ON A CLASS OF ANALYTIC FUNCTION RELATED TO SCHWARZ LEMMA

BÜLENT NAFLI ÖRNEK

Abstract. In this paper, we plan to introduce the class of the analytic functions called $P(b)$ and to investigate the various properties of the functions belonging this class. The modulus of the second coefficient $c_2$ in the expansion of $f(z) = z + c_2z^2 + ...$ belonging to the given class will be estimated from above. Also, we estimate a modulus of the second angular derivative of $f(z)$ function at the boundary point $\alpha$ with $f'(\alpha) = 1 - b$, $b \in \mathbb{C}$, by taking into account their first nonzero two Maclaurin coefficients.

1. Introduction

Let $k$ be an analytic function in the unit disc $U = \{z : |z| < 1\}$, $k(0) = 0$ and $k : U \to U$. In accordance with the classical Schwarz lemma, for any point $z$ in the unit disc $U$, we have $|k(z)| \leq |z|$ for all $z \in U$ and $|k'(0)| \leq 1$. In addition, if the equality $|k(z)| = |z|$ holds for any $z \neq 0$, or $|k'(0)| = 1$, then $k$ is a rotation; that is $k(z) = ze^{i\theta}$, $\theta$ real ([5], p.329). Schwarz lemma has important applications in engineering [12, 13]. In this study, the Shwarz lemma will be presented for the following class $P(b)$ which will be given.

Let $\mathcal{A}$ denote the class of functions $f(z) = z + \sum_{p=2}^{\infty} c_pz^p$ that are analytic in $U$. Also, let $\mathcal{P}(b)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\sum_{p=2}^{\infty} p|c_p| \leq |b|,$$

where $0 \neq b \in \mathbb{C}$. In this paper, we study some of the properties of the classes $\mathcal{P}(b)$. Namely, the modulus of the second coefficient $c_2$ in the expansion of $f(z) = z + c_2z^2 + ...$ belonging to the given class will be estimated from above. That is, we obtain the bound of $c_2$. 

Received by the editors November 28, 2021. Accepted January 17, 2022.

2010 Mathematics Subject Classification. 30C80.

Key words and phrases. analytic function, Schwarz lemma.
Let \( f \in \mathcal{P}(b) \) and consider the following function

\[
\varphi(z) = \frac{f'(z) - 1}{f'(z) + 2b - 1}.
\]

It is an analytic function in \( U \) and \( \varphi(0) = 0 \). Now, let us show that \(|\varphi(z)| < 1\) in \( U \). Now let us check the difference of the modules of the numerator and denominator of the function \( \varphi(z) \) given in (1.2). Therefore, we take

\[
|f'(z) - 1| - |f'(z) + 2b - 1| = \left| \sum_{p=2}^{\infty} p|c_p|z^{p-1} \right| - \left| 2b + \sum_{p=2}^{\infty} p|c_p|z^{p-1} \right|
\]

\[
\leq \sum_{p=2}^{\infty} p|c_p||z|^{p-1} - 2|b| + \sum_{p=2}^{\infty} p|c_p| |z|^{p-1} < 2 \sum_{p=2}^{\infty} p|c_p| - 2|b|.
\]

Since \( \sum_{p=2}^{\infty} p|c_p| \leq |b| \), we obtain

\[
|f'(z) - 1| - |f'(z) + 2b - 1| \leq 0
\]

and

\[
\left| \frac{f'(z) - 1}{f'(z) + 2b - 1} \right| < 1.
\]

Therefore, from the Schwarz lemma, we obtain

\[
\varphi(z) = \frac{f'(z) - 1}{f'(z) + 2b - 1} = \frac{2c_2z + 3c_3z^2 + \ldots}{2c_2z + 3c_3z^2 + \ldots + 2b}
\]

\[
\varphi(z) = \frac{2c_2 + 3c_3z + \ldots}{2c_2z + 3c_3z^2 + \ldots + 2b}
\]

\[
|\varphi'(0)| = \left| \frac{c_2}{b} \right| \leq 1
\]

and

\[ |c_2| \leq |b| \cdot \]

We thus obtain the following lemma.

