elements of degree g. Notice that R_e is a ring with $1 \in R_e$.

Let R be a G-graded ring with nonzero unity 1 and M a left R-module. We say that M is a G-graded R-module if $M = \bigoplus_{g\in G} M_g$, where M_g is an abelian subgroup of M, and $R_gM_h \subseteq M_{gh}$ for every $g, h \in G$. The support of M, $\operatorname{supp}(M, G)$, and $h(M)$ are defined similarly to $\operatorname{supp}(R, G)$ and $h(R)$, respectively. Also, the elements of M_g are called homogeneous elements of degree g.

We say that a ring R is trivially G-graded if $R_g = 0$ for every $g \neq e$ and $R_e = R$. The trivial gradation of a module M by G is defined in a similar way.

A G-graded ring R is first strong, if $R_gR_h = R_{gh}$ for all $g, h \in \operatorname{supp}(R, G)$ or equivalently if $1 \in R_gR_{g^{-1}}$ for all $g \in \operatorname{supp}(R, G)$. It is not difficult to see if R is first strong, then $\operatorname{supp}(R, G)$ is a subgroup of G (see [8]). If $\operatorname{supp}(R, G) = G$ and R is first strong, we say R is strong (see [6]).

A G-graded R-module M is called first strongly graded if $supp(R, G)$ is a subgroup of G and $R_g\, M_h = M_{gh}$ for every $(g, h) \in \operatorname{supp}(R, G) \times G$. If $\operatorname{supp}(R, G) = G$, we obtain the definition of strongly graded modules [7].

To avoid repetition, we assume that all underlying rings and modules are non-trivial, and all modules are left modules.

3. Decompositions of graded maximal submodules

In this section, we present two decompositions of the gr-maximal submodules. These decompositions allow us to prove many theorems, generalize different theorems, and reprove old theorems about gr-maximal submodules and ideals.

Theorem 3.1. *Let M be a G-graded R-module and N a G-graded R-submodule of M. If there exists $h \in \operatorname{supp}(M, G)$ such that $N = \left(\underset{g\in G-\{h\}}{\oplus} M_g\right) \oplus K$ where K is a maximal R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$, then N is a gr-maximal R-submodule of M.*

Proof. Assume $N = \left(\underset{g\in G-\{h\}}{\oplus} M_g\right) \oplus K$ where K is a maximal R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$ and $h \in \operatorname{supp}(M,G)$. Then N is a G-graded R-submodule of M. Let A be a G-graded R-submodule of M such that $N \subsetneqq A \subseteq M$. For every $g \in G-\{h\}$ we have $N_g = M_g$, which yields $A_g = M_g$. Also, we have $K \subsetneqq A_h \subseteq M_h$. However, K is a maximal R_e-submodule of M_h. So, $A_h = M_h$ and hence $A = M$. As a result, N is a gr-maximal R-submodule of M. □

The next theorem is a partial converse of Theorem 3.1. The proof requires the following lemma which has an easy proof.

Lemma 3.2. *Let M be a G-graded R-module and L an R_e-submodule of M_h. Then $\underset{g\neq h}{\oplus} M_g \oplus L$ is a graded R-submodule of M if and only if L contains $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$.*

Theorem 3.3. *Let M be a G-graded R-module and N a gr-maximal R-submodule of M such that there exists $h \in \operatorname{supp}(M,G)$ with $N_h \neq M_h$ and contains $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$, then N_h is a maximal R_e-submodule of M_h. Moreover, $N = \left(\underset{g\in G-\{h\}}{\oplus} M_g\right) \oplus N_h$.*

Proof. Suppose L is a maximal R_e-submodule of M_h such that $N_h \subsetneqq L \subseteq M_h$. By Lemma 3.2, $\underset{g\neq h}{\oplus} M_g \oplus L$ is a graded R-submodule of M such that $N \subsetneqq \underset{g\neq h}{\oplus} M_g \oplus L \subseteq M$. Thus, $N = \underset{g\neq h}{\oplus} M_g \oplus L$ and hence $N_h = L$. So, N_h is a gr-maximal R-submodule of M_h. □

Definition 3.4. Let R be a G-graded R-module. A gr-maximal R-submodule N of M of the form $N = \left(\underset{g\in G-\{h\}}{\oplus} M_g\right) \oplus K$ where K is a maximal R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$ and $h \in \operatorname{supp}(M,G)$ is called a gr-maximal R-submodule of degree h.

It is obvious that there is a one-to-one correspondence between the gr-maximal R-submodules of degree h and the maximal R_e-submodules of M_h that contain $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$.

