TOEPLITZ AND HANKEL OPERATORS
WITH CARLESON MEASURE SYMBOLS

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Abstract. In this paper, we introduce Toeplitz operators and Hankel operators with complex Borel measures on the closed unit disk. When a positive measure \( \mu \) on \((-1, 1)\) is a Carleson measure, it is known that the corresponding Hankel matrix is bounded and vice versa. We show that for a positive measure \( \mu \) on \( \mathbb{D} \), \( \mu \) is a Carleson measure if and only if the Toeplitz operator with symbol \( \mu \) is a densely defined bounded linear operator. We also study Hankel operators of Hilbert–Schmidt class.

1. Introduction

Let \( \mathbb{D} \) and \( \mathbb{T} \) denote the open unit disk and the unit circle in the complex plane, respectively. A Toeplitz operator with bounded symbol is a compression to \( H^2 \) of a multiplication operator on \( L^2(\mathbb{T}) \). Toeplitz operators were introduced by O. Toeplitz [22,23] and interesting properties of them have been studied by many authors (cf. [2,3,14,20,24], etc.). In addition, Toeplitz operators have been studied in various function spaces other than \( H^2 \) (cf. [1,10,19,21]). Research on Toeplitz operators with operator-valued symbols can be found in the papers [6–9]. The author [17] has investigated Toeplitz operators with symbols of complex Borel measures on \( \mathbb{T} \). In this paper, we define Toeplitz operators and Hankel operators on \( H^2 \) whose symbols are complex Borel measures on \( \mathbb{T} \). In this paper, we define Toeplitz operators and Hankel operators on \( H^2 \) whose symbols are complex Borel measures on the closed unit disk \( \overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T} \).

The Hardy space \( H^2 \) is the class of analytic functions on \( \mathbb{D} \) whose Taylor coefficients are square summable. The \( H^2 \)-functions also can be viewed as square integrable functions on \( \mathbb{T} \) via nontangential limit. We refer the reader to the texts [11], [15], and [16] for details of Hardy spaces. Throughout this paper we use \( \| \cdot \|_2 \) and \( \langle \cdot, \cdot \rangle \) to denote the norm and the inner product in \( H^2 \), respectively.

Received November 5, 2020; Accepted February 3, 2021.

2010 Mathematics Subject Classification. Primary 47B35, 47L60, 28A25.

Key words and phrases. Toeplitz operators, Hankel operators, densely defined operators, Carleson measures.

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Let $M(\mathbb{D})$ denote the space of complex Borel measures on $\mathbb{D}$. For $\mu \in M(\mathbb{D})$ and for $n, k \in \mathbb{N}_0$, define the $(n, k)$-moment of $\mu$ by

$$
\mu_{n,k} = \int_D z^n \overline{z}^k \, d\mu(z).
$$

If $k = 0$, we simply write $\mu_n = \mu_{n,0}$. Observe that

$$
|\mu_{n,k}| \leq \int_D |z|^{n+k} \, |d\mu(z)| \leq \|\mu\|.
$$

Hence the double sequence $\{\mu_{n,k}\}$ is bounded. Note that every complex Borel measure on $\mathbb{D}$ is completely determined by its moments. To see this, suppose that $\mu$ and $\nu$ are complex Borel measures on $\mathbb{D}$ such that $\mu_{n,k} = \nu_{n,k}$ for every $n, k \in \mathbb{N}_0$. Then

$$
\int_D f \, d\mu = \int_D f \, d\nu
$$

whenever $f = p(z, \overline{z})$ is a trigonometric polynomial. Since the trigonometric polynomials are dense in $C(\mathbb{D})$ with respect to the supremum norm, the identity (1) holds for every $f \in C(\mathbb{D})$. In view of the Riesz representation theorem, this shows that the measure $\mu - \nu$ is a linear functional on $C(\mathbb{D})$ which is zero. It follows that $\mu - \nu = 0$, i.e., $\mu = \nu$.

Let $m_2$ be the normalized Lebesgue measure on $\mathbb{D}$ so that $m_2(\mathbb{D}) = 1$. Then, for every $n, k \in \mathbb{N}_0$,

$$
(m_2)_{n,k} = \int_D z^n \overline{z}^k \, dm_2(z) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^{n+k+1} e^{i(n-k)} \, d\theta \, dr.
$$

Thus $(m_2)_{n,k} = \frac{1}{n+1}$ if $n = k$, and $(m_2)_{n,k} = 0$ otherwise. On the other hand, the moments of the unit mass $\delta_0$ concentrated at the point $z = 0$ is

$$
(\delta_0)_{n,k} = \begin{cases} 
1 & (n = k = 0), \\
0 & \text{(otherwise)}.
\end{cases}
$$

