THE QUADRATIC HYPONORMALITY OF ONE-STEP EXTENSION OF THE BERGMAN-TYPE SHIFT

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ABSTRACT. Let $p > 1$ and $\alpha^{[p]}(x) = \sqrt{x}, \sqrt{\frac{p-1}{p^2-1}}, \sqrt{\frac{p-2}{p^2-2}}, \ldots$, with $0 < x \leq \frac{p}{p^2-1}$. In [10], the authors considered the subnormality, $n$-hyponormality and positive quadratic hyponormality of $W_{\alpha^{[p]}(x)}$. By continuing to study, in this paper, we give a sufficient condition of quadratic hyponormality of $W_{\alpha^{[p]}(x)}$. Finally, we give an example to characterize the gaps of $W_{\alpha^{[p]}(x)}$ distinctively.

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1. Introduction

Let $T$ be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We recall some basic definitions of some classes of operators. We say that $T$ is normal if $T^*T = TT^*$; hyponormal if $T^*T \geq TT^*$, and subnormal if $T$ has a normal extension. For $S, T \in B(\mathcal{H})$, let $[S, T] := ST - TS$. We say that an $n$-tuple $T = (T_1, \ldots, T_n)$ of bounded linear operators on $B(\mathcal{H})$ is hyponormal if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of $n$ copies of $\mathcal{H}$. For any $k \in \mathbb{N}$, we say $T \in B(\mathcal{H})$ is (strongly) $k$-hyponormal if $(I, T, \ldots, T^k)$ is hyponormal. It is well-known that $T$ is subnormal if and only if $T$ is $k$-hyponormal for all $k \in \mathbb{N}$. An operator $T$ in $B(\mathcal{H})$ is said to be weakly $n$-hyponormal if $p(T)$ is hyponormal for any polynomial $p$ with degree less than or equal to $n$. And an operator $T$ is polynomially hyponormal if $p(T)$ is hyponormal for every polynomial $p$. In particular, the quadratical hyponormality (i.e. weak 2-hyponormality) of weight shift has been considered in detail in [1], [2], [4] and [7].
Recall that let \( \alpha := \{\alpha_n\}_{n=0}^{\infty} \) be a bounded sequence in the set \( \mathbb{R}_+ \). The (uni-
lateral) weighted shift \( W_\alpha \) acting on \( \ell^2(\mathbb{N}_0) \), with an orthonormal basis \( \{e_i\}_{i=0}^{\infty} \),
is defined by \( W_\alpha e_n := \alpha_n e_{n+1} \) for all \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). It follows straight-
forward that \( W_\alpha \) is hyponormal if and only if the weight sequence \( \{\alpha_n\}_{n=0}^{\infty} \) is
non-decreasing.

If a weight sequence \( \alpha = \{\alpha_n\}_{n=0}^{\infty} \) is given by \( \alpha_n = \sqrt{\frac{n+1}{n+2}} (n \geq 0) \), then
the corresponding weighted shift is called the Bergman shift. Let \( x > 0 \) and
\( \alpha(x) : \alpha_0 = \sqrt{x}, \alpha_n = \sqrt{\frac{n+2}{n+3}} (n \geq 1) \). The k-hyponormality, subnormality and
quadratic hyponormality of \( W_\alpha(x) \) were considered in detail in [3], [4], [5], [6], [7]
and [9] etc. In [8], the authors considered the backward extension of Bergman-
type shift \( \alpha^{[p]}(x) = \sqrt{x}, \sqrt{\frac{p}{2}}, \sqrt{\frac{p-1}{3p-2}}, \ldots, \) with \( p > 1 \). Furthermore, let
\( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p > 1 \) and \( \alpha^{[m,p]}(x) = \sqrt{x}, \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \sqrt{\frac{m+2}{m+3}}, \ldots \).
By continuing to study, in this paper, we give a sufficient condition of quadratic
hyponormality of \( W_{\alpha^{[m,p]}(x)} \), which extends all the results on Bergman
weighted shift \( W_\alpha(x) \) with \( m \in \mathbb{N} \), and \( \alpha(x) = \sqrt{x}, \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \sqrt{\frac{m+2}{m+3}}, \ldots \).