Corollary 3.5. *If R is not strongly graded and $\sum\limits_{g\in G-\{e\}} R_gR_{g^{-1}} \subsetneqq R_e$, then R has a gr-maximal ideal of degree e.*

Proof. Applying Zorn's lemma on the partial ordered set

$$\{I : I \text{ is an ideal of } R_e \text{ containing } \sum_{g\in G-\{e\}} R_gR_{g^{-1}}\},$$

we get that R_e has a maximal ideal containing $\sum_{g\in G-\{e\}} R_g R_{g^{-1}}$. Thus, by Theorem 3.1 R has a gr-maximal ideal of degree e. ☐

Theorem 3.6. *Let M be a G-graded R-module and N a gr-maximal R-submodule of M. There is at most one homogeneous component $N_h \neq M_h$ of N with the property that N_h contains $\sum_{g\in G-\{h\}} R_{hg^{-1}} M_g$.*

Proof. Suppose that there exist at least two different components $N_h \neq M_h$ and $N_i \neq M_i$ of N, where $h, i \in \operatorname{supp}(M, G)$, containing $\sum_{g\in G-\{h\}} R_{hg^{-1}} M_g$ and $\sum_{g\in G-\{i\}} R_{ig^{-1}} M_g$, respectively. By Lemma 3.2, $K = \bigoplus_{g\neq i} M_g \oplus \sum_{g\in G-\{i\}} R_{ig^{-1}} M_g$ is a G-graded R-submodule of M such that $N \subsetneqq K \subsetneqq M$, which contradicts the fact that N is gr-maximal. ☐

Definition 3.7 ([6])**.** Let M be a G-graded R-module (resp. an R-module) and N a G-graded R-submodule (resp. a submodule) of M. We say N is a graded simple or gr-simple (resp. simple) submodule of M, if $\{0\}$ and M are the only graded submodules (resp. submodules) of M.

Example 3.8. Let M be a non-simple R-module. Give R the trivial gradation by $\mathbb{Z}_2$ and $M \oplus M$ the gradation $M_0 = M \oplus 0$ and $M_1 = 0 \oplus M$. Since $R_1 M_0 = R_1 M_1 = 0$, Theorems 3.3 and 3.6 guarantee that the gr-maximal submodules should be of some degree. Let $\Delta = \{(x, x) : x \in M\}$. Then Δ is a maximal R-submodule but not a gr-maximal R-submodule because it cannot be decomposed according to Theorem 3.3.

Definition 3.9. An R-module (resp. G-graded R-module) is called max-nested (resp. gr-max-nested) if every submodule (resp. graded submodule) is included in a maximal submodule (resp. gr-maximal submodule).

Max-nested modules cover a wide class of modules such as rings with unity, finitely generated modules, Noetherian modules, multiplication modules,... etc (same for gr-max-nested modules).

Theorem 3.10. *Let M be a G-graded R-module such that for every $h \in \operatorname{supp}(M, G)$, $\sum_{g\in G-\{h\}} R_{hg^{-1}} M_g$ is included in every nonzero R_e-submodule of M_h if M_h is not a simple R_e-module and $\sum_{g\in G-\{h\}} R_{hg^{-1}} M_g = 0$ if M_h is simple. Then M is gr-max-nested if and only if M_h is max-nested, for every $h \in G$ and the only gr-maximal R-submodules of M are the gr-maximal submodules of degree g, for all $g \in \operatorname{supp}(M, G)$.*

Proof. Assume M is a gr-max-nested module. Without loss of generality, assume M is not gr-simple. On one hand, let N be a gr-maximal R-submodule of M. We distinguish among four cases.

Case 1: Suppose N_g is 0 for every $g \in H$, where $\emptyset \neq H \subsetneqq \text{supp}(M,G)$ and $N_g = M_g$ for every $g \in \text{supp}(M,G) - H$ and $|H| > 1$. Let $h \in H$. Then $N \subsetneqq \underset{g\neq h}{\oplus} M_g \subsetneqq M$. Since $\underset{g\neq h}{\oplus} M_g$ is a G-graded R-submodule of M, we get that N is not gr-maximal which is a contradiction. So this case is rejected.

Case 2: Suppose N_g is 0 for every $g \in H$, where $\emptyset \neq H \subsetneqq \text{supp}(M,G)$, $N_g = M_g$ for every $g \in \text{supp}(M,G) - H$, $|H| = 1$, say $H = \{h\}$, and M_h is not a simple R_e-module. Let $L \subsetneqq M_h$ be a nonzero R_e-submodule of M_h. Then $N \subsetneqq \underset{g\neq h}{\oplus} M_g \oplus L \subsetneqq M$. Since $\underset{g\neq h}{\oplus} M_g \oplus L$ is a G-graded R-submodule of M, we get that N is not gr-maximal which is a contradiction. So, again, this case is rejected.

Case 3: Suppose N_g is 0 for every $g \in H$, where $\emptyset \neq H \subsetneqq \text{supp}(M,G)$, $N_g = M_g$ for every $g \in \text{supp}(M,G) - H$, $|H| = 1$, say $H = \{h\}$, and M_h is a simple R_e-module. Since $\{0\}$ is a maximal R_e-submodule of M_h, by assumptions, we obtain N is gr-maximal of degree h. This case is accepted.

Case 4: If there exists $h \in \text{supp}(M,G)$ such that $0 \neq N_h \neq M_h$. By the assumption, $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g \subseteq N_h$. Theorem 3.3 implies N is gr-maximal of degree h. This case is accepted.

The cases show that a gr-maximal submodule should be of a specific degree.