Let $C_A(\mathbb{D})$ be the disk algebra, i.e., the set of all continuous functions on $\mathbb{D}$ which are analytic in $\mathbb{D}$. For $f \in C_A(\mathbb{D})$, define a function $T_{\mu}f$ on $\mathbb{D}$ by

$$
(T_{\mu}f)(z) := \int_D f(w) \frac{1}{1 - wz} \, d\mu(w) \quad (z \in \mathbb{D}).
$$

Note that, for each $z \in \mathbb{D}$, the series $\sum_{n=0}^{\infty} \frac{1}{1-z^n}$ converges uniformly on $\mathbb{D}$. It follows that

$$
T_{\mu}f(z) = \int_D f(w) \sum_{n=0}^{\infty} \frac{w^n z^n}{1 - wz} \, d\mu(w)
$$

$$
= \sum_{n=0}^{\infty} \int_D f(w) w^n \, d\mu(w) z^n = \sum_{n=0}^{\infty} (f \cdot \mu)_{0,n} z^n.
$$
Therefore the function $T_\mu f$ is analytic in $D$. If $T_\mu f$ belongs to the Hardy space $H^2$, we say that $f \in D(T_\mu)$. That is, we define

$$D(T_\mu) = \{ f \in C_A(D) : T_\mu f \in H^2 \}.$$ 

It is easy to see that $D(T_\mu)$ is a linear subspace of $H^2$. The mapping $T_\mu$ is a linear operator $H^2$ with domain $D(T_\mu)$.

Similarly, we define a linear operator $H_\mu$ on $H^2$ with domain

$$D(H_\mu) = \{ f \in C_A(D) : H_\mu f \in H^2 \},$$

where

$$H_\mu f(z) := \int_D \frac{f(w)}{1-wz} \, d\mu(w) \quad (z \in D).$$

**Definition.** The linear operator $T_\mu$ is called the **Toeplitz operator with symbol $\mu$**. The linear operator $H_\mu$ is called the **Hankel operator with symbol $\mu$**.

If $\varphi \in L^\infty$, the classical Toeplitz operator $T_\varphi$ on $H^2$ is given by

$$(T_\varphi f)(z) = P(\varphi f)(z) = \int_T \frac{f(\zeta)}{1-\zeta z} \varphi(\zeta) \, dm(\zeta) \quad (f \in H^2),$$

where $P$ is the orthogonal projection of $L^2$ onto $H^2$ and $m$ is the normalized Lebesgue measure on $T$. The identity (2) is a generalization of the above identity. Similarly, the identity (4) is a generalization of the identity for the Hankel operator $H_\varphi$:

$$(H_\varphi f)(z) = \int_T \frac{\zeta f(\zeta)}{1-\zeta z} \varphi(\zeta) \, dm(\zeta) \quad (f \in H^2).$$

(For notational convenience, we divided the integrand in (4) by the variable $w$.) Note also that if $\text{supp } \mu \subseteq [-1, 1]$, then $T_\mu = H_\mu$.

Properties of the operator $T_\mu$ when $\text{supp } \mu \subseteq T$ have been studied in the paper [17]. Some of them also hold for $T_\mu$ and $H_\mu$. For example, for the domain $D = D(T_\mu), D(H_\mu)$, one of the following holds:

(i) $D = \{0\}$.
(ii) $D$ is dense in $H^2$.
(iii) $\text{cl}_{H^2} D = \theta H^2$, where $\theta$ is a singular inner function.

In this paper we focus on the boundedness of Toeplitz operators $T_\mu$ and the Hilbert–Schmidt class of the Hankel operators $H_\mu$. In Section 2, we will show that $T_\mu$ is densely defined bounded linear operator if and only if $\mu$ is a Carleson measure. In Section 3, we provide a general sufficient condition for Hankel operators to belong to the Hilbert–Schmidt class.
2. The boundedness of $T_{\mu}$

Let $T(\mu)$ be the infinite matrix whose entries are the moments of $\mu \in M(\mathbb{D})$:

$$T(\mu) := \begin{bmatrix} 
\mu_{0,0} & \mu_{1,0} & \mu_{2,0} & \cdots \\
\mu_{0,1} & \mu_{1,1} & \mu_{2,1} & \cdots \\
\mu_{0,2} & \mu_{1,2} & \mu_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{bmatrix}$$

(5)

The moment matrix $T(\mu)$ corresponds to $T_{\mu}$ in some sense by (3). If the support of $\mu$ is contained in $T$, then

$$\mu_{n,k} = \int T z^n \overline{z}^k d\mu(z) = \int T z^{n-k} d\mu(z)$$

for every $n, k \in \mathbb{N}_0$. Hence the matrix $T(\mu)$ is a Toeplitz matrix. On the other hand, if the support of $\mu$ is contained in the segment $(-1, 1)$, then

$$\mu_{n,k} = \int_{(-1,1)} x^n x^k d\mu(x) = \int_{(-1,1)} x^{n+k} d\mu(x)$$

for every $n, k \in \mathbb{N}_0$. Hence the matrix $T(\mu)$ is a Hankel matrix.

