Commun. Korean Math. Soc. **37** (2022), No. 1, pp. 91–103
https://doi.org/10.4134/CKMS.c200418
pISSN: 1225-1763 / eISSN: 2234-3024

TOEPLITZ AND HANKEL OPERATORS WITH CARLESON MEASURE SYMBOLS

Jaehui Park

Abstract. In this paper, we introduce Toeplitz operators and Hankel operators with complex Borel measures on the closed unit disk. When a positive measure μ on $(-1, 1)$ is a Carleson measure, it is known that the corresponding Hankel matrix is bounded and vice versa. We show that for a positive measure μ on $\mathbb{D}$, μ is a Carleson measure if and only if the Toeplitz operator with symbol μ is a densely defined bounded linear operator. We also study Hankel operators of Hilbert–Schmidt class.

1. Introduction

Let $\mathbb{D}$ and $\mathbb{T}$ denote the open unit disk and the unit circle in the complex plane, respectively. A Toeplitz operator with bounded symbol is a compression to H^2 of a multiplication operator on $L^2(\mathbb{T})$. Toeplitz operators were introduced by O. Toeplitz [22,23] and interesting properties of them have been studied by many authors (cf. [2,3,14,20,24], etc.). In addition, Toeplitz operators have been studied in various function spaces other than H^2 (cf. [1,10,19,21]). Research on Toeplitz operators with operator-valued symbols can be found in the papers [6–9]. The author [17] has investigated Toeplitz operators with symbols of complex Borel measures on $\mathbb{T}$. In this paper, we define Toeplitz operators and Hankel operators on H^2 whose symbols are complex Borel measures on the closed unit disk $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$.

The Hardy space H^2 is the class of analytic functions on $\mathbb{D}$ whose Taylor coefficients are square summable. The H^2-functions also can be viewed as square integrable functions on $\mathbb{T}$ via nontangential limit. We refer the reader to the texts [11], [15], and [16] for details of Hardy spaces. Throughout this paper we use $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$ to denote the norm and the inner product in H^2, respectively.

Received November 5, 2020; Accepted February 3, 2021.
2010 *Mathematics Subject Classification.* Primary 47B35, 47L60, 28A25.
Key words and phrases. Toeplitz operators, Hankel operators, densely defined operators, Carleson measures.

Let $M(\overline{\mathbb{D}})$ denote the space of complex Borel measures on $\overline{\mathbb{D}}$. For $\mu \in M(\overline{\mathbb{D}})$ and for $n, k \in \mathbb{N}_0$, define the (n, k)-moment of μ by

$$\mu_{n,k} = \int_{\overline{\mathbb{D}}} z^n \overline{z}^k \, d\mu(z).$$

If $k = 0$, we simply write $\mu_n = \mu_{n,0}$. Observe that

$$|\mu_{n,k}| \leq \int_{\overline{\mathbb{D}}} |z|^{n+k} \, d|\mu|(z) \leq \|\mu\|.$$

Hence the double sequence $\{\mu_{n,k}\}$ is bounded. Note that every complex Borel measure on $\overline{\mathbb{D}}$ is completely determined by its moments. To see this, suppose that μ and ν are complex Borel measures on $\overline{\mathbb{D}}$ such that $\mu_{n,k} = \nu_{n,k}$ for every $n, k \in \mathbb{N}_0$. Then

$$\int_{\overline{\mathbb{D}}} f \, d\mu = \int_{\overline{\mathbb{D}}} f \, d\nu \tag{1}$$

whenever $f = p(z, \overline{z})$ is a trigonometric polynomial. Since the trigonometric polynomials are dense in $C(\overline{\mathbb{D}})$ with respect to the supremum norm, the identity (1) holds for every $f \in C(\overline{\mathbb{D}})$. In view of the Riesz representation theorem, this shows that the measure $\mu - \nu$ is a linear functional on $C(\overline{\mathbb{D}})$ which is zero. It follows that $\mu - \nu = 0$, i.e., $\mu = \nu$.

Let m_2 be the normalized Lebesgue measure on $\overline{\mathbb{D}}$ so that $m_2(\overline{\mathbb{D}}) = 1$. Then, for every $n, k \in \mathbb{N}_0$,

$$(m_2)_{n,k} = \int_{\mathbb{D}} z^n \overline{z}^k \, dm_2(z) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^{n+k+1} e^{i(n-k)} \, d\theta dr.$$

Thus $(m_2)_{n,k} = \frac{1}{n+1}$ if $n = k$, and $(m_2)_{n,k} = 0$ otherwise. On the other hand, the moments of the unit mass δ_0 concentrated at the point $z = 0$ is

$$(\delta_0)_{n,k} = \begin{cases} 1 & (n = k = 0), \\ 0 & (\text{otherwise}). \end{cases}$$

Let $C_A(\mathbb{D})$ be the disk algebra, i.e., the set of all continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$. For $f \in C_A(\mathbb{D})$, define a function $\mathcal{T}_\mu f$ on $\mathbb{D}$ by

$$(\mathcal{T}_\mu f)(z) := \int_{\overline{\mathbb{D}}} \frac{f(w)}{1 - \overline{w}z} \, d\mu(w) \qquad (z \in \mathbb{D}). \tag{2}$$