On the other hand, let $h \in \text{supp}(M,g)$ and $L \subsetneqq M_h$ be a nonzero R_e-submodule of M_h. We have $K = \underset{g\neq h}{\oplus} M_g \oplus L$ is a G-graded R-submodule of M such that $K \neq M$. By the assumption, there exists a gr-maximal submodule N of M such that $N \supseteq K$. Thus, N is gr-maximal of degree h and hence N_h is a maximal R_e-submodule of M_h containing L. From this we conclude that M_h is a max-nested R_e-module, for every $h \in \text{supp}(M,G)$.

For the converse, assume M_h is max-nested, for every $h \in G$ and the only gr-maximal R-submodules of M are the gr-maximal submodules of degree g, for all $g \in \text{supp}(M,G)$. Let $0 \neq L \subsetneqq M$ be a graded R-submodule. We distinguish among four cases:

Case 1: Suppose L_g is 0 for every $g \in H$, where $\emptyset \neq H \subsetneqq \text{supp}(M,G)$ and $L_g = M_g$ for every $g \in \text{supp}(M,G) - H$ and $|H| > 1$. Let $h \in H$. Then $L \subseteq N = \underset{g\neq h}{\oplus} M_g \oplus K$, where $K = 0$ if M_h is simple and K is a maximal R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g$ if M_h is not simple. Since N is a G-graded R-submodule of M, we get that N is a gr-maximal submodule containing L.

Case 2: Suppose L_g is 0 for every $g \in H$, where $\emptyset \neq H \subsetneqq \text{supp}(M,G)$, $L_g = M_g$ for every $g \in \text{supp}(M,G) - H$, $|H| = 1$, say $H = \{h\}$, and M_h is not a simple R_e-module. Since M_h is max-nested, there exists $K \subsetneqq M_h$ a maximal nonzero R_e-submodule of M_h. Then $N = \underset{g\neq h}{\oplus} M_g \oplus K$ is a G-graded R-submodule of M of degree h containing L.

Case 3: Suppose L_g is 0 for every $g \in H$, where $\emptyset \neq H \subsetneqq \mathrm{supp}(M, G)$, $L_g = M_g$ for every $g \in \mathrm{supp}(M, G) - H$, $|H| = 1$, say $H = \{h\}$, and M_h is a simple R_e-module. Since $\{0\}$ is a maximal R_e-submodule of M_h, by assumptions, we obtain L itself is gr-maximal of degree h.

Case 4: If there exists $h \in \mathrm{supp}(M, G)$ such that $0 \neq L_h \neq M_h$. Since M_h is max-nested, $L_h \subseteq K$ where K is a maximal R_e-submodule of M_h. By assumptions and Theorem 3.3, $N = \underset{g \neq h}{\oplus} M_g \oplus K$ is a gr-maximal R-submodule of M of degree h containing L.

From the cases above, we conclude that M is a gr-max-nested module. $\square$

As an application of Theorem 3.10, we have the following examples.

Example 3.11. Let F be a field accommodated with the trivial gradation by $\mathbb{Z}$. The vector space $F[x]$ over F is graded by $(F[x])_n = Fx^n$ if $n = 0, 1, \ldots$ and $(F[x])_n = 0$ if $n = -1, -2, \ldots$. Notice that $(F[x])_n$, $n = 0, 1, 2, \ldots$ is a simple F-module due to being a vector subspace of dimension 1. The vector space $F[x]$ is not strongly graded, therefore there is a chance of the existence of gr-maximal subspaces. Since $\underset{n \neq j}{\oplus} F_{n-j} F x^n = 0$, where $j = 0, 1, \ldots$. According to Theorem 3.10 the gr-maximal subspaces are $M_j = \oplus \sum_{\substack{n=0 \\ n \neq j}}^{\infty} Fx^n$, where $j = 0, 1, \ldots$. Actually, M_j is a gr-maximal subspace of degree j.

Example 3.12. Consider the abelian group $R = \mathbb{Z}$ as a $\mathbb{Z}$-graded module with the trivial gradation and the abelian group $M = Z_{p^2} \oplus Z_{q^2}$ where p and q are different prime numbers, as a $\mathbb{Z}$-graded $\mathbb{Z}$-module with the trivial gradation. There are only two gr-maximal submodules of M and they are of degree 0, namely $Z_p \oplus Z_{q^2}$ and $Z_{p^2} \oplus Z_q$. Notice the satisfaction of the two gr-maximal submodules for the conditions in Theorem 3.1.

The proof of the following theorem is easy.

Theorem 3.13. *A G-graded R-module M has at least one gr-maximal R-submodule of degree h if and only if M_h has at least one maximal R_e-submodule containing $\sum_{g \neq h} R_{hg^{-1}} M_g$. Further, if $\sum_{g \neq h} R_{hg^{-1}} M_g$ is a maximal R_e-submodule of M_h, then M has a unique gr-maximal R-submodule of degree h.*

Proof. The proof is straight forward from Theorem 3.1. $\square$

In [2, Lemma 2.7], the authors proved that if M is a gr-finitely generated R-module (i.e., M is spanned by finite number of homogeneous elements), then M has a gr-maximal R-submodule. Next, we give another condition that guarantees the existence of gr-maximal R-modules.