Another matrix we consider is the infinite Hankel matrix

$$H(\mu) := \begin{bmatrix} 
\mu_0 & \mu_1 & \mu_2 & \cdots \\
\mu_1 & \mu_2 & \mu_3 & \cdots \\
\mu_2 & \mu_3 & \mu_4 & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{bmatrix},$$

(6)

which corresponds to $H_{\mu}$. Recall that $\mu_n = \mu_{n,0}$.

A linear operator $T_{\mu}$ may not be bounded.

**Example 2.1.** (a) Suppose that $\alpha \in \mathbb{D}$. Let $\mu = \delta_{\alpha}$ be the unit mass concentrated at the point $\alpha \in \mathbb{D}$. If $f \in C_A(\mathbb{D})$, then

$$T_{\mu}f(z) = \int_{\mathbb{D}} f(w) \frac{d\mu(w)}{1 - \overline{\alpha} w} = \frac{f(\alpha)}{1 - \overline{\alpha} z} \quad (z \in \mathbb{D}).$$

Note that the function $k_{\alpha}(z) = \frac{1}{1 - \overline{\alpha} z}$ is the reproducing kernel function for $H^2$. Then

$$T_{\mu}f = \langle f, k_{\alpha} \rangle k_{\alpha} = (k_{\alpha} \otimes k_{\alpha})f.$$ 

In particular, $T_{\mu}f \in H^2$. Therefore $D(T_{\mu}) = C_A(\mathbb{D})$ and $T_{\mu}$ is a restriction of the rank one projection $k_{\alpha} \otimes k_{\alpha}$ to $C_A(\mathbb{D})$. The matrix representation of $T_{\mu}$ is

$$T(\mu) = \begin{bmatrix} 
1 & \alpha & \alpha^2 & \cdots \\
\overline{\alpha} & \overline{\alpha} \alpha & \overline{\alpha} \alpha^2 & \cdots \\
\overline{\alpha}^2 & \overline{\alpha}^2 \alpha & \overline{\alpha}^2 \alpha^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{bmatrix}$$
(b) Consider the function

$$\varphi(x) = \frac{1}{2\sqrt{1-x}} \quad (0 \leq x < 1).$$

Let $m_1$ denote the Lebesgue measure on $[0, 1)$. Since

$$\int_{[0,1)} |\varphi| \, dm_1 = \int_0^1 \frac{1}{2\sqrt{1-x}} \, dx = \int_0^1 \frac{1}{2\sqrt{y}} \, dy = 1,$$

the function $\varphi$ belongs to $L^1(m_1)$. Hence $\mu := \varphi \cdot m_1$ is a finite positive Borel measure on $D$. For each $n \in \mathbb{N}_0$,

$$\mu_n = \int_0^1 \frac{x^n}{2\sqrt{1-x}} \, dx = \int_0^1 \frac{(1-y)^n}{2\sqrt{y}} \, dy = \int_0^1 (1-x^2)^n \, dx.$$

If $n \geq 1$, by integration by parts,

$$\mu_n = 2n \int_0^1 x^2 (1-x^2)^{n-1} \, dx = 2n \int_0^1 (1-(1-x^2))(1-x^2)^{n-1} \, dx = 2n(\mu_{n-1} - \mu_n).$$

Hence we have

$$\mu_0 = 1, \quad \mu_n = \frac{2n}{2n+1} \mu_{n-1} \quad (n = 1, 2, 3, \ldots).$$

By using the induction, we can show that

$$\frac{1}{2n+1} \leq \mu_n^2 \leq \frac{1}{n+1}$$

for every $n \in \mathbb{N}_0$. Hence $\{\mu_n\} \notin \ell^2$. Note that the domain $D(T_\mu)$ does not contain all polynomials. Indeed, if $f_n(z) = z^n$, then

$$T_\mu f_n(z) = \int_0^1 \frac{\varphi(x) z^n}{1-zx} \, d\mu(x) = \sum_{k=0}^\infty \mu_{n+k} z^k,$$

which does not belong to $H^2$ because $\{\mu_{n+k}\}_{k \geq 0} \notin \ell^2$. Hence $z^n \notin D(T_\mu)$ for any $n \in \mathbb{N}_0$. On the other hand, if $p_n(z) = 1 - z^n$, then