Note that, for each $z \in \mathbb{D}$, the series $\frac{1}{1-\overline{w}z} = \sum_{n=0}^\infty \overline{w}^n z^n$ converges uniformly on $\overline{\mathbb{D}}$. It follows that

$$\begin{aligned} \mathcal{T}_\mu f(z) &= \int_{\overline{\mathbb{D}}} f(w) \sum_{n=0}^\infty \overline{w}^n z^n \, d\mu(w) \\ &= \sum_{n=0}^\infty \int_{\overline{\mathbb{D}}} f(w) \overline{w}^n \, d\mu(w) z^n = \sum_{n=0}^\infty (f \cdot \mu)_{0,n} z^n. \end{aligned} \tag{3}$$

Therefore the function $\mathcal{T}_\mu f$ is analytic in $\mathbb{D}$. If $\mathcal{T}_\mu f$ belongs to the Hardy space H^2, we say that $f \in \mathcal{D}(\mathcal{T}_\mu)$. That is, we define

$$\mathcal{D}(\mathcal{T}_\mu) = \{f \in C_A(\mathbb{D}) : \mathcal{T}_\mu f \in H^2\}.$$

It is easy to see that $\mathcal{D}(\mathcal{T}_\mu)$ is a linear subspace of H^2. The mapping $\mathcal{T}_\mu$ is a linear operator H^2 with domain $\mathcal{D}(\mathcal{T}_\mu)$.

Similarly, we define a linear operator $\mathcal{H}_\mu$ on H^2 with domain

$$\mathcal{D}(\mathcal{H}_\mu) = \{f \in C_A(\mathbb{D}) : \mathcal{H}_\mu f \in H^2\},$$

where

$$(\mathcal{H}_\mu f)(z) := \int_{\mathbb{D}} \frac{f(w)}{1 - wz}\, d\mu(w) \qquad (z \in \mathbb{D}). \tag{4}$$

Definition. The linear operator $\mathcal{T}_\mu$ is called the *Toeplitz operator with symbol* μ. The linear operator $\mathcal{H}_\mu$ is called the *Hankel operator with symbol* μ.

If $\varphi \in L^\infty$, the classical Toeplitz operator T_φ on H^2 is given by

$$(T_\varphi f)(z) = P(\varphi f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} \varphi(\zeta)\, dm(\zeta) \qquad (f \in H^2),$$

where P is the orthogonal projection of L^2 onto H^2 and m is the normalized Lebesgue measure on $\mathbb{T}$. The identity (2) is a generalization of the above identity. Similarly, the identity (4) is a generalization of the identity for the Hankel operator H_φ:

$$(H_\varphi f)(z) = \int_{\mathbb{T}} \frac{\zeta f(\zeta)}{1 - \zeta z} \varphi(\zeta)\, dm(\zeta) \qquad (f \in H^2).$$

(For notational convenience, we divided the integrand in (4) by the variable w.) Note also that if $\operatorname{supp} \mu \subseteq [-1, 1]$, then $\mathcal{T}_\mu = \mathcal{H}_\mu$.

Properties of the operator $\mathcal{T}_\mu$ when $\operatorname{supp} \mu \subseteq \mathbb{T}$ have been studied in the paper [17]. Some of them also hold for $\mathcal{T}_\mu$ and $\mathcal{H}_\mu$. For example, for the domain $\mathcal{D} = \mathcal{D}(\mathcal{T}_\mu), \mathcal{D}(\mathcal{H}_\mu)$, one of the following holds:

(i) $\mathcal{D} = \{0\}$.
(ii) $\mathcal{D}$ is dense in H^2.
(iii) $\mathrm{cl}_{H^2} \mathcal{D} = \theta H^2$, where θ is a singular inner function.

In this paper we focus on the boundedness of Toeplitz operators $\mathcal{T}_\mu$ and the Hilbert–Schmidt class of the Hankel operators $\mathcal{H}_\mu$. In Section 2, we will show that $\mathcal{T}_\mu$ is densely defined bounded linear operator if and only if μ is a Carleson measure. In Section 3, we provide a general sufficient condition for Hankel operators to belong to the Hilbert–Schmidt class.

2. The boundedness of $\mathcal{T}_\mu$

Let $T(\mu)$ be the infinite matrix whose entries are the moments of $\mu \in M(\overline{\mathbb{D}})$:

$$T(\mu) := \begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \mu_{2,0} & \cdots \\ \mu_{0,1} & \mu_{1,1} & \mu_{2,1} & \cdots \\ \mu_{0,2} & \mu_{1,2} & \mu_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

The moment matrix $T(\mu)$ corresponds to $\mathcal{T}_\mu$ in some sense by (3). If the support of μ is contained in $\mathbb{T}$, then

$$\mu_{n,k} = \int_{\mathbb{T}} z^n \overline{z}^k \, d\mu(z) = \int_{\mathbb{T}} z^{n-k} \, d\mu(z)$$

for every $n, k \in \mathbb{N}_0$. Hence the matrix $T(\mu)$ is a Toeplitz matrix. On the other hand, if the support of μ is contained in the segment $(-1, 1)$, then

$$\mu_{n,k} = \int_{(-1,1)} x^n x^k \, d\mu(x) = \int_{(-1,1)} x^{n+k} \, d\mu(x)$$

for every $n, k \in \mathbb{N}_0$. Hence the matrix $T(\mu)$ is a Hankel matrix.

Another matrix we consider is the infinite Hankel matrix

$$H(\mu) := \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{6}$$

which corresponds to $\mathcal{H}_\mu$. Recall that $\mu_n = \mu_{n,0}$.

A linear operator $\mathcal{T}_\mu$ may not be bounded.