**Lemma 1.1.** If \( f \in \mathcal{P}(b) \), then we have the inequality

\[
|c_2| \leq |b|.
\]
Since \( r(z) \) function satisfies the conditions of the Schwarz lemma, we obtain

\[
\begin{align*}
   r(z) &= \frac{f'(z) - 1}{f'(z) + 2b - 1}\frac{1}{\prod_{i=1}^{n} \frac{z-s_i}{1-s_i}} \\
   &= \frac{2c_2z + 3c_3z^2 + ...}{2c_2z + 3c_3z^2 + ... + 2b\prod_{i=1}^{n} \frac{z-s_i}{1-s_i}}.
\end{align*}
\]

We thus obtain the following lemma.

**Lemma 1.2.** Let \( f \in \mathcal{P}(b) \) and \( s_1, s_2, ..., s_n \) be critical points of the function \( f(z) - z \) in \( U \) that are different from zero. Then we have the inequality

\[
|r'(0)| = \frac{|c_2|}{|b|\prod_{i=1}^{n} |s_i|} \leq 1
\]

and

\[
|c_2| \leq |b|\prod_{i=1}^{n} |s_i|.
\]

We thus obtain the following lemma.

**Lemma 1.3.** If \( k(z) \) extends continuously to some boundary point \( \alpha \in \partial U = \{z : |z| = 1\} \) with \( |\alpha| = 1 \), and if \( |k(\alpha)| = 1 \) and \( k'(\alpha) \) exists, then

\[
|k'(\alpha)| \geq \frac{2}{1 + |k'(0)|}
\]

and

\[
|k'(\alpha)| \geq 1.
\]

Moreover, the equality in (1.4) holds if and only if \( k(z) = z \frac{z^n + a}{1 + az} \) for some \( a \in (-1, 0] \).

Also, the equality in (1.3) holds if and only if \( k(z) = ze^{i\theta} \).
Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 6, 7, 8, 9, 10, 11].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [14]).

**Lemma 1.4 (Julia-Wolff lemma).** Let $k$ be an analytic function in $U$, $k(0) = 0$ and $k(U) \subset U$. If, in addition, the function $k$ has an angular limit $k(\alpha)$ at $\alpha \in \partial U$, $|k(\alpha)| = 1$, then the angular derivative $k'(\alpha)$ exists and $1 \leq |k'(\alpha)| \leq \infty$.

**Corollary 1.5.** The analytic function $k$ has a finite angular derivative $k'(\alpha)$ if and only if $k'$ has the finite angular limit $k'(\alpha)$ at $\alpha \in \partial U$.

## 2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{P}(b)$ class. Also, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

**Theorem 2.1.** Let $f \in \mathcal{P}(b)$. Assume that, for some $\alpha \in \partial U$, $f$ has an angular limit $f(\alpha)$ at the points $\alpha$, $f'(\alpha) = 1 - b$. Then we have the inequality

\begin{equation}
|f''(\alpha)| \geq \frac{|b|}{2}.
\end{equation}

**Proof.** Consider the function

$$
\varphi(z) = \frac{f'(z) - 1}{f'(z) + 2b - 1}.
$$

Also, since $f'(\alpha) = 1 - b$, we have $|\varphi(\alpha)| = 1$. Therefore, from (1.5), we obtain

$$
1 \leq |\varphi'(\alpha)| = \frac{2|f''(\alpha)||b|}{|b|^2} = \frac{2|f''(\alpha)|}{|b|}
$$

and

$$
|f''(\alpha)| \geq \frac{|b|}{2}.
$$

The inequality (2.1) can be strengthened from below by taking into account, $c_2 = \frac{f''(0)}{2}$, the first coefficient of the expansion of the function $f(z) = z + c_2 z^2 + c_3 z^3 + \ldots$. 
Theorem 2.2. Under the same assumptions as in Theorem 2.1, we have

\[ |f''(\alpha)| \geq \frac{4|b|^2}{2|b| + |f''(0)|}. \tag{2.2} \]