All of the calculations in this paper were taken by using the software Scientific

2. Preliminaries and Notations

We know that a weighted shift \( W_\alpha \) is quadratically hyponormal if \( W_\alpha + sW_\alpha^2 \)
is hyponormal for arbitrary complex number \( s([7]) \), that is,
\[
M(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2] \geq 0
\]
for arbitrary complex number \( s \). We let \( \{e_i\}_{i=0}^{\infty} \) be an orthonormal basis for
\( \ell^2(\mathbb{N}_0) \) and
\[
M_n(s) := P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n,
\]
where \( P_n \) is the orthogonal projection onto the subspace generated by \( \{e_i\}_{i=0}^{n} \).
Then \( M_n(s) \) has the following form
\[
M_n(s) = \begin{pmatrix}
\rho_0 & \kappa_0 & 0 & \cdots & 0 & 0 \\
\gamma_0 & \rho_1 & \kappa_1 & 0 & \cdots & 0 \\
0 & \gamma_1 & \rho_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \rho_{n-1} & \kappa_{n-1} \\
0 & 0 & 0 & \cdots & \kappa_{n-1} & \rho_n
\end{pmatrix},
\]
where

\[
\begin{align*}
\rho_n & := \sigma_n + |s|^2 \delta_n, \\
\kappa_n & := s \sqrt{\theta_n}, \\
\sigma_n & := \alpha_n^2 - \alpha_{n-1}^2, \\
\delta_n & := \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_n^2, \\
\phi_n & := \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2),
\end{align*}
\]

for any nonnegative integer \( n \) and \( \alpha_n := 0 \) for negative integer \( n \).

Hence, \( W \) is quadratically hyponormal if and only if \( M_n(s) \geq 0 \) for arbitrary complex number \( s \) and \( n \in \mathbb{N}_0 \). Let \( t := |s|^2 \) and \( d_n(t) := \det M_n(t) \) which is a polynomial in \( t \) of degree \( n + 1 \), with Maclaurin expansion \( d_n(t) := \sum_{k=0}^{n+1} \theta_{n,k} t^k \).

It is easy to find that \( d_n(t) \) satisfies

\[
\begin{align*}
d_0(t) &= \rho_0, \\
d_1(t) &= \rho_0 \rho_1 - |\kappa_0|^2, \\
d_{n+2}(t) &= \rho_{n+2} d_{n+1}(t) - |\kappa_{n+1}|^2 d_n(t), \quad (n \geq 0). 
\end{align*}
\]

Also we can get the followings

\[
\begin{align*}
\theta_{n,0} &= \sigma_0 \cdots \sigma_n, \\
\theta_{n,n+1} &= \delta_0 \cdots \delta_n, \\
\theta_{1,1} &= \sigma_1 \delta_0 + \sigma_0 \delta_1 - \phi_0, \\
\theta_{n+2,k} &= \sigma_{n+2} \theta_{n+1,k} + \delta_{n+2} \theta_{n+1,k-1} - \phi_{n+1} \theta_{n,k-1}, 
\end{align*}
\]

for \( n \geq 0 \) and \( k \geq 1 \).

**Lemma 1.** \( \theta_{n,1} = \sigma_0 \cdots \sigma_{n-2} \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2) \geq 0 \), for all \( n \geq 1 \).

### 3. Key Lemmas

In this section, we consider an one-step extension \( W_{\alpha^{[p]}(x)} \) of the Bergman-type shift, where

\[
\alpha^{[p]}(x) : \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \ldots
\]

where \( p > 1 \) and \( 0 < x \leq \frac{p}{2p-1} \). We have \( \theta_{n,k} \geq 0 \) for all \( 0 \leq n \leq 4 \) and \( 0 \leq k \leq 4 \) with \( 0 \leq k \leq n + 1 \) except for \( \theta_{4,3} \).

\[
\begin{align*}
\theta_{0,0} &= x > 0, \\
\theta_{0,1} &= \frac{p}{2p-1} x > 0, \\
\theta_{1,0} &= x \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{1,1} &= \frac{xp}{2p-1} \left( \frac{3p-2}{4p-3} - x \right) > 0, \\
\theta_{1,2} &= \frac{x^2}{(3p-2)(4p-3)} > 0,
\end{align*}
\]
Lemma 3. Let \( \theta_{n,k} \geq 0 \) for all \( n \geq 4, k \geq 4 \).