$$T_\mu p_n(z) = \sum_{k=0}^\infty (\mu_k - \mu_{n+k}) z^k.$$

Since $\mu_k - \mu_{n+k} \leq \frac{\mu_k}{2n}$, the sequence $\{\mu_k - \mu_{n+k}\}_{k \geq 0}$ belongs to $\ell^2$. Hence $T_\mu p_n \in H^2$, i.e., $p_n \in D(T_\mu)$. Observe that $\|p_n\|^2_2 = 2$, but

$$\|T_\mu p_n\|^2_2 = \sum_{k=0}^\infty |\mu_k - \mu_{n+k}|^2 \to \infty$$

as $n \to \infty$. This shows that $T_\mu$ is unbounded.
If $\mu$ is a complex Borel measure on $\mathbb{D}$, we may write $\mu = \mu_1 + \mu_2$, where $\mu_1$ and $\mu_2$ are complex Borel measures on $\mathbb{D}$ which are concentrated on $\mathbb{T}$ and $\mathbb{D}$, respectively. Then $T_\mu f = T_{\mu_1} f + T_{\mu_2} f$ for $f \in C_A(\mathbb{D})$. In the case of $\text{supp} \mu \subseteq \mathbb{T}$, the following is known (see e.g., [26]):

**Theorem 2.2.** Let $\mu \in M(\mathbb{T})$. The followings are equivalent:

1. $\mu$ is a compatible measure, i.e., $\int_\mathbb{T} |f|^2 \, d\mu \leq c \int_\mathbb{T} |f|^2 \, dm$ for all $f \in C_A(\mathbb{D})$.
2. $\mathcal{D}(T_\mu)$ contains all polynomials and $T_\mu$ is bounded on $\mathcal{D}(T_\mu)$.

In the remainder of this paper we will focus on the case of measures concentrated in $\mathbb{D}$ and investigate the boundedness of $T_\mu$. A compatible measure is replaced by a positive Carleson measure. A complex Borel measure $\mu$ on $\mathbb{D}$ is called a Carleson measure if there exists a constant $c > 0$ such that

$$|\mu|(S_{\theta_0, h}) \leq c \cdot h$$

for every sector $S_{\theta_0, h} = \{re^{i\theta} : 1 - h \leq r < 1, |\theta_0 - \theta| \leq h\}$. The Carleson imbedding theorem (cf. [4], [13]) shows that a complex Borel measure $\mu$ on $\mathbb{D}$ is a Carleson measure if and only if there exists a constant $c > 0$ such that

$$\int_\mathbb{D} |f|^2 \, d|\mu| \leq c \cdot \|f\|^2_2$$

for every $f \in H^2$, or equivalently, the identical imbedding operator $I_\mu$ from $H^2$ into $L^2(\mathbb{D}, |\mu|)$, given by

$$I_\mu f = f \quad (f \in H^2),$$

is bounded. If

$$\lim_{h \to 0} \frac{|\mu|(S_{\theta_0, h})}{h} = 0,$$

the measure $\mu$ is called a vanishing Carleson measure. In this case $I_\mu$ becomes a compact operator.

An interesting relation between Hankel matrices and Carleson measures was studies by [25] (see also [18]): An infinite Hankel matrix $\{\alpha_{j+k}\}_{j,k \geq 0}$ determines a bounded operator on $\ell^2$ if and only if there exists a Carleson measure $\mu$ on $\mathbb{D}$ such that $\alpha_j = \int_\mathbb{D} w^j \, d\mu(w)$ for all $j \geq 0$. As a result, for a measure $\mu$ on the segment $(-1, 1)$, the moment matrix $T(\mu)$ is bounded if and only if $\mu$ is a Carleson measure. In particular, we can see that $T_\mu$ is bounded.

We extend this result to the case when $\mu$ is a positive measures on $\mathbb{D}$. To do this, we first observe the following lemma.

**Lemma 2.3.** Let $\mu \in M(\mathbb{D})$. Then

$$\langle T_\mu f, g \rangle = \int_\mathbb{D} f \bar{g} \, d\mu$$

for every $f \in \mathcal{D}(T_\mu)$ and $g \in C_A(\mathbb{D})$. 

Proof. The proof of the lemma for measures on $\mathbb{T}$ can be found in [17]. The proof of the lemma for measures on $\mathbb{D}$ is exactly same. For the sake of completeness, we give the proof.