Proof. Let \( w(z) \) function be the same as (1.2). So, from (1.4), we obtain

\[ \frac{2}{1 + |\varphi'(0)|} \leq |\varphi'(\alpha)| = \frac{2|f''(\alpha)|}{|b|}. \]

Since

\[ |\varphi'(0)| = \left| \frac{c_2}{b} \right|, \]

we take

\[ \frac{2}{1 + \left| \frac{c_2}{b} \right|} \geq \frac{2|f''(\alpha)|}{|b|}, \]

\[ |f''(\alpha)| \geq \frac{2|b|^2}{|b| + \left| \frac{f''(0)}{2} \right|}, \]

and

\[ |f''(\alpha)| \geq \frac{4|b|^2}{2|b| + |f''(0)|}. \]

\[ \square \]

The inequality (2.2) can be strengthened as below by taking into account \( c_3 = \frac{f'''(0)}{3!} \) which is the coefficient in the expansion of the function \( f(z) = z + c_2 z^2 + c_3 z^3 + \ldots \).

Theorem 2.3. Let \( f \in \mathcal{P}(b) \). Assume that, for some \( \alpha \in \partial U \), \( f \) has an angular limit \( f(\alpha) \) at the points \( \alpha \), \( f'(\alpha) = 1 - b \). Then we have the inequality

\[ |f''(\alpha)| \geq \frac{|b|}{2} \left( 1 + \frac{4(|b| - |c_2|)^2}{2(|b|^2 - |c_2|^2) + |3bc_3 - 2c_3^2|} \right). \tag{2.3} \]

Proof. Let \( \varphi(z) \) be the same as in the proof of Theorem 2.1 and \( h(z) = z \). By the maximum principle, for each \( z \in U \), we have the inequality \( |\varphi(z)| \leq |h(z)| \). So,

\[
\begin{align*}
v(z) &= \frac{\varphi(z)}{h(z)} = \frac{1}{z} \left( \frac{f'(z) - 1}{f'(z) + 2b - 1} \right) \\
&= \frac{1}{z} \frac{2c_2 z + 3c_3 z^2 + \ldots}{2c_2 z + 3c_3 z^2 + \ldots + 2b} \\
&= \frac{2c_2 + 3c_3 z + \ldots}{2c_2 z + 3c_3 z^2 + \ldots + 2b}
\end{align*}
\]
is analytic function in $U$ and $|v(z)| \leq 1$ for $z \in U$. In particular, we have

\begin{equation}
|v(0)| = \frac{|c_2|}{|b|} \leq 1
\end{equation}

and

\begin{equation}
|v'(0)| = \frac{|3bc_3 - 2c_2^2|}{2|b|^2}.
\end{equation}

The auxiliary function

\[ t(z) = \frac{v(z) - v(0)}{1 - v(0)v(z)} \]

is analytic in $U$, $t(0) = 0$, $|t(z)| < 1$ for $|z| < 1$ and $|t(\alpha)| = 1$ for $\alpha \in \partial U$. From (1.4), we obtain

\[ \frac{2}{1 + |t'(0)|} \leq |t'(\alpha)| = \frac{1 - |v(0)|^2}{|1 - v(0)v(\alpha)|^2} |v'(\alpha)| \]

\[ \leq \frac{1 + |v(0)|}{1 - |v(0)|} \left\{ |v'(\alpha)| - |b'(\alpha)| \right\} \]

\[ = |b| + |c_2| \left( \frac{2|f''(\alpha)|}{|b|} - 1 \right). \]