Example 2.1. (a) Suppose that $\alpha \in \mathbb{D}$. Let $\mu = \delta_\alpha$ be the unit mass concentrated at the point $\alpha \in \mathbb{D}$. If $f \in C_A(\mathbb{D})$, then

$$\mathcal{T}_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{1 - \overline{w}z} \, d\mu(w) = \frac{f(\alpha)}{1 - \overline{\alpha}z} \qquad (z \in \mathbb{D}).$$

Note that the function $k_\alpha(z) = \frac{1}{1-\overline{\alpha}z}$ is the reproducing kernel function for H^2. Then

$$\mathcal{T}_\mu f = \langle f, k_\alpha \rangle k_\alpha = (k_\alpha \otimes k_\alpha) f.$$

In particular, $\mathcal{T}_\mu f \in H^2$. Therefore $\mathcal{D}(\mathcal{T}_\mu) = C_A(\mathbb{D})$ and $\mathcal{T}_\mu$ is a restriction of the rank one projection $k_\alpha \otimes k_\alpha$ to $C_A(\mathbb{D})$. The matrix representation of $\mathcal{T}_\mu$ is

$$T(\mu) = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots \\ \overline{\alpha} & \overline{\alpha}\alpha & \overline{\alpha}\alpha^2 & \cdots \\ \overline{\alpha}^2 & \overline{\alpha}^2\alpha & \overline{\alpha}^2\alpha^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(b) Consider the function

$$\varphi(x) = \frac{1}{2\sqrt{1-x}} \qquad (0 \le x < 1).$$

Let m_1 denote the Lebesgue measure on $[0,1)$. Since

$$\int_{[0,1)} |\varphi|\, dm_1 = \int_0^1 \frac{1}{2\sqrt{1-x}}\, dx = \int_0^1 \frac{1}{2\sqrt{y}}\, dy = 1,$$

the function φ belongs to $L^1(m_1)$. Hence $\mu := \varphi \cdot m_1$ is a finite positive Borel measure on $\mathbb{D}$. For each $n \in \mathbb{N}_0$,

$$\mu_n = \int_0^1 \frac{x^n}{2\sqrt{1-x}}\, dx = \int_0^1 \frac{(1-y)^n}{2\sqrt{y}}\, dy = \int_0^1 (1-x^2)^n\, dx.$$

If $n \ge 1$, by integration by parts,

$$\begin{aligned} \mu_n &= 2n \int_0^1 x^2(1-x^2)^{n-1}\, dx \\ &= 2n \int_0^1 (1-(1-x^2))(1-x^2)^{n-1}\, dx = 2n(\mu_{n-1} - \mu_n). \end{aligned}$$

Hence we have

$$\mu_0 = 1, \quad \mu_n = \frac{2n}{2n+1}\mu_{n-1} \qquad (n = 1, 2, 3, \dots).$$

By using the induction, we can show that

$$\frac{1}{2n+1} \le \mu_n^2 \le \frac{1}{n+1}$$

for every $n \in \mathbb{N}_0$. Hence $\{\mu_n\} \notin \ell^2$. Note that the domain $\mathcal{D}(\mathcal{T}_\mu)$ does not contain all polynomials. Indeed, if $f_n(z) = z^n$, then

$$\mathcal{T}_\mu f_n(z) = \int_0^1 \frac{\varphi(x)x^n}{1-xz}\, d\mu(x) = \sum_{k=0}^\infty \mu_{n+k} z^k,$$

which does not belong to H^2 because $\{\mu_{n+k}\}_{k\ge0} \notin \ell^2$. Hence $z^n \notin \mathcal{D}(\mathcal{T}_\mu)$ for any $n \in \mathbb{N}_0$. On the other hand, if $p_n(z) = 1 - z^n$, then

$$\mathcal{T}_\mu p_n(z) = \sum_{k=0}^\infty (\mu_k - \mu_{n+k}) z^k.$$

Since $\mu_k - \mu_{n+k} \le \frac{\mu_k}{2k}$, the sequence $\{\mu_k - \mu_{n+k}\}_{k\ge0}$ belongs to ℓ^2. Hence $\mathcal{T}_\mu p_n \in H^2$, i.e., $p_n \in \mathcal{D}(\mathcal{T}_\mu)$. Observe that $\|p_n\|_2^2 = 2$, but

$$\|\mathcal{T}_\mu p_n\|_2^2 = \sum_{k=0}^\infty |\mu_k - \mu_{n+k}|^2 \to \infty$$

as $n \to \infty$. This shows that $\mathcal{T}_\mu$ is unbounded.

If μ is a complex Borel measure on $\overline{\mathbb{D}}$, we may write $\mu = \mu_1 + \mu_2$, where μ_1 and μ_2 are complex Borel measures on $\overline{\mathbb{D}}$ which are concentrated on $\mathbb{T}$ and $\mathbb{D}$, respectively. Then $\mathcal{T}_\mu f = \mathcal{T}_{\mu_1} f + \mathcal{T}_{\mu_2} f$ for $f \in C_A(\mathbb{D})$. In the case of $\operatorname{supp} \mu \subseteq \mathbb{T}$, the following is known (see e.g., [26]):

Theorem 2.2. *Let $\mu \in M(\mathbb{T})$. The followings are equivalent*:

(a) *μ is a compatible measure, i.e., $\int_{\mathbb{T}} |f|^2 \, d\mu \leq c \int_{\mathbb{T}} |f|^2 \, dm$ for all $f \in C_A(\mathbb{D})$.*