Suppose that $f \in D(T_\mu)$ and $g \in C_A(\mathbb{D})$, so that $T_\mu f \in H^2$. Write $T_\mu f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$. Then

$$\langle T_\mu f, g \rangle = \sum_{n=0}^{\infty} a_n b_n.$$

By (3), for each $z \in \mathbb{D}$,

$$(T_\mu f)(z) = \sum_{n=0}^{\infty} \left[ \int_{\mathbb{T}} f(w) \overline{w^n} d\mu(w) \right] z^n.$$

Hence we have

$$a_n = \int_{\mathbb{T}} f(w) \overline{w^n} d\mu(w) \quad (n = 0, 1, 2, \ldots).$$

Observe that, for each $0 < r < 1$,

$$g_r = \sum_{n=0}^{\infty} b_n r^n z^n \in C_A(\mathbb{D}).$$

It follows that

$$\langle T_\mu f, g_r \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n r^n} = \sum_{n=0}^{\infty} \int_{\mathbb{T}} f(w) \overline{w^n} \overline{w^n r^n} d\mu(w)$$

$$= \int_{\mathbb{T}} f(w) \sum_{n=0}^{\infty} b_n r^n w^n d\mu(w) = \int_{\mathbb{D}} f(w) g_r(w) d\mu(w).$$

If we let $r \to 1$, then $\|g - g_r\|_{\infty} \to 0$, and hence $\langle T_\mu, g_r \rangle \to \langle T_\mu, g \rangle$ and $\int_{\mathbb{T}} f(z) d\mu \to \int_{\mathbb{D}} f(z) d\mu$. This completes the proof of the lemma. □

Now we have:

**Theorem 2.4.** Let $\mu$ be a positive finite Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

(a) $\mu$ is a Carleson measure.

(b) $T_\mu$ is densely defined and bounded on its domain.

Proof. (a) $\Rightarrow$ (b). Suppose that $\mu$ is a Carleson measure. Then there exists a constant $c > 0$ such that

$$\int_{\mathbb{D}} |f| \, d\mu \leq c \|f\|_2 \|g\|_2$$

for every $f, g \in C_A(\mathbb{D})$. Let $n \in \mathbb{N}_0$ and let $f(z) = z^n$. Then

$$T_\mu f(z) = \int_{\mathbb{D}} \frac{w^n}{1 - wz} \, d\mu(w) = \sum_{j=0}^{\infty} \int_{\mathbb{D}} w^n \overline{w^j} \, d\mu(w)z^j = \sum_{j=0}^{\infty} \mu_{n,j} z^j.$$

This completes the proof of the lemma.
For each \( k \in \mathbb{N}_0 \), put \( p_k(z) = \sum_{j=0}^k \mu_{n,j} z^j \). Then
\[
\int_{\mathbb{D}} |p_k|^2 d\mu = \int_{\mathbb{D}} z^n \sum_{j=0}^k \mu_{n,j} z^j d\mu(z) = \sum_{j=0}^k |\mu_{n,j}|^2 = \|p_k\|_2^2.
\]
Since \( \int_{\mathbb{D}} |p_k|^2 d\mu \leq c\|f\|_2\|p_k\|_2 \), it follows that \( \|p_k\|_2 \leq c\|f\|_2 \). Hence
\[
\|T_\mu f\|_2^2 = \sum_{j=0}^\infty |\mu_{n,j}|^2 = \lim_{k \to \infty} \|p_k\|_2^2 \leq c\|f\|_2 < \infty.
\]
Therefore, \( T_\mu f \in H^2 \), i.e., \( f \in \mathcal{D}(T_\mu) \). We have shown that \( \mathcal{D}(T_\mu) \) contains every monomial \( z^n \). Since \( \mathcal{D}(T_\mu) \) is a linear space, it contains all polynomials. Hence \( \mathcal{D}(T_\mu) \) is dense in \( H^2 \) and \( T_\mu \) is bounded on \( \mathcal{D}(T_\mu) \).

(b) \( \Rightarrow \) (a). Suppose that \( \mathcal{D}(T_\mu) \) is dense in \( H^2 \) and \( T_\mu \) is bounded on \( \mathcal{D}(T_\mu) \). By Lemma 2.3, for every \( f \in \mathcal{D}(T_\mu) \),
\[
\int_{\mathbb{D}} |f|^2 d\mu = \|\langle T_\mu f, f \rangle\| \leq \|T_\mu\||f||f||^2.
\]
Define \( I_\mu : \mathcal{D}(T_\mu) \to L^2(\mathbb{D}, \mu) \) by \( I_\mu f = f \) for \( f \in \mathcal{D}(T_\mu) \). By the above inequality, we may extend \( I_\mu \) to a bounded operator on \( H^2 \) with bound \( \|T_\mu\|^{1/2} \).