Theorem 3.14. *Let M be a G-graded R-module such that a homogeneous component M_h of M is a finitely generated R_e-module, where $h \in \mathrm{supp}(M, G)$ such that $\sum_{g \in G - \{h\}} R_{hg^{-1}} M_g \subsetneqq M_h$, then M has a gr-maximal R-submodule.*

Proof. Assume $M_h = R_e x_1 + \cdots + R_e x_n$, where $x_1, \ldots, x_n$ are a minimal number of elements of M_h that span M_h. Since $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g \subsetneq M_h$, then not all of $x_1, \ldots, x_n$ belong to $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g$. Without loss of generality, assume $x_1 \notin \sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g$. Let $N = R_e x_2 + \cdots + R_e x_n$. Then $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g \subseteq N$ and $N \neq M_h$. Now we apply Zorn's Lemma to the set $W = \{K : K \text{ is an } R_e\text{-submodule of } M_h \text{ containing } N \text{ and } x_1 \notin K\}$. $W \neq \emptyset$ because $N \in W$. If Λ is a chain of W, then $\bigcup_{K\in\Lambda} K$ is an R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g$. Since $x_1 \notin K$ for every $K \in \Lambda$, we obtain $x_1 \notin \bigcup_{K\in\Lambda} K$. Thus, $\bigcup_{K\in\Lambda} K$ is an upper bound of Λ in W. By Zorn's Lemma, W has a maximal element, name it A. The submodule A is a maximal R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g$. To see this, assume L is an R_e-submodule such that $A \subseteq L$. Then $N \subseteq L$ because $N \subseteq A$ and $x_1 \notin L$ because $L \neq M_h$. So, $L \in W$. However, A is a maximal element of W. Therefore $L \subseteq A$ and hence $L = A$. By Theorem 3.1, M has a gr-maximal R-submodule of degree h. □

A gr-maximal submodule needs not to be gr-maximal of some degree as shown in the following theorem.

Theorem 3.15. *Strongly G-graded R-modules do not possess gr-maximal R-submodules of any degree.*

Proof. If $h \in G = \operatorname{supp}(M, G)$, and K is an R_e-submodule of M_h such that $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}} M_g \subseteq K$, then $\sum\limits_{g\in G-\{h\}} M_h \subseteq K \subseteq M_h$ which gives $K = M_h$. Therefore, a strongly graded R-module does not possess gr-maximal R-submodules of any degree. □

Actually, the gr-maximal submodules of graded modules over first strongly graded rings have a different decomposition from gr-maximal submodules of a specific degree as demonstrated in the next work.

Theorem 3.16 ([7])**.** *Let R be a G-graded ring. Then R is first strong if and only if every G-graded R-module is first strong.*

If M is a G-graded R-module such that $M_e = 0$, we can relabel the components of M to produce a new gradation to M such that the e-component is nonzero [5]. So, in the next theorem we can assume, without loss of generality, that $M_e \neq 0$.

Lemma 3.17. *Suppose R is a first strongly G-graded ring and M a G-graded R-module. Then, K is a maximal R_e-submodule of M_g if and only if $R_h K$ is a maximal R_e-submodule of M_{hg}, for every $g \in \operatorname{supp}(M, G)$ and $h \in \operatorname{supp}(R, G)$.*

Proof. Let $g \in \operatorname{supp}(M,G)$ and $h \in \operatorname{supp}(R,G)$ and K be a maximal R_e-submodule of M_g. Let A be an R_e-submodule of M_{hg} such that $R_h K \subsetneqq A \subseteq M_{hg}$. Thus

$$\begin{aligned} R_{h^{-1}} R_h K \subsetneq R_{h^{-1}} A \subseteq R_{h^{-1}} M_{hg} &\Rightarrow R_e K \subsetneqq R_{h^{-1}} A \subseteq M_g \\ &\Rightarrow R_{h^{-1}} A = M_g \\ &\Rightarrow A = M_{hg}. \end{aligned}$$

We deuce that $R_h K$ is a maximal R_e-submodule of M_{hg}. The converse is proved in the same manner. $\square$

Theorem 3.18. *Let R be a first strongly G-graded ring, M a G-graded R-module with $\operatorname{supp}(R,G) = \operatorname{supp}(M,G)$, and N a G-graded R-submodule of M. Then, N is gr-maximal if and only if $N = \underset{g\in G}{\oplus} N_g$ where N_g is a gr-maximal R_e-submodule of M for every $g \in \operatorname{supp}(M,G)$, and zero otherwise.*

Proof. Assume N is gr-maximal. There exists $h \in \operatorname{supp}(M,G)$ such that $N_h \neq M_h$. Fix $g \in \operatorname{supp}(M,G)$. If $N_g = M_g$, then $R_{hg^{-1}} N_g = R_{hg^{-1}} M_g$ which implies by Theorem 3.16 that $N_h = M_h$ which is a contradiction. Thus, $N_g \neq M_g$ for every $g \in \operatorname{supp}(M,G)$. Let $g,h \in \operatorname{supp}(M,G)$ and assume L is an R_e-submodule of M_h such that $N_h \subsetneqq L \subseteq M_h$. By Theorem 3.16, $N_{gh} = R_g N_h \subsetneqq R_g L \subseteq R_g M_h = M_{gh}$. Hence, $N \subsetneqq \underset{g\in G}{\oplus} R_g L \subseteq M$. Since $\underset{g\in G}{\oplus} R_g L$ is a G-graded R-submodule of M and N is gr-maximal, we obtain that $\underset{g\in G}{\oplus} R_g L = M$ which in turn implies that $L = R_e L = M_h$. Therefore, N_h is a maximal R_e-submodule of M_h. Lemma 3.17 implies N_h is a maximal R_e-submodule of M_h, for each $h \in \operatorname{supp}(M,G)$.