(b) *$\mathcal{D}(\mathcal{T}_\mu)$ contains all polynomials and $\mathcal{T}_\mu$ is bounded on $\mathcal{D}(\mathcal{T}_\mu)$.*

In the remainder of this paper we will focus on the case of measures concentrated in $\mathbb{D}$ and investigate the boundedness of $\mathcal{T}_\mu$. A compatible measure is replaced by a positive Carleson measure. A complex Borel measure μ on $\mathbb{D}$ is called a Carleson measure if there exists a constant $c > 0$ such that

$$|\mu|(S_{\theta_0,h}) \leq c \cdot h$$

for every sector $S_{\theta_0,h} = \{re^{i\theta} : 1 - h \leq r < 1, \, |\theta_0 - \theta| \leq h\}$. The Carleson imbedding theorem (cf. [4], [13]) shows that a complex Borel measure μ on $\mathbb{D}$ is a Carleson measure if and only if there exists a constant $c > 0$ such that

$$\int_{\mathbb{D}} |f|^2 \, d|\mu| \leq c \cdot \|f\|_2^2$$

for every $f \in H^2$, or equivalently, the identical imbedding operator I_μ from H^2 into $L^2(\mathbb{D}, |\mu|)$, given by

$$I_\mu f = f \qquad (f \in H^2),$$

is bounded. If

$$\lim_{h \to 0} \frac{|\mu|(S_{\theta_0,h})}{h} = 0,$$

the measure μ is called a vanishing Carleson measure. In this case I_μ becomes a compact operator.

An interesting relation between Hankel matrices and Carleson measures was studies by [25] (see also [18]): An infinite Hankel matrix $\{\alpha_{j+k}\}_{j,k \geq 0}$ determines a bounded operator on ℓ^2 if and only if there exists a Carleson measure μ on $\mathbb{D}$ such that $\alpha_j = \int_{\mathbb{D}} w^j \, d\mu(w)$ for all $j \geq 0$. As a result, for a measure μ on the segment $(-1, 1)$, the moment matrix $T(\mu)$ is bounded if and only if μ is a Carleson measure. In particular, we can see that $\mathcal{T}_\mu$ is bounded.

We extend this result to the case when μ is a positive measures on $\mathbb{D}$. To do this, we first observe the following lemma.

Lemma 2.3. *Let $\mu \in M(\overline{\mathbb{D}})$. Then*

$$\langle \mathcal{T}_\mu f, g \rangle = \int_{\overline{\mathbb{D}}} f \overline{g} \, d\mu$$

for every $f \in \mathcal{D}(\mathcal{T}_\mu)$ and $g \in C_A(\mathbb{D})$.

Proof. The proof of the lemma for measures on $\mathbb{T}$ can be found in [17]. The proof of the lemma for measures on $\overline{\mathbb{D}}$ is exactly same. For the sake of completeness, we give the proof.

Suppose that $f \in \mathcal{D}(\mathcal{T}_\mu)$ and $g \in C_A(\mathbb{D})$, so that $\mathcal{T}_\mu f \in H^2$. Write $\mathcal{T}_\mu f = \sum_{n=0}^\infty a_n z^n$ and $g = \sum_{n=0}^\infty b_n z^n$. Then

$$\langle \mathcal{T}_\mu f, g\rangle = \sum_{n=0}^\infty a_n \overline{b_n}.$$

By (3), for each $z \in \mathbb{D}$,

$$(\mathcal{T}_\mu f)(z) = \sum_{n=0}^\infty \left[\int_{\overline{\mathbb{D}}} f(w)\overline{w}^n \, d\mu(w) \right] z^n.$$

Hence we have

$$a_n = \int_{\overline{\mathbb{D}}} f(w)\overline{w}^n \, d\mu(w) \qquad (n = 0, 1, 2, \dots).$$

Observe that, for each $0 < r < 1$,

$$g_r = \sum_{n=0}^\infty b_n r^n z^n \in C_A(\mathbb{D}).$$

It follows that

$$\begin{aligned}\langle \mathcal{T}_\mu f, g_r\rangle = \sum_{n=0}^\infty a_n \overline{b_n} r^n &= \sum_{n=0}^\infty \int_{\overline{\mathbb{D}}} f(w)\overline{w}^n \overline{b_n} r^n \, d\mu(w) \\ &= \int_{\overline{\mathbb{D}}} f(w) \overline{\sum_{n=0}^\infty b_n r^n w^n} \, d\mu(w) = \int_{\overline{\mathbb{D}}} f(w)\overline{g_r(w)} \, d\mu(w).\end{aligned}$$

If we let $r \to 1$, then $\|g - g_r\|_\infty \to 0$, and hence $\langle \mathcal{T}_\mu, g_r\rangle \to \langle \mathcal{T}_\mu, g\rangle$ and $\int_{\overline{\mathbb{D}}} f\overline{g_r}\, d\mu \to \int_{\overline{\mathbb{D}}} f\overline{g}\, d\mu$. This completes the proof of the lemma. □

Now we have:

Theorem 2.4. *Let μ be a positive finite Borel measure on $\mathbb{D}$. Then the following statements are equivalent*:

(a) *μ is a Carleson measure.*
(b) *$\mathcal{T}_\mu$ is densely defined and bounded on its domain.*