Since

\[ t'(z) = \frac{1 - |v(0)|^2}{(1 - v(0)v(z))^2} v'(z) \]

and

\[ |t'(0)| = \frac{|v'(0)|}{1 - |v(0)|^2} = \frac{|3bc_3 - 2c_2^2|}{2|b|^2} = \frac{|3bc_3 - 2c_2^2|}{2(|b|^2 - |c_2|^2)}, \]

we obtain

\[ \frac{2}{1 + \frac{|3bc_3 - 2c_2^2|}{2(|b|^2 - |c_2|^2)}} \leq \frac{|b| + |c_2|}{|b| - |c_2|} \left( \frac{2|f''(\alpha)|}{|b|} - 1 \right), \]

\[ \frac{4 \left(|b|^2 - |c_2|^2\right)}{2 \left(|b|^2 - |c_2|^2\right) + |3bc_3 - 2c_2^2|} \frac{|b| - |c_2|}{|b|} \leq \frac{2|f''(\alpha)|}{|b|} - 1 \]

and

\[ |f''(\alpha)| \geq \frac{|b|}{2} \left( 1 + \frac{4 (|b| - |c_2|)^2}{2 (|b|^2 - |c_2|^2) + |3bc_3 - 2c_2^2|} \right). \]

\[\square\]
If \( f(z) - z \) have critical points different from \( z = 0 \), taking into account these critical points, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

**Theorem 2.4.** Let \( f \in P(b) \) and \( s_1, s_2, ..., s_n \) be critical points of the function \( f(z) - z \) in \( D \) that are different from zero. Assume that, for some \( \alpha \in \partial U \), \( f \) has an angular limit \( f(\alpha) \) at the points \( \alpha \), \( f'(\alpha) = 1 - b \). Then we have the inequality

\[
(2.5) \ |f''(\alpha)| \geq \frac{|b|}{2} \left( 1 + \sum_{i=1}^{n} \frac{1 - |s_i|^2}{|1 - s_i|^2} \right) + \frac{4 \left( |b| \prod_{i=1}^{n} |s_i| - |c_2| \right)^2}{2 \left( |b| \prod_{i=1}^{n} |s_i|^2 - |c_2|^2 \right) + \prod_{i=1}^{n} |s_i| \left( 3bc_3 - 2c_2^2 + 2bc_2 \sum_{i=1}^{n} \frac{1 - |s_i|^2}{s_i} \right)}.
\]

**Proof.** Let \( \varphi(z) \) be as in (1.2) and \( s_1, s_2, ..., s_n \) be critical points of the function \( f(z) - z \) in \( U \) that are different from zero. Also, consider the function \( B(z) = z \prod_{i=1}^{n} \frac{z - s_i}{1 - \overline{s_i}z} \).

By the maximum principle for each \( z \in U \), we have

\[
|\varphi(z)| \leq |B(z)|.
\]

Consider the function

\[
n(z) = \frac{\varphi(z)}{B(z)} = \left( \frac{f'(z) - 1}{f'(z) + 2b - 1} \right) \frac{1}{z \prod_{i=1}^{n} \frac{z - s_i}{1 - \overline{s_i}z}} = \frac{2c_2z + 3c_3z^2 + ...}{2c_2z + 3c_3z^2 + ... + 2b \prod_{i=1}^{n} \frac{z - s_i}{1 - \overline{s_i}z}} = \frac{2c_2 + 3c_3z + ...}{2c_2z + 3c_3z^2 + ... + 2b \prod_{i=1}^{n} \frac{z - s_i}{1 - \overline{s_i}z}}.
\]

\( n(z) \) is analytic in \( U \) and \( |n(z)| < 1 \) for \( |z| < 1 \). In particular, we have

\[
|n(0)| = \frac{|c_2|}{|b| \prod_{i=1}^{n} |s_i|}.
\]
and

\[ |n'(0)| = \frac{|3bc_3 - 2c_2^2 + 2bc_3 \sum_{i=1}^{n} \frac{1-|s_i|^2}{s_i}|}{2|b|^2 \prod_{i=1}^{n} |s_i|}. \]

The auxiliary function

\[ g(z) = \frac{n(z) - n(0)}{1 - n(0)n(z)} \]

is analytic in \( U \), \(|g(z)| < 1 \) for \(|z| < 1 \) and \( g(0) = 0 \). For \( \alpha \in \partial U \) and \( f'(\alpha) = 1 - b \) we take \(|g(\alpha)| = 1 \).