For the converse, assume $N = \underset{g\in G}{\oplus} N_g$ where N_g is a gr-maximal R_e-submodule of M for every $g \in \operatorname{supp}(M,G)$ and zero, otherwise. Let L be a G-graded R-submodule of M such that $N \subsetneqq L \subseteq M$. Then for each $g \in \operatorname{supp}(M,G)$, we have $N_g \subsetneqq L_g \subseteq M_g$. By assumption, $L_g = M_g$ for each $g \in \operatorname{supp}(M,G)$ and this yields $L = M$. thus, N is gr-maximal. $\square$

The following corollary follows directly from Theorem 3.18.

Corollary 3.19. *Let R be a strongly G-graded ring and M a G-graded R-module. Then a G-graded R-submodule N of M is gr-maximal if and only if $N = \underset{g\in G}{\oplus} N_g$ where N_g is a gr-maximal R_e-submodule of M for every $g \in G$.*

A linear homomorphism $f : M \to M'$ from a G-graded R-module M to a G-graded R-module M' is said to be a gr-homomorphism of degree $i \in G$ if $f(M_g) \subseteq M'_{gi}$. The proof of the next theorem is straightforward from Theorem 3.1.

Lemma 3.20. *Let M be a G-graded R-module and X a maximal R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$. Then RX is gr-maximal of degree h if and only if $R_gX = M_{gh}$ for every $g \neq e$.*

Proof. Apply Theorem 3.1. □

Definition 3.21. Let R be a G-graded ring, and M a G-graded R-module. We say M is strongly graded at $h \in G$, if $R_gM_h = M_{gh}$ for every $g \in G$.

A graded module which is strongly graded at e is called a flexible graded module (see [9]).

Theorem 3.22. *Let R be a G-graded ring, M a G-graded R-module, X an R_e-submodule of M_h containing $\sum\limits_{g\in G-\{h\}} R_{hg^{-1}}M_g$, where $h \in G$. If RX is a gr-maximal R-submodule of degree h, then M is strongly graded at h.*

Proof. Let $g \in G$ and $g \neq e$. By Lemma 3.20, we have $M_{gh} \supseteq R_gM_h \supseteq R_gX = M_{gh}$. Thus, $R_gM_h = M_{gh}$. Since $R_eM_h = M_h$, we obtain that M is strongly graded at h. □

The following corollary is a direct consequence of the previous theorem.

Corollary 3.23. *If M is a G-graded R-module that contains a gr-maximal R-submodule of degree e of the form RX, where X is an R_e-submodule of M_e, then M is a flexible module.*

Theorem 3.24. *If $f : M \to \bar{M}$ is a gr-epimorphism of degree e, and N is a gr-maximal R-submodule of M of degree h such that $f(N) \neq \bar{M}$, then $f(N)$ is a gr-maximal R-submodule of $\bar{M}$ of degree h.*

Proof. By the assumption, $f(N_g) = f(M_g) = \bar{M}_g$ for every $g \in G - \{h\}$. Since $f(N) \neq \bar{M}$, we obtain $f(N_h) \neq M_h$ and it is not difficult to see that $f(N_h)$ is a maximal R_e-submodule of $\bar{M}_h$. Further,

$$\sum_{g\in G-\{h\}} R_{hg^{-1}}\bar{M}_g = \sum_{g\in G-\{h\}} R_{hg^{-1}}f(M_g) = f\left(\sum_{g\in G-\{h\}} R_{hg^{-1}}M_g\right) \subseteq f(N_h).$$

By Theorem 3.1, $f(N)$ is a gr-maximal R-submodule of $\bar{M}$ of degree h. □

Theorem 3.25. *Let M be a G-graded R-module and N a graded R-submodule of M. If L is a gr-maximal R-submodule of degree $h \in G$, then $\frac{L+N}{N}$ is a gr-maximal R-submodule of $\frac{M}{N}$ of degree h.*

Proof. The proof follows directly by Theorem 3.24 where $f : M \to \frac{M}{N}$ is the natural epimorphism. □

Assume $M = \bigoplus\limits_{g\in G} M_g$ is a G-graded R-module. Given the trivial gradation to R_e by G, then $M' = \bigoplus\limits_{g\in G} M_g$ is a G-graded R_e-module. We call this gradation the gradation induced by the original gradation on M as an R_e-submodule.

Theorem 3.26. *Let M be a G-graded R-module and K a maximal R_e-submodule of M_h. Then $(\underset{g\neq h}{\oplus} M_g) \oplus K$ is a gr-maximal R_e-submodule of M of degree h.*

Proof. The proof follows from Theorem 3.1. □

The following corollary is a straightforward consequence of Theorem 3.26.