\[
\begin{align*}
\theta_{2,0} &= \frac{(p-1)^2}{(3p-2)(2p-1)} x \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{2,1} &= \frac{2(p-1)^2}{(4p-3)(3p-2)} x \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{2,2} &= xp (p - 1)^2 (4p - 3) (3p - 2)(2p - 1) > 0, \\
\theta_{2,3} &= xp (p - 1)^2 \left( \frac{4p^2 - 3p}{2p - 1} \right) > 0, \\
\theta_{3,0} &= \frac{(4p-3)(3p-2)(2p-1)}{x(p-1)^4} \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{3,1} &= \frac{2(p-1)^4}{(5p-4)(4p-3)(3p-2)(2p-1)} x \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{3,2} &= \frac{p(4p-3)(3p-2)(2p-1)}{(5p-4)(4p-3)(3p-2)(2p-1)} > 0, \\
\theta_{3,3} &= \frac{(4p-3)(3p-2)(2p-1)^2}{(5p-4)(4p-3)(3p-2)(2p-1)} x \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{3,4} &= \frac{(4p-3)(3p-2)(2p-1)^2}{(5p-4)(4p-3)(3p-2)(2p-1)} > 0, \\
\theta_{4,0} &= \frac{(5p-4)(3p-2)(2p-1)}{2x(p-1)^6} \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{4,1} &= \frac{2(p-1)^6}{(6p-5)(5p-4)(4p-3)(3p-2)(2p-1)} x \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{4,2} &= \frac{(6p-5)(5p-4)(4p-3)(3p-2)(2p-1)}{(5p-4)(4p-3)(3p-2)(2p-1)} \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{4,3} &= \frac{4(p-1)^6x}{(6p-5)(5p-4)(4p-3)(3p-2)(2p-1)} \left( \frac{p}{2p-1} - x \right) \geq 0, \\
\theta_{4,4} &= \frac{16xp^2(p-1)^6}{(6p-5)(5p-4)(4p-3)(3p-2)(2p-1)} \left( \frac{p}{2p-1} - x \right) \geq 0.
\end{align*}
\]

Considering the \( W_\alpha^{[p]}(x) \), we can obtain the following lemmas.

Lemma 2. Let \( \alpha^{[p]}(x) \) be as in (2). Then \( \theta_{n,2} \geq 0 \) for all \( n \geq 1 \).

Proof. For \( n \geq 2 \), by (1) we have

\[
\begin{align*}
\delta_{n+2} \theta_{n+1,1} - \phi_{n+1} \theta_{n,1} &= \delta_{n+2} \sigma_0 \cdots \sigma_n \alpha_n^2 (\alpha_n^2 - \alpha_n^2) - \phi_{n+1} \sigma_0 \cdots \sigma_{n-1} \alpha_n^2 (\alpha_n^2 - \alpha_n^2) \\
&= \sigma_0 \cdots \sigma_n (\delta_{n+2} \sigma_0 \cdots \sigma_n \alpha_n^2 (\alpha_n^2 - \alpha_n^2) - \phi_{n+1} \sigma_0 \cdots \sigma_n \alpha_n^2 (\alpha_n^2 - \alpha_n^2)) \\
&= \frac{24 (p-1)^6 \sigma_0 \cdots \sigma_n}{(\Delta + 4p - 3)(\Delta + 2p - 1)^2 (\Delta + p)^2 (\Delta + 3p - 2)^2} \geq 0,
\end{align*}
\]

with \( \Delta = n(p - 1) \). It follows that if \( \theta_{n+1,2} \geq 0 \), then for \( n \geq 2 \),

\[
\theta_{n+2,2} = \upsilon_{n+2} \theta_{n+1,2} + \delta_{n+2} \theta_{n+1,1} - \phi_{n+1} \theta_{n,1} \geq 0.
\]

Since \( \theta_{n,2} \geq 0 \) for \( n = 1, 2, 3 \) with \( 0 < x \leq \frac{p}{2p-1} \) and \( p > 1 \), we can get \( \theta_{n,2} \geq 0 \) for all \( n \geq 1 \). \( \square \)

Lemma 3. Let \( \alpha^{[p]}(x) \) be as in (2). Then \( \theta_{n,k} = \delta_n \theta_{n-1,k-1} \) for all \( n \geq 4, k \geq 4 \).
Proof. Clearly, $\sigma_{n+1} \delta_n = \phi_n$ ([10], Lemma 5.1), for all $n \geq 3$. So for all $n \geq 4$, it is simple that

$$\theta_{n,k} = \sigma_n \theta_{n-1,k} + \delta_n \theta_{n-1,k-1} - \phi_{n-1} \theta_{n-2,k-1}$$

$$= \delta_n \theta_{n-1,k-1} - \phi_{n-1} \theta_{n-2,k-1} + \sigma_n \left[ \delta_n \theta_{n-2,k} + \delta_{n-1} \theta_{n-2,k-1} - \phi_{n-2} \theta_{n-3,k-1} \right]$$

$$= \delta_n \theta_{n-1,k-1} + \sigma_n \left[ \delta_n \theta_{n-2,k} - \phi_{n-2} \theta_{n-3,k-1} \right]$$

$$= \delta_n \theta_{n-1,k-1} + \sigma_n \cdots \sigma_4 h_k,$$

with $h_k := \sigma_3 \theta_{2,k} - \phi_2 \theta_{1,k-1}, \ k \geq 1.$

Since $h_k = 0$ for all $k \geq 4$. Thus $\theta_{n,k} = \delta_n \theta_{n-1,k-1}$ for all $n \geq 4, k \geq 4$. \qed