From (1.4), we obtain

\[
\frac{2}{1 + |g'(0)|} \leq |g'(\alpha)| = \frac{1 - |n(0)|^2}{|1 - n(0)n(\alpha)|} |n'(\alpha)|
\]

\[
\leq \frac{1 + |n(0)|}{1 - |n(0)|} (|g'(\alpha)| - |B'(\alpha)|).
\]

It can be seen that

\[ |g'(0)| = \frac{|n'(0)|}{1 - |n(0)|^2} \]

and

\[
|g'(0)| = \frac{|3bc_3 - 2c_2^2 + 2bc_3 \sum_{i=1}^{n} \frac{1-|s_i|^2}{s_i}|}{2|b|^2 \prod_{i=1}^{n} |s_i|}
\]

\[
= \frac{|3bc_3 - 2c_2^2 + 2bc_3 \sum_{i=1}^{n} \frac{1-|s_i|^2}{s_i}|}{2 \left( \frac{|b| \prod_{i=1}^{n} |s_i|}{|c_2|^2} \right)^2}
\]

Also, we have

\[ |B'(\alpha)| = 1 + \sum_{i=1}^{n} \frac{1 - |s_i|^2}{|1 - s_i|^2}, \quad \alpha \in \partial U. \]
Therefore, we obtain

\[
1 + \frac{1}{n} \prod_{i=1}^{n} |s_i| \left( 2 \left( \frac{3bc_3 - 2c_2^2 + 2bc_2 \sum_{i=1}^{n} \frac{1-|s_i|^2}{s_i}}{|b| \prod_{i=1}^{n} |s_i| - |c_2|} \right) \right)
\]

\[
\leq \frac{|b| \prod_{i=1}^{n} |s_i| + |c_2|}{|b| \prod_{i=1}^{n} |s_i| - |c_2|} \left( \frac{2 |f''(\alpha)|}{|b|} - 1 - \sum_{i=1}^{n} \frac{1 - |s_i|^2}{|1 - s_i|^2} \right),
\]

\[
4 \left( \left( \frac{|b| \prod_{i=1}^{n} |s_i|}{|b| \prod_{i=1}^{n} |s_i| - |c_2|} \right)^2 - |c_2|^2 \right)
\]

\[
2 \left( \left( \frac{|b| \prod_{i=1}^{n} |s_i|}{|b| \prod_{i=1}^{n} |s_i| - |c_2|} \right)^2 - |c_2|^2 \right) + \prod_{i=1}^{n} |s_i| \left( 3bc_3 - 2c_2^2 + 2bc_2 \sum_{i=1}^{n} \frac{1-|s_i|^2}{s_i} \right)
\]

\[
\leq \frac{|b| \prod_{i=1}^{n} |s_i| + |c_2|}{|b| \prod_{i=1}^{n} |s_i| - |c_2|} \left( \frac{2 |f''(\alpha)|}{|b|} - 1 - \sum_{i=1}^{n} \frac{1 - |s_i|^2}{|1 - s_i|^2} \right),
\]

\[
4 \left( \left( \frac{|b| \prod_{i=1}^{n} |s_i|}{|b| \prod_{i=1}^{n} |s_i| - |c_2|} \right)^2 - |c_2|^2 \right)
\]

and so, we get inequality (2.5).