Then, for every \( f \in H^2 \), we have
\[
\int_{\mathbb{D}} |I_\mu f|^2 d\mu \leq \|T_\mu\||f||f||^2.
\]
Now let \( f \in H^2 \) and let \( \{f_n\} \) be a sequence in \( \mathcal{D}(T_\mu) \) which converges to \( f \). Then \( f_n(z) \to f(z) \) for every \( z \in \mathbb{D} \). On the other hand, since \( I_\mu \) is bounded, we have \( I_\mu f_n = f_n \to I_\mu f \) in \( L^2(\mathbb{D}, \mu) \). It follows from Fatou’s lemma that
\[
\int_{\mathbb{D}} |I_\mu f - f_n|^2 d\mu \leq \liminf_{n \to \infty} \int_{\mathbb{D}} |I_\mu f - f_n|^2 d\mu = \liminf_{n \to \infty} \|I_\mu f - f_n\|_{L^2(\mathbb{D}, \mu)}^2 = 0.
\]
Thus \( I_\mu f = f \) a.e. \( [\mu] \). Hence we have \( \int_{\mathbb{D}} |f|^2 d\mu \leq \|T_\mu\||f||f||^2 \) for every \( f \in H^2 \), i.e., \( \mu \) is a Carleson measure. \( \square \)

Remark 2.5. A similar argument shows that \( \mathcal{H}_\mu \) is densely defined and bounded on its domain whenever \( \mu \) is a Carleson measure. For the converse, however, even in the case of \( \mathcal{D}(\mathcal{H}_\mu) = C_A(\mathbb{D}) \), we can only guarantee that there exists a Carleson measure \( \nu \) such that \( \mu_n = \nu_n \) for \( n \in \mathbb{N} \).

3. The Hilbert–Schmidt class of \( \mathcal{H}_\mu \)

For \( 1 \leq p \leq \infty \), let \( S_p \) denote the Schatten \( p \)-class of operators on \( H^2 \) (or \( \ell^2 \)). If \( p = 1 \), the following is known [18]: For \( \mu \in M(\mathbb{D}) \), \( \mathcal{H}(\mu) \in S_1 \) if and only if \( H(\mu) = H(\nu) \) for some finite complex measure \( \nu \) such that
\[
\int_{\mathbb{D}} \frac{1}{1 - |w|^2} \, d\mu(w) < \infty.
\]
In particular, if $\mu$ is a measure on $(-1,1)$ and $H(\mu) \in S_1$, then $\mu$ satisfies
\[ \int_{(-1,1)} \frac{1}{1 - t^2} \, d\mu(t) < \infty. \]

Note that if $\mu$ is a complex measure on $\mathbb{D}$ satisfying (7), then $\mu$ is a vanishing Carleson measure.

**Question 3.1.** Under what conditions on $\mu$ does $H(\mu)$ belong to the Hilbert–Schmidt class $S_2$ (or $S_2$)?

If $\mu$ is a positive Borel measure on $[0,1)$, answers to the question are given by [5] and [12]:

**Theorem 3.2** ([5]). Assume $1 < p < \infty$ and let $\mu$ be a positive Borel measure on $[0,1)$. Then, $H(\mu) \in S_p$ if and only if $\sum_{n=0}^{\infty} (n+1)^{p-1} \hat{\mu}(n) < \infty$.

**Theorem 3.3** ([12]). Let $\mu$ be a finite positive Borel measure on $[0,1)$ and suppose that $H(\mu)$ is bounded on $H^2$. Then $H(\mu) \in S_2$ if and only if
\[ \int_{[0,1]} \frac{\mu([t,1))}{(1-t^2)^2} \, d\mu(t) < \infty. \]

By using this, we can find measures $\mu$ such that $H_\mu \in S_2 \setminus S_1$ or $H_\mu \in S_\infty \setminus S_2$, e.g., $\mu := \sum_{n \geq 1} c_n \delta_{\lambda_n}$, where $c_n = 2^{-n}$, $\lambda_n = 1 - n \cdot 2^{-n}$.

**Remark 3.4.** (a) Theorem 3.2 also holds for a positive Borel measure on $(-1,1)$. To see this, define $\mu'(E) := \mu(-E)$ for $E \subseteq (-1,1)$. Then $\mu'(n) = (-1)^n \mu_n$.

Define $\hat{\mu} := \mu_{[0,1)} + \mu'_{[0,1)}$. (Here, if $\mu_{[0,1)}$ is the measure on $[0,1)$ given by $\mu_{[0,1)}(E) = \mu(E \cap [0,1)$.) Then (i) $\hat{\mu}$ is a measure supported on $[0,1)$; (ii) $\hat{\mu}_n = \mu_n = |\mu_n|$, if $n$ is even; and (iii) $\hat{\mu}_n = \int_{(-1,1]} |t^n| \, d\mu \geq |\mu_n|$, if $n$ is odd.