Corollary 3.27. *Let M be a G-graded R-module and L a G-graded R-submodule of M. If L is a gr-maximal R-submodule of M of degree h, then L is a gr-maximal R_e-submodule of M' of degree h.*

The converse of the previous corollary is not necessarily true as shown in the next example.

Example 3.28. Let G be a nontrivial group with identity e, and R a strongly G-graded ring with unity. Then, by Theorem 3.15, the ring R does not have gr-maximal ideals of degree e. On the other hand, since R_e has a maximal ideal, Theorem 3.26 asserts that R' has a gr-maximal R_e-submodule of degree e.

Theorem 3.29. *Let $f : M \to \bar{M}$ be a gr-epimorphism of degree e. If* $\mathrm{Ker}(f)$ *is gr-maximal of some degree, then $\bar{M}$ is a trivially graded simple R-module.*

Proof. Suppose that $\mathrm{Ker}(f)$ is gr-maximal of some degree. Then $\{0\}$ is a gr-maximal R-submodule of $\bar{M}$ of the same degree. By Theorem 3.1, $\bar{M}$ is gr-simple and trivially graded by G. □

Theorem 3.30. *Let M be a G-graded R-module and N a gr-maximal submodule of M of degree h. Then $\frac{M}{N}$ is a gr-simple R-module which is isomorphic to $\frac{M_h}{N_h}$ as an R_e-modules.*

Proof. We have $\frac{M}{N} = \frac{\underset{g\in G}{\oplus} M_g}{\underset{g\in G}{\oplus} N_g} \cong \underset{g\in G}{\oplus} \frac{M_g}{N_g}$. Since $M_g = N_g$ for every $g \neq h$, we get $\frac{M}{N} \cong \frac{M_h}{N_h}$, where the symbol $\cong$ means "isomorphic as an R_e-modules". □

Let M be a G-graded R-module and N a G-graded R-submodule. We define the set $(N :_R M)$ by $(N :_R M) = \{r \in R : rM \subseteq N\}$. The set $(N :_R M)$ is a graded ideal of R (see [3]).

Theorem 3.31. *Let M be a G-graded R-module and N a graded R-submodule of M. If N and $(N :_R M)$ are gr-maximal of some degrees, then the degrees coincide.*

Proof. Assume N is a gr-maximal R-submodule of degree h. Then $N = \left(\underset{g\in G-\{h\}}{\oplus} M_g\right) \oplus K$ where K is a maximal R_e-submodule of M_h containing $\sum_{g\in G-\{h\}} R_{hg^{-1}} M_g$ and $h \in \mathrm{supp}(M, G)$. Also, assume $(N :_R M)$ is gr-maximal of degree $\sigma \in \text{supp}(R, G)$. Then $(N :_R M) = \left(\underset{g \in G - \{\sigma\}}{\oplus} R_g\right) \oplus (N :_R M)_\sigma$ where $(N :_R M)_\sigma$ is a maximal R_e-submodule of R_σ containing $\sum_{g \in G - \{\sigma\}} R_{\sigma g^{-1}} R_g$. If $\sigma \neq h$ we obtain

$$R_\sigma M = \underset{g \neq h}{\oplus} R_\sigma M_{\sigma^{-1} g} \oplus R_\sigma M_{\sigma^{-1} h} \subseteq \underset{g \neq h}{\oplus} M_g \oplus K = N,$$

which implies $(N :_R M)_\sigma = R_\sigma$ which contradicts that $(N :_R M)_\sigma$ is a maximal R_e-submodule of R_σ. Therefore, $\sigma = h$. □

Let S be a subset of a left R-module M. The annihilator of S in R is defined to be $Ann_R(S) = \{r \in R : rs = 0, \ \forall s \in S\}$. If S is a graded submodule of M, then $Ann_R(S)$ is a graded ideal of R [3]. It is easy to see that an R_e-submodule X of a component M_h of graded R-module M is a G-graded R-submodule of M if and only if $\underset{g \neq e}{\oplus} R_g \subseteq Ann_R(X)$.

Definition 3.32. Let M be a G-graded R-module and N a G-graded R-submodule of M. We say N is a graded direct summand (or gr-direct summand) of M, if there exists a graded R-submodule L of M such that $M = N \oplus L$.

The following theorem states that a gr-maximal submodule of some degree cannot be a gr-direct summand in a graded module unless a very strict restriction is applied.

Theorem 3.33. *Let M be a G-graded R-module, and N a maximal G-graded R-submodule of M of degree $h \in G$. Then*

(1) *If N is a gr-direct summand of M, say $M = N \oplus L$, then L is a simple R_e-submodule of M_h and $\underset{g \neq e}{\oplus} R_g \subseteq Ann_R(L_h)$ (or equivalently, a gr-simple R-submodule of M).*
(2) *N a gr-direct summand of M if and only if N_h is a direct summand of M_h.*

Proof. (1) Assume $M = N \oplus L$, where L is a graded R-submodule of M. Then $M_g = N_g \oplus L_g$ for every $g \in G$. If $g \neq h$, then $N_g = M_g$ which yields $L_g = 0$. Hence, $L = L_h$ which means L is an R_e-submodule of M_h. Since L is a G-graded submodule, it follows from the paragraph before Definition 3.32 that $\underset{g \neq e}{\oplus} R_g \subseteq Ann_R(L_h)$. The fact that L is a simple R_e-submodule of M_h (or equivalently, a gr-simple R-submodule of M) comes from the fact that $\frac{M}{N}$ is isomorphic to L as an R_e-modules (or equivalently as a graded modules with isomorphism degree equal to e).