Proof. (a) ⇒ (b). Suppose that μ is a Carleson measure. Then there exists a constant $c > 0$ such that

$$\int_{\mathbb{D}} |fg| \, d\mu \le c\|f\|_2\|g\|_2$$

for every $f, g \in C_A(\mathbb{D})$. Let $n \in \mathbb{N}_0$ and let $f(z) = z^n$. Then

$$\mathcal{T}_\mu f(z) = \int_{\mathbb{D}} \frac{w^n}{1 - \overline{w}z} \, d\mu(w) = \sum_{j=0}^\infty \int_{\mathbb{D}} w^n \overline{w}^j \, d\mu(w) z^j = \sum_{j=0}^\infty \mu_{n,j} z^j.$$

For each $k \in \mathbb{N}_0$, put $p_k(z) = \sum_{j=0}^{k} \mu_{n,j} z^j$. Then

$$\int_{\mathbb{D}} f\overline{p_k}\, d\mu = \int_{\mathbb{D}} z^n \sum_{j=0}^{k} \overline{\mu_{n,j}}\, \overline{z}^j\, d\mu(z) = \sum_{j=0}^{k} \overline{\mu_{n,j}}\mu_{n,j} = \sum_{j=0}^{k} |\mu_{n,j}|^2 = \|p_k\|_2^2.$$

Since $|\int_{\mathbb{D}} f\overline{p_k}\, d\mu| \le c\|f\|_2\|p_k\|_2$, it follows that $\|p_k\|_2 \le c\|f\|_2$. Hence

$$\|\mathcal{T}_\mu f\|_2^2 = \sum_{j=0}^{\infty} |\mu_{n,j}|^2 = \lim_{k\to\infty} \|p_k\|_2^2 \le c\|f\|_2 < \infty.$$

Therefore, $\mathcal{T}_\mu f \in H^2$, i.e., $f \in \mathcal{D}(\mathcal{T}_\mu)$. We have shown that $\mathcal{D}(T_\mu)$ contains every monomial z^n. Since $\mathcal{D}(\mathcal{T}_\mu)$ is a linear space, it contains all polynomials. Hence $\mathcal{D}(\mathcal{T}_\mu)$ is dense in H^2 and $\mathcal{T}_\mu$ is bounded on $\mathcal{D}(\mathcal{T}_\mu)$.

(b) $\Rightarrow$ (a). Suppose that $\mathcal{D}(\mathcal{T}_\mu)$ is dense in H^2 and $\mathcal{T}_\mu$ is bounded on $\mathcal{D}(\mathcal{T}_\mu)$. By Lemma 2.3, for every $f \in \mathcal{D}(\mathcal{T}_\mu)$,

$$\int_{\mathbb{D}} |f|^2\, d\mu = |\langle \mathcal{T}_\mu f, f\rangle| \le \|\mathcal{T}_\mu\|\|f\|_2^2.$$

Define $I_\mu : \mathcal{D}(\mathcal{T}_\mu) \to L^2(\mathbb{D},\mu)$ by $I_\mu f = f$ for $f \in \mathcal{D}(\mathcal{T}_\mu)$. By the above inequality, we may extend I_μ to a bounded operator on H^2 with bound $\|\mathcal{T}_\mu\|^{1/2}$. Then, for every $f \in H^2$, we have

$$\int_{\mathbb{D}} |I_\mu f|^2\, d\mu \le \|\mathcal{T}_\mu\|\|f\|_2^2.$$

Now let $f \in H^2$ and let $\{f_n\}$ be a sequence in $\mathcal{D}(\mathcal{T}_\mu)$ which converges to f. Then $f_n(z) \to f(z)$ for every $z \in \mathbb{D}$. On the other hand, since I_μ is bounded, we have $I_\mu f_n (= f_n) \to I_\mu f$ in $L^2(\mathbb{D},\mu)$. It follows from Fatou's lemma that

$$\begin{aligned}\int_{\mathbb{D}} |I_\mu f - f|^2\, d\mu &\le \liminf_{n\to\infty} \int_{\mathbb{D}} |I_\mu f - f_n|^2\, d\mu \\ &= \liminf_{n\to\infty} \|I_\mu f - f_n\|_{L^2(\mathbb{D},\mu)}^2 = 0.\end{aligned}$$

Thus $I_\mu f = f$ a.e. $[\mu]$. Hence we have $\int_{\mathbb{D}} |f|^2\, d\mu \le \|\mathcal{T}_\mu\|\|f\|_2^2$ for every $f \in H^2$, i.e., μ is a Carleson measure. □

Remark 2.5. A similar argument shows that $\mathcal{H}_\mu$ is densely defined and bounded on its domain whenever μ is a Carleson measure. For the converse, however, even in the case of $\mathcal{D}(\mathcal{H}_\mu) = C_A(\mathbb{D})$, we can only guarantee that there exists a Carleson measure ν such that $\mu_n = \nu_n$ for $n \in \mathbb{N}$.

3. The Hilbert–Schmidt class of $\mathcal{H}_\mu$

For $1 \le p \le \infty$, let S_p denote the Schatten p-class of operators on H^2 (or ℓ^2). If $p = 1$, the following is known [18]: For $\mu \in M(\mathbb{D})$, $H(\mu) \in S_1$ if and only if $H(\mu) = H(\nu)$ for some finite complex measure ν such that

$$\int_{\mathbb{D}} \frac{1}{1-|w|^2}\, d\mu(w) < \infty. \tag{7}$$

In particular, if μ is a measure on $(-1,1)$ and $H(\mu) \in S_1$, then μ satisfies

$$\int_{(-1,1)} \frac{1}{1-t^2}\, d\mu(t) < \infty.$$

Note that if μ is a complex measure on $\mathbb{D}$ satisfying (7), then μ is a vanishing Carleson measure.