If $H(\mu) \in S_p$, then it is easy to show that $H(\hat{\mu}) \in S_p$. Hence, by Theorem 3.2, $\sum_{n=0}^{\infty} (n+1)^{p-1} |\mu_n|^p < \infty$. Conversely, suppose that $\sum_{n=0}^{\infty} (n+1)^{p-1} |\mu_n|^p < \infty$. Put
\[ a_n := \int_{[0,1]} t^n \, d\mu_{[0,1)} \quad \text{and} \quad b_n := \int_{(0,1)} t^n \, d\mu'_{[0,1)}. \]

Then $a_n + b_n = \mu_n$ whenever $n$ is even, so
\[ \sum_{n, \text{even}} (n+1)^{p-1} a_n^p < \infty \quad \text{and} \quad \sum_{n, \text{even}} (n+1)^{p-1} b_n^p < \infty. \]

Since $\{a_n\}$ is a decreasing sequence of nonnegative numbers, it follows that $\sum_{n=0}^{\infty} (n+1)^{p-1} a_n^p < \infty$. By Theorem 3.2, we have $H(\mu_{[0,1)}) \in S_p$. In the same way, $H(\mu'_{[0,1)}) \in S_p$.

Observe that $b_n = (-1)^n \int_{(-1,0)} t^n \, d\mu$. Thus $H(\mu_{(-1,0)}) = U \mathcal{H}_{\mu_{[0,1)}} U \in S_p$, where $U$ is the unitary map which maps $e_n$ to $(-1)^n e_n$.

Therefore $\mathcal{H}_{\mu} = \mathcal{H}_{\mu_{[0,1)}} + \mathcal{H}_{\mu_{(-1,0)}} \in S_p$. 

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(b) By Theorem 3.2, we obtain
\[ H(\mu) \in S_3 \iff \sum_{n=0}^{\infty} (n+1)^2 \hat{\mu}(n)^3 < \infty. \]

Observe that
\[ \sum_{n=0}^{\infty} (n+1)^2 \hat{\mu}(n)^3 \approx \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} \hat{\mu}(n)^3 = \sum_{i,j,k} \hat{\mu}(i+j+k)^3, \]
\[ \sum_{i,j,k} \hat{\mu}(i+j+k)^3 = \int_{[0,1]} \int_{[0,1]} \int_{[0,1]} \frac{1}{1-\tau su} \ d\mu(u)\mu(s)\mu(t) \approx \int_{[0,1]} \frac{\mu([t,1])^2}{(1-t)^3} \ d\mu(t). \]

Therefore
\[ H(\mu) \in S_3 \iff \int_{[0,1]} \frac{\mu([t,1])^2}{(1-t)^3} \ d\mu(t) < \infty. \]

In a similar manner, it may be true that, for \( p = 1, 2, 3, \ldots \),
\[ H(\mu) \in S_p \iff \int_{[0,1]} \frac{\mu([t,1])^{p-1}}{(1-t)^p} \ d\mu(t) < \infty. \]

Now we try to extend Theorem 3.3 to a measure on \( \mathbb{D} \). Since \( S_1 \subseteq S_2 \), the condition on \( \mu \) must be weaker than (7). For \( 0 < t < 1 \), define
\[ \mathbb{D}_t = \{ z : |z| < t \}, \quad \mathbb{T}_t = \{ z : |z| = t \}, \quad \mathbb{A}_t = \{ z : t < |z| < 1 \}. \]

Note that \( \mathbb{D}_t = \mathbb{D}_t \cup \mathbb{T}_t, \mathbb{T}_t = \mathbb{A}_t \cup \mathbb{T}_t, \) and \( \mathbb{D} = \mathbb{D}_t \cup \mathbb{A}_t \cup \mathbb{T}_t. \) We first consider the positive measure on \( \mathbb{D} \) such that

(8) \[ \int_{\mathbb{D}} \frac{\mu(\mathbb{A}_t)}{(1-|z|)^2} \ d\mu(z) < \infty. \]

**Proposition 3.5.** If \( \mu \geq 0 \) on \( \mathbb{D} \) satisfies (8), then \( \mu \) is a vanishing Carleson measure on \( \mathbb{D} \).