(2) The proof of this part follows directly from the proof of the previous part. □

Part (1) of Theorem 3.33 yields the following corollaries.

Corollary 3.34. *Let M be a G-graded R-module such that M_h has no nonzero proper simple R_e-submodule. Then M does not have a gr-direct summand gr-maximal submodule of degree h.*

In the next corollary, an element $m \neq 0$ of a left R-module M is torsion free if $rm = 0$ implies $r = 0$.

Corollary 3.35. *Let M be a torsion free G-graded R-module. Then M does not have a gr-direct summand gr-maximal submodule of any degree.*

Example 3.36. In Example 3.11 and according to either of the above corollaries, $F[x]$ does not possess a gr-maximal F-subspace which is a gr-direct summand of $F[x]$.

Example 3.37. Consider the ring $R = \mathbb{Z}$ as a $\mathbb{Z}_2$-graded module with the trivial gradation and the abelian group $M = Z_{p^2} \oplus Z_{q^2}$ where p and q are different prime numbers, as a $\mathbb{Z}_2$-graded $\mathbb{Z}$-module with the gradation $M_0 = Z_{p^2}$ and $M_1 = Z_{q^2}$. There are only two gr-maximal submodules of M, namely $Z_p \oplus Z_{q^2}$ of degree 0 and $Z_{p^2} \oplus Z_q$ of degree 1. By Theorem 3.33, both gr-maximal submodules are gr-direct summands of M.

Definition 3.38. Let M be a G-graded R-module. A G-graded R-submodule N of M is called graded essential or graded large (alternatively, gr-essential or gr-large) if $N \cap L \neq 0$ for every graded R-submodule $L \neq 0$ of M.

Recall that a maximal submodule is either a direct summand or a large submodule. We prove the same result for maximal graded modules of a degree.

Theorem 3.39. *A gr-maximal submodule of some degree is either gr-large or gr-direct summand.*

Proof. Let M be a G-graded R-module, N a gr-maximal R-submodule of M of degree h. If the maximal submodule N_h is a direct summand R_e-submodule of M_h, Theorem 3.33 implies N is a gr-direct summand of M. Assume N_h is a large R_e-submodule of M_h and $L \neq 0$ a graded R-submodule of M. If $L_h \neq 0$, then $L_h \cap N_h \neq 0$ and hence $N \cap L \neq 0$. If $L_g \neq 0$ for some $g \neq h$, then $L_g \cap N_g = L_g \cap M_g = L_g \neq 0$. Thus, $N \cap L \neq 0$. So, N is gr-large. □

4. Decompositions of the graded Jacobson radical and units

The following section is devoted to present decompositions of the gr-Jacobson radical of a graded module in terms of the Jacobson radical of its components and use these decompositions to develop different results.

The set of all gr-maximal R-submodules of degree g will be denoted by $\mathfrak{M}_g$. Recall that the Jacobson radical $J(M)$ (resp. the graded Jacobson radical $J_{gr}(M)$ or briefly the gr-Jacobson radical) of M is defined to be the intersection of the maximal R-submodules of M (resp. the intersection of all gr-maximal R-submodules of M).

Definition 4.1. Let M be a G-graded R-module. We define the Jacobson radical of M of degree $h \in G$, denoted by $F(M_h)$, to be the intersection of all maximal R_e-submodules of M_h containing $\sum_{g\in G-\{h\}} R_{hg^{-1}}M_g$.

Directly from Definition 4.1 we obtain that $F(M_h)$ is an R_e-submodule of M_h and $J(M_h) \subseteq F(M_h)$. Moreover, if M_h does not contain maximal R_e-submodules containing $\sum_{g\in G-\{h\}} R_{hg^{-1}}M_g$, then $F(M_h) = M_h$.

Theorem 4.2. *Let M be a G-graded R-module whose gr-maximal modules possess degrees. Then $J_{gr}(M) = \bigoplus_{g\in G} F(M_h)$.*

Proof. As a matter of fact,

$$J_{gr}(M) = \bigcap_{g\in G} \bigcap_{L\in\mathfrak{M}_g} L = \bigcap_{g\in G} \left(\bigoplus_{\sigma\in G-\{g\}} M_\sigma \oplus F(M_g) \right) = \bigoplus_{g\in G} F(M_g).$$
□

Corollary 4.3. *Let M be a G-graded R-module whose gr-maximal modules possess degrees. Assume that $F(M_g) = J(M_g)$ for every $g \in \operatorname{supp}(M, G)$. Then, $J_{gr}(M) = \bigoplus_{g\in G} J(M_g)$, where $J(M_g)$ is the Jacobson radical of M_g and $J(M_g) = M_g$ if M_g has no maximal R_e-submodules.*

There were many contributions by mathematicians to find out the conditions that guarantee the equality $J_{gr}(R)\cap R_e = J(R_e)$. The following corollary gives a new condition.