Question 3.1. Under what conditions on μ does $H(\mu)$ belong to the Hilbert–Schmidt class S_2 (or S_p)?

If μ is a positive Borel measure on $[0,1)$, answers to the question are given by [5] and [12]:

Theorem 3.2 ([5])**.** *Assume* $1 < p < \infty$ *and let* μ *be a positive Borel measure on* $[0,1)$*. Then,* $H(\mu) \in S_p$ *if and only if* $\sum_{n=0}^{\infty}(n+1)^{p-1}\hat{\mu}(n)^p < \infty$.

Theorem 3.3 ([12])**.** *Let* μ *be a finite positive Borel measure on* $[0,1)$ *and suppose that* $H(\mu)$ *is bounded on* H^2*. Then* $H(\mu) \in S_2$ *if and only if*

$$\int_{[0,1)} \frac{\mu([t,1))}{(1-t)^2}\, d\mu(t) < \infty.$$

By using this, we can find measures μ such that $\mathcal{H}_\mu \in S_2 \backslash S_1$ or $\mathcal{H}_\mu \in S_\infty \backslash S_2$, e.g., $\mu := \sum_{n \geq 1} c_n \delta_{\lambda_n}$, where $c_n = 2^{-n}$, $\lambda_n = 1 - n \cdot 2^{-n}$.

Remark 3.4. (a) Theorem 3.2 also holds for a positive Borel measure on $(-1,1)$. To see this, define $\mu'(E) := \mu(-E)$ for $E \subseteq (-1,1)$. Then $\mu'_n = (-1)^n \mu_n$. Define $\tilde{\mu} := \mu_{[0,1)} + \mu'_{(0,1)}$. (Here, if $\mu_{[0,1)}$ is the measure on $[0,1)$ given by $\mu_{[0,1)}(E) = \mu(E \cap [0,1)$.) Then (i) $\tilde{\mu}$ is a measure supported on $[0,1)$; (ii) $\tilde{\mu}_n = \mu_n = |\mu_n|$, if n is even; and (iii) $\tilde{\mu}_n = \int_{(-1,1)} |t^n|\, d\mu \geq |\mu_n|$, if n is odd.

If $H(\mu) \in S_p$, then it is easy to show that $H(\tilde{\mu}) \in S_p$. Hence, by Theorem 3.2, $\sum_{n=0}^{\infty}(n+1)^{p-1}|\mu_n|^p < \infty$. Conversely, suppose that $\sum_{n=0}^{\infty}(n+1)^{p-1}|\mu_n|^p < \infty$. Put

$$a_n := \int_{[0,1)} t^n\, d\mu_{[0,1)} \quad \text{and} \quad b_n := \int_{(0,1)} t^n\, d\mu'_{(0,1)}.$$

Then $a_n + b_n = \mu_n$ whenever n is even, so

$$\sum_{n:\text{even}} (n+1)^{p-1} a_n^p < \infty \quad \text{and} \quad \sum_{n:\text{even}} (n+1)^{p-1} b_n^p < \infty.$$

Since $\{a_n\}$ is a decreasing sequence of nonnegative numbers, it follows that $\sum_n (n+1)^{p-1} a_n^p < \infty$. By Theorem 3.2, we have $H(\mu_{[0,1)}) \in S_p$. In the same way, $H(\mu'_{(0,1)}) \in S_p$. Observe that $b_n = (-1)^n \int_{(-1,0)} t^n\, d\mu$. Thus $H(\mu_{(-1,0)}) = U\mathcal{H}_{\mu'_{(0,1)}}U \in S_p$, where U is the unitary map which maps e_n to $(-1)^n e_n$. Therefore $\mathcal{H}_\mu = \mathcal{H}_{\mu_{[0,1)}} + \mathcal{H}_{\mu_{(-1,0)}} \in S_p$.

(b) By Theorem 3.2, we obtain

$$H(\mu) \in S_3 \iff \sum_{n=0}^{\infty} (n+1)^2 \hat{\mu}(n)^3 < \infty.$$

Observe that

$$\sum_{n=0}^{\infty} (n+1)^2 \hat{\mu}(n)^3 \approx \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} \hat{\mu}(n)^3 = \sum_{i,j,k} \hat{\mu}(i+j+k)^3,$$

$$\begin{aligned} \sum_{i,j,k} \hat{\mu}(i+j+k)^3 &= \int_{[0,1)} \int_{[0,1)} \int_{[0,1)} \frac{1}{(1-tsu)^3} \, d\mu(u)\mu(s)\mu(t) \\ &\approx \int_{[0,1)} \frac{\mu([t,1))^2}{(1-t)^3} \, d\mu(t). \end{aligned}$$

Therefore

$$H(\mu) \in S_3 \iff \int_{[0,1)} \frac{\mu([t,1))^2}{(1-t)^3} \, d\mu(t) < \infty.$$

In a similar manner, it may be true that, for $p = 1, 2, 3, \ldots$,

$$H(\mu) \in S_p \iff \int_{[0,1)} \frac{\mu([t,1))^{p-1}}{(1-t)^p} \, d\mu(t) < \infty.$$