If \( f(z) - z \) has no critical points different from \( z = 0 \) in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

**Theorem 2.5.** Let \( f \in \mathcal{P}(b) \), \( f(z) - z \) has no critical points in \( U \) except \( z = 0 \) and \( c_2 > 0 \). Assume that, for some \( \alpha \in \partial U \), \( f \) has an angular limit \( f(\alpha) \) at the points \( \alpha \), \( f'(\alpha) = 1 - b \). Then we have the inequality

\[
|f''(\alpha)| \geq \frac{|b|}{2} \left( 1 - \frac{4c_2 |b| c_2 \ln \left( \frac{c_2}{|b|} \right)}{4c_2 |b| \ln \left( \frac{c_2}{|b|} \right) - \left( 3bc_3 - 2c_2^3 \right)} \right).
\]
Proof. Let \( c_2 > 0 \) in the expression of the function \( f(z) \). Having in mind the inequality (2.4) and the function \( f(z) - z \) has no critical points in \( U \) except \( z = 0 \), we denote by \( \ln v(z) \) the analytic branch of the logarithm normed by the condition

\[
\ln v(0) = \ln \left( \frac{c_2}{|b|} \right) < 0.
\]

The auxiliary function

\[
d(z) = \frac{\ln v(z) - \ln v(0)}{\ln v(z) + \ln v(0)}
\]

is analytic in the unit disc \( U \), \( |d(z)| < 1 \), \( d(0) = 0 \) and \( |d(\alpha)| = 1 \) for \( \alpha \in \partial U \).

From (1.4), we obtain

\[
\frac{2}{1 + |d'(0)|} \leq |d'(\alpha)| = \left| \frac{2 \ln v(0)}{\ln v(\alpha) + \ln v(0)} \right| \left| \frac{\varphi'(\alpha)}{v(\alpha)} \right| = \frac{-2 \ln v(0)}{\ln^2 v(0) + \arg^2 v(\alpha)} \left( |\varphi'(\alpha)| - 1 \right).
\]

Replacing \( \arg^2 v(\alpha) \) by zero, then

\[
1 - \frac{1}{1 - \frac{|3bc_3 - 2c_2^2|}{2b^2 c_2 \ln \left( \frac{c_2}{|b|} \right)}} \leq -1 \left\{ \frac{2 |f''(\alpha)|}{|b|} - 1 \right\}
\]

and

\[
1 - \frac{4 |b| c_2 \ln^2 \left( \frac{c_2}{|b|} \right)}{4 |b| c_2 \ln \left( \frac{c_2}{|b|} \right) - 3bc_3 - 2c_2^2} \leq \frac{2 |f''(\alpha)|}{|b|}.
\]

Thus, we obtain the inequality (2.6). \( \square \)

The following theorem shows the relationship between the coefficients \( c_2 \) and \( c_3 \) in the Maclaurin expansion of the \( f(z) = z + c_2 z^2 + c_3 z^3 + \cdots \) function.

**Theorem 2.6.** Let \( f \in \mathcal{P}(b) \), \( f(z) - z \) has no critical points in \( U \) except \( z = 0 \) and \( c_2 > 0 \). Then we have the inequality

\[
|3bc_3 - 2c_2^2| \leq 4 \left| \frac{bc_2 \ln \left( \frac{c_2}{|b|} \right)}{|b|} \right|.
\]

**Proof.** Let \( d(z) \) be the same as in the proof of Theorem 5. Here, \( d(z) \) is analytic in the unit disc \( U \), \( |d(z)| < 1 \), \( d(0) = 0 \). Therefore, the function \( d(z) \) satisfies the
assumptions of the Schwarz Lemma. Thus, we obtain

\[ 1 \geq |d'(0)| = \frac{|2 \ln v(0)|}{|\ln v(0) + \ln v(0)|^2} \left| \frac{v'(0)}{v(0)} \right| \\
= \frac{-1}{2 \ln v(0)} \left| \frac{v'(0)}{v(0)} \right| \left| \frac{3bc_3 - 2c_2^2}{2bc_3} \right| \\
= -\frac{2c_2}{|b|^2} \ln \left( \frac{c_2}{|b|} \right) \]

and

\[ |3bc_3 - 2c_2^2| \leq 4 \left| bc_2 \ln \left( \frac{c_2}{|b|} \right) \right|. \]

\[ \square \]

REFERENCES