Corollary 4.4. *The following statements are true*:

(1) *Let M be a G-graded R-module whose gr-maximal modules possess degrees. Then, $J_{gr}(M) \cap M_h = F(M_h)$.*
(2) *Let R be a G-graded ring whose gr-maximal ideals possess degrees and such that $F(R_e) = J(R_e)$. Then $J_{gr}(R) \cap R_e = J(R_e)$.*

Proof. (1) The proof is directly obtained from Theorem 4.2.
(2) Since R_e is a ring with unity, $\mathfrak{M}_e \neq \emptyset$. By (1), $J_{gr}(R)\cap R_e = J(R_e)$. □

Let M be a G-graded R-module. Denote by M' the module M as a G-graded R_e-module with the gradation $M'_g = M_g$ for every $g \in G$ and R_e has the trivial gradation by G (the gradation induced by the original gradation of M by G).

Theorem 4.5. *Let M be a G-graded R-module whose gr-maximal modules possess degrees. Then*

(1) *$F(M'_g) = J(M'_g)$ for every $g \in G$.*
(2) *$J_{gr}(M') \cap M'_g = J(M'_g)$ for every $g \in G$.*
(3) *$J_{gr}(M') = \underset{g\in G}{\oplus} J(M'_g)$, where $J(M'_g) = M'_g$ if M'_g is empty of maximal R_e-submodules.*

Proof. Apply Corollaries 4.3 and 4.4. □

Theorem 4.6. *Let R be a first strongly G-graded ring, M a G-graded R-module with* $\operatorname{supp}(R,G)=\operatorname{supp}(M,G)$. *Then* $J_{gr}(M)=\bigoplus_{\sigma\in\operatorname{supp}(M,G)} J(M_\sigma)$.

Proof. Let $\mathfrak{M}_{gr}$ be the set of all gr-maximal submodules of M. By Theorem 3.18 we have

$$J_{gr}(M)=\bigcap_{L\in\mathfrak{M}_{gr}} L=\bigcap_{L\in\mathfrak{M}_{gr}}\left(\bigoplus_{\sigma\in\operatorname{supp}(M,G)} N_\sigma\right)=\bigoplus_{\sigma\in\operatorname{supp}(M,G)} J(M_\sigma),$$

where N_σ is a maximal R_e-submodule of M_σ. □

Next, we describe the units of graded rings and give them an explicit form when the graded ring is a gr-local ring whose unique gr-maximal ideal is of degree e.

Lemma 4.7. *Let R be a G-graded ring. Then R has a gr-maximal ideal of degree e if and only if* $\sum_{g\neq e} R_gR_{g^{-1}}\subsetneqq R_e$

Proof. If R has a gr-maximal ideal J of degree e, by Theorem 3.1 J_e is a maximal ideal of R_e containing $\sum_{g\neq e} R_gR_{g^{-1}}$. Therefore, $\sum_{g\neq e} R_gR_{g^{-1}}\subsetneqq R_e$. Conversely, assume $\sum_{g\neq e} R_gR_{g^{-1}}\subsetneqq R_e$. Since R_e has a unity, there exists a maximal ideal I of R_e containing the proper ideal $\sum_{g\neq e} R_gR_{g^{-1}}$. Now, the ideal $\underset{g\neq e}{\oplus} R_g\oplus I$ is a gr-maximal ideal of degree e by Theorem 3.1. □

Theorem 4.8. *Let R be a G-graded ring. If R has a homogeneous unit of degree different from e, then R has no gr-maximal ideals of degree e.*

Proof. Assume u is a homogeneous unit of degree $h\neq e$. Then u^{-1} is homogeneous of degree h^{-1}. Thus, $1=uu^{-1}\in R_hR_{h^{-1}}$. This implies $\sum_{g\neq e} R_gR_{g^{-1}}=R_e$. By Lemma 4.7, R cannot include gr-maximal ideals of degree e. □

Corollary 4.9. *Let R be a G-graded ring. If R has at least one gr-maximal ideal of degree e (i.e., $F(R_e)\neq R_e$), then all homogeneous units have degree e.*

Proof. Take the contrapositive of Theorem 4.8. □

Theorem 4.10. *Let R be a gr-local ring whose unique gr-maximal ideal K has degree e. Then an element $r=\sum_{g\in G} r_g\in R$ is a unit if and only if r_e is a unit and r_g is not a unit for every $g\neq e$.*

Proof. Assume $r=\sum_{g\in G} r_g\in R$ is a unit. Then $r\notin K$. By Theorem 3.1, $r_e\notin K_e$. Since K_e is the unique maximal ideal of R_e, we obtain that r_e is a unit. For the converse, assume r_e is a unit. By Corollary 4.9, r_g is not a unit for every $g \neq e$ and hence $r_g \in J(R)$ for every $g \neq e$. Thus, $\sum\limits_{g \neq e} r_g \in J(R)$ which yields $1 + r_e^{-1} \sum\limits_{g \neq e} r_g \in U(R)$. Now, $r = r_e \left(1 + r_e^{-1} \sum\limits_{g \neq e} r_g\right) \in U(R)$. □

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