**Proof.** Observe that
\[ \int_{\mathbb{A}_t} \mu(\mathbb{A}_t) \ d\mu(z) = \int_{\mathbb{A}_t} \int_{\mathbb{D}} \chi_{\mathbb{A}_t}(w) \ d\mu(w) \ d\mu(z) = \int_{\mathbb{D}} \int_{\mathbb{A}_t} \chi_{\mathbb{A}_t}(w) \ d\mu(w) \ d\mu(z) = \int_{\mathbb{D}} \mu(\mathbb{A}_t \cap \mathbb{D}(w)) \ d\mu(w) \]
\[ = \int_{\mathbb{D}} \mu(\mathbb{A}_t \cap \mathbb{D}(w)) \ d\mu(w) = \int_{\mathbb{A}_t} \mu(\mathbb{A}_t \cap \mathbb{D}(w)) \ d\mu(w). \]

Hence
\[ 2 \int_{\mathbb{A}_t} \mu(\mathbb{A}_t) \ d\mu(z) = \int_{\mathbb{A}_t} \mu(\mathbb{A}_t) \ d\mu(z) + \int_{\mathbb{A}_t} \mu(\mathbb{A}_t \cap \mathbb{D}(w)) \ d\mu(z) \]
\[ = \int_{\mathbb{A}_t} \mu(\mathbb{A}_t) \ d\mu(z) + \int_{\mathbb{A}_t} \mu(\mathbb{T}_t) \ d\mu(z). \]
\[ \mu(\mathcal{A}_s)^2 + \int_{\mathbb{T}} \mu(\mathbb{T}|z|) \, d\mu(z). \]

In particular,
\[ \mu(\mathcal{A}_s)^2 \leq 2 \int_{\mathbb{T}} \mu(\mathcal{A}_{|z|}) \, d\mu(z). \]

Let \( \epsilon > 0 \). Then there exists \( s_0 > 0 \) such that \( s \geq s_0 \) implies
\[ \int_{\mathbb{T}} \mu(\mathcal{A}_{|z|}) \frac{1}{(1-|z|)^2} \, d\mu(z) < \epsilon. \]

It follows from (9) that
\[ 2\epsilon > 2 \int_{\mathbb{T}} \mu(\mathcal{A}_{|z|}) \frac{1}{(1-|z|)^2} \, d\mu(z) \geq \frac{2}{(1-s)^2} \int_{\mathbb{T}} \mu(\mathcal{A}_{|z|}) \, d\mu(z) \geq \frac{\mu(\mathcal{A}_s)^2}{(1-s)^2} \geq \frac{\mu(S_{\theta,1-s})^2}{(1-s)^2} \]
for every \( \theta \). This shows that \( \mu \) is a vanishing Carleson measure. \qed

**Theorem 3.6.** If a positive measure \( \mu \) on \( \mathbb{D} \) satisfies (8), then \( H(\mu) \in S_2 \).

**Proof.** Suppose that \( \mu \) satisfies the above condition. Since
\[ \|H(\mu)\|_{S^2} = \sum_{i,j=0}^{\infty} |\hat{\mu}(i,j)|^2 \]
\[ \leq \sum_{i,j=0}^{\infty} \int_{\mathbb{D}} \int_{\mathbb{D}} (|z||w|)^{i+j} \, d\mu(z) \, d\mu(w) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{d\mu(z) \, d\mu(w)}{(1-|z||w|)^2}, \]
it suffices to show that the last integral is finite. Observe that for any positive measurable function \( f(z,w) \), we have
\[ \int_{\mathbb{D}} \int_{\mathbb{D}} f(z,w) \, d\mu(w) \, d\mu(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} f(z,w) \chi_{\mathbb{D}|z|}(w) \, d\mu(w) \, d\mu(z) \]
\[ = \int_{\mathbb{D}} \int_{\mathbb{D}} f(z,w) \chi_{\mathbb{A}|w|}(z) \, d\mu(z) \, d\mu(w) \]
\[ = \int_{\mathbb{D}} \int_{\mathbb{D}} f(z,w) \, d\mu(z) \, d\mu(w). \]

Hence
\[ \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{d\mu(z) \, d\mu(w)}{(1-|z||w|)^2} = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{d\mu(z) \, d\mu(w)}{(1-|z||w|)^2} + \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{d\mu(z) \, d\mu(w)}{(1-|z||w|)^2} \]
\[ \leq \int_{\mathbb{D}} \frac{\mu(\mathbb{T}_{|z|})}{(1-|z|)^2} \, d\mu(z) + \int_{\mathbb{D}} \frac{\mu(\mathcal{A}_{|z|})}{(1-|z|)^2} \, d\mu(z) \]
\[ \leq 2 \cdot \int_{\mathbb{D}} \frac{\mu(\mathcal{A}_{|z|})}{(1-|z|)^2} \, d\mu(z) < \infty. \] \qed
Note that the converse is not true: If $m_2$ is a Lebesgue measure on $\mathbb{D}$, then $H(m_2)$ is of finite rank, but

$$\int_{\mathbb{D}} \frac{\mu(\overline{A}_z)}{(1-|z|^2)} \, dm_2(z) = \int_0^{2\pi} \int_0^1 \frac{\pi(1-r^2)}{(1-r^2)^2} r \, dr \, d\theta = \infty.$$ 

References


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