Now we try to extend Theorem 3.3 to a measure on $\mathbb{D}$. Since $S_1 \subseteq S_2$, the condition on μ must be weaker than (7). For $0 < t < 1$, define

$$\mathbb{D}_t = \{z : |z| < t\}, \quad \mathbb{T}_t = \{z : |z| = t\}, \quad A_t = \{z : t < |z| < 1\}.$$

Note that $\overline{\mathbb{D}_t} = \mathbb{D}_t \cup \mathbb{T}_t$, $\overline{A_t} = A_t \cup \mathbb{T}_t$, and $\mathbb{D} = \mathbb{D}_t \cup A_t \cup \mathbb{T}_t$. We first consider the positive measure on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} \frac{\mu(\overline{A}_{|z|})}{(1-|z|)^2} \, d\mu(z) < \infty. \tag{8}$$

Proposition 3.5. *If $\mu \geq 0$ on $\mathbb{D}$ satisfies* (8)*, then μ is a vanishing Carleson measure on $\mathbb{D}$.*

Proof. Observe that

$$\begin{aligned} \int_{\overline{A}_s} \mu(\overline{A}_{|z|}) \, d\mu(z) &= \int_{\overline{A}_s} \int_{\mathbb{D}} \chi_{\overline{A}_{|z|}}(w) \, d\mu(w) \, d\mu(z) = \int_{\mathbb{D}} \int_{\overline{A}_s} \chi_{\overline{\mathbb{D}}_{|w|}}(z) \, d\mu(z) \, d\mu(w) \\ &= \int_{\mathbb{D}} \mu(\overline{A}_s \cap \overline{\mathbb{D}}_{|w|}) \, d\mu(w) = \int_{\overline{A}_s} \mu(\overline{A}_s \cap \overline{\mathbb{D}}_{|w|}) \, d\mu(w). \end{aligned}$$

Hence

$$\begin{aligned} 2 \int_{\overline{A}_s} \mu(\overline{A}_{|z|}) \, d\mu(z) &= \int_{\overline{A}_s} \mu(\overline{A}_{|z|}) \, d\mu(z) + \int_{\overline{A}_s} \mu(\overline{A}_s \cap \overline{\mathbb{D}}_{|z|}) \, d\mu(z) \\ &= \int_{\overline{A}_s} \mu(\overline{A}_s) \, d\mu(z) + \int_{\overline{A}_s} \mu(\mathbb{T}_{|z|}) \, d\mu(z) \end{aligned}$$

$$= \mu(\overline{A}_s)^2 + \int_{\overline{A}_s} \mu(\mathbb{T}_{|z|})\, d\mu(z).$$

In particular,

$$\mu(\overline{A}_s)^2 \leq 2 \int_{\overline{A}_s} \mu(\overline{A}_{|z|})\, d\mu(z). \tag{9}$$

Let $\epsilon > 0$. Then there exists $s_0 > 0$ such that $s \geq s_0$ implies

$$\int_{\overline{A}_s} \frac{\mu(\overline{A}_{|z|})}{(1-|z|)^2}\, d\mu(z) < \epsilon.$$

It follows from (9) that

$$\begin{aligned} 2\epsilon &> 2 \int_{\overline{A}_s} \frac{\mu(\overline{A}_{|z|})}{(1-|z|)^2}\, d\mu(z) \\ &\geq \frac{2}{(1-s)^2} \int_{\overline{A}_s} \mu(\overline{A}_{|z|})\, d\mu(z) \geq \frac{\mu(\overline{A}_s)^2}{(1-s)^2} \geq \frac{\mu(S_{\theta,1-s})^2}{(1-s)^2} \end{aligned}$$

for every θ. This shows that μ is a vanishing Carleson measure. □

Theorem 3.6. *If a positive measure μ on $\mathbb{D}$ satisfies* (8)*, then $H(\mu) \in S_2$.*

Proof. Suppose that μ satisfies the above condition. Since

$$\begin{aligned} \|H(\mu)\|_{S^2} &= \sum_{i,j=0}^{\infty} |\hat{\mu}(i+j)|^2 \\ &\leq \sum_{i,j=0}^{\infty} \int_{\mathbb{D}} \int_{\mathbb{D}} (|z||w|)^{i+j}\, d\mu(z)\, d\mu(w) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{d\mu(z)\, d\mu(w)}{(1-|z||w|)^2}, \end{aligned}$$

it suffices to show that the last integral is finite. Observe that for any positive measurable function $f(z,w)$, we have

$$\begin{aligned} \int_{\mathbb{D}} \int_{\mathbb{D}_{|z|}} f(z,w)\, d\mu(w)\, d\mu(z) &= \int_{\mathbb{D}} \int_{\mathbb{D}} f(z,w) \chi_{\mathbb{D}_{|z|}}(w)\, d\mu(w)\, d\mu(z) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} f(z,w) \chi_{A_{|w|}}(z)\, d\mu(z)\, d\mu(w) \\ &= \int_{\mathbb{D}} \int_{A_{|w|}} f(z,w)\, d\mu(z)\, d\mu(w). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{d\mu(z)\, d\mu(w)}{(1-|z||w|)^2} &= \int_{\mathbb{D}} \int_{A_{|z|}} \frac{d\mu(z)\, d\mu(w)}{(1-|z||w|)^2} + \int_{\mathbb{D}} \int_{\overline{A}_{|z|}} \frac{d\mu(z)\, d\mu(w)}{(1-|z||w|)^2} \\ &\leq \int_{\mathbb{D}} \frac{A_{|z|}}{(1-|z|)^2} d\,\mu(z) + \int_{\mathbb{D}} \frac{\mu(\overline{A}_{|z|})}{(1-|z|)^2}\, d\mu(z) \\ &\leq 2 \cdot \int_{\mathbb{D}} \frac{\mu(\overline{A}_{|z|})}{(1-|z|)^2}\, d\mu(z) < \infty. \end{aligned}$$

□

Note that the converse is not true: If m_2 is a Lebesgue measure on $\mathbb{D}$, then $H(m_2)$ is of finite rank, but

$$\int_{\mathbb{D}} \frac{\mu(\overline{A}_{|z|})}{(1-|z|^2)}\, dm_2(z) = \int_0^{2\pi} \int_0^1 \frac{\pi(1-r^2)}{(1-r)^2} r\, dr d\theta = \infty.$$

References

[1] S. Axler, J. B. Conway, and G. McDonald, *Toeplitz operators on Bergman spaces*, Canadian J. Math. **34** (1982), no. 2, 466–483. https://doi.org/10.4153/CJM-1982-031-1

[2] A. Brown and R. G. Douglas, *Partially isometric Toeplitz operators*, Proc. Amer. Math. Soc. **16** (1965), 681–682. https://doi.org/10.2307/2033903

[3] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/64), 89–102. https://doi.org/10.1515/crll.1964.213.89

[4] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559. https://doi.org/10.2307/1970375

[5] C. Chatzifountas, D. Girela, and J. Peláez, *A generalized Hilbert matrix acting on Hardy spaces*, J. Math. Anal. Appl. **413** (2014), no. 1, 154–168. https://doi.org/10.1016/j.jmaa.2013.11.046

[6] R. E. Curto, I. S. Hwang, D. Kang, and W. Y. Lee, *Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols*, Adv. Math. **255** (2014), 562–585. https://doi.org/10.1016/j.aim.2014.01.008

[7] R. E. Curto, I. S. Hwang, and W. Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230** (2012), no. 4-6, 2094–2151. https://doi.org/10.1016/j.aim.2012.04.019

[8] R. E. Curto, I. S. Hwang, and W. Y. Lee, *Hyponormality of bounded-type Toeplitz operators*, Math. Nachr. **287** (2014), no. 11-12, 1207–1222. https://doi.org/10.1002/mana.201300200

[9] R. E. Curto, I. S. Hwang, and W. Y. Lee, *Matrix functions of bounded type: an interplay between function theory and operator theory*, Mem. Amer. Math. Soc. **260** (2019), no. 1253, v+100 pp. https://doi.org/10.1090/memo/1253

[10] J. J. Duistermaat and Y. J. Lee, *Toeplitz operators on the Dirichlet space*, J. Math. Anal. Appl. **300** (2004), no. 1, 54–67. https://doi.org/10.1016/j.jmaa.2004.05.031

[11] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970.

[12] P. Galanopoulos and J. Peláez, *A Hankel matrix acting on Hardy and Bergman spaces*, Studia Math. **200** (2010), no. 3, 201–220. https://doi.org/10.4064/sm200-3-1

[13] J. B. Garnett, *Bounded Analytic Functions*, revised first edition, Graduate Texts in Mathematics, 236, Springer, New York, 2007. https://doi.org/10.1007/0-387-49763-3

[14] P. Hartman and A. Wintner, *The spectra of Toeplitz's matrices*, Amer. J. Math. **76** (1954), 867–882. https://doi.org/10.2307/2372661

[15] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[16] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.

[17] J. Park, *Toeplitz operators whose symbols are Borel measures*, J. Funct. Space **2021** (2021), Art. ID 5599823, 11pp. https://doi.org/10.1155/2021/5599823

[18] V. V. Peller, *Hankel Operators and Their Applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003. https://doi.org/10.1007/978-0-387-21681-2

[19] R. Rochberg and Z. J. Wu, *Toeplitz operators on Dirichlet spaces*, Integral Equations Operator Theory **15** (1992), no. 2, 325–342. https://doi.org/10.1007/BF01204241

[20] M. Rosenblum, *Self-adjoint Toeplitz operators and associated orthonormal functions*, Proc. Amer. Math. Soc. **13** (1962), 590–595. https://doi.org/10.2307/2034831

[21] K. Stroethoff, *Hankel and Toeplitz operators on the Fock space*, Michigan Math. J. **39** (1992), no. 1, 3–16. https://doi.org/10.1307/mmj/1029004449

[22] O. Toeplitz, *Zur Theorie der quadratische Formen von unendlichvielen Veräinderlichen*, Göttinger Nachrichten (1910), 489–506.

[23] O. Toeplitz, *Über die Fourier'sche Entwicklung positiver Funktionen*, Rend. Circ. Mat. Palermo **32** (1911), 191–192.

[24] H. Widom, *On the spectrum of a Toeplitz operator*, Pacific J. Math. **14** (1964), 365–375. https://projecteuclid.org/euclid.pjm/1103034389

[25] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121** (1966), 1–35. https://doi.org/10.2307/1994330

[26] D. R. Yafaev, *Toeplitz versus Hankel*: *semibounded operators*, Opuscula Math. **38** (2018), no. 4, 573–590. https://doi.org/10.7494/opmath.2018.38.4.573

Jaehui Park
Research Institute of Mathematics
Seoul National University
Seoul 08826, Korea
Email address: nephenjia@snu.ac.kr