Lemma 4. Let $\alpha[p](x)$ be as in (2). If $\theta_{n,3} \geq 0$, then $\theta_{n+1,3} \geq 0$ for $n \geq 4$.

Proof. Since ([10], Lemma 5.1) $\delta_{n+1} \sigma_n > \phi_n$, and for all $n \geq 4$,

$$\delta_{n+1} \theta_{n,2} - \phi_n \theta_{n-1,2}$$

$$= \delta_{n+1} \left( \sigma_n \theta_{n-1,2} + \delta_n \theta_{n-1,1} - \phi_{n-1} \theta_{n-2,1} \right) - \phi_n \theta_{n-1,2}$$

$$= (\delta_{n+1} \sigma_n - \phi_n) \theta_{n-1,2} + \delta_{n+1} (\delta_n \theta_{n-1,1} - \phi_{n-1} \theta_{n-2,1}) \geq 0,$$

and $\delta_n \theta_{n-1,1} - \phi_{n-1} \theta_{n-2,1} \geq 0$ by the proof of Lemma 2. Therefore if $\theta_{n,3} \geq 0$, then

$$\theta_{n+1,3} = \sigma_{n+1} \theta_{n,3} + \delta_{n+1} \theta_{n,2} - \phi_n \theta_{n-1,2} \geq 0$$

for all $n \geq 4$. \qed

Through Lemma 1, Lemma 2, Lemma 3 and Lemma 4, it follows that $\theta_{n,k} \geq 0$ for all $n, k \geq 0$ with $0 \leq k \leq n + 1$ if and only if $\theta_{n,3} \geq 0$ for all $n \geq 4$, or equivalently $\theta_{4,3} \geq 0$. See Fig. 1 below.

![Figure 1: The positivity of $\theta_{n,i}$.](image_url)
Proposition 5([10]). Let $a^{[p]}(x)$ be as in (2).

(a) If $1 < p \leq \frac{25+\sqrt{241}}{12}$, then $W_{a^{[p]}(x)}$ is positively quadratically hyponormal if and only if $0 < x \leq \frac{2p-1}{p^2}$.

(b) If $p > \frac{25+\sqrt{241}}{12}$, then $W_{a^{[p]}(x)}$ is positively quadratically hyponormal if and only if $0 < x \leq \xi_1 := \frac{44p^4-98p^3+71p^2-16p}{94p^4-277p^3+312p^2-160p+32}$.

Remark. When $1 < p \leq \frac{25+\sqrt{241}}{12}$, $\theta_{4,3} \geq 0 \iff 0 < x \leq \frac{p}{2p-1}$ and when $p > \frac{25+\sqrt{241}}{12}$, $\theta_{4,3} \geq 0 \iff 0 < x \leq \xi_1$.

According to ([10]), it has the other interesting results.

Proposition 6. Let $a^{[p]}(x)$ be as in (2).

(a) $W_{a^{[p]}(x)}$ is subnormal if and only if $0 < x \leq \frac{1}{p}$.

(b) $W_{a^{[p]}(x)}$ is $n$-hyponormal if and only if $0 < x \leq \frac{1}{p} \prod_{i=1}^{n}(p-(i-1))^2(p-1)^{2n}$.

4. The Quadratic Hyponormality of $W_{a^{[p]}(x)}$

Let $a^{[p]}(x)$ be as in (2). Proposition 5 obtained equivalent condition of positive quadratical hyponormality of $W_{a^{[p]}(x)}$. In this section we give a sufficient condition of the quadratical hyponormality of $W_{a^{[p]}(x)}$. Let

$$
\begin{align*}
\xi_0 &:= \frac{p}{2p-1}, \\
\xi_1 &:= \frac{44p^4-98p^3+71p^2-16p}{94p^4-277p^3+312p^2-160p+32}, \\
\xi_2 &:= \frac{151p^4-478p^3+576p^2-312p+64}{856p^4-2791p^3+3418p^2-1857p^2+376p}, \\
\xi_3 &:= \frac{1609p^4-7126p^3+11335p^2-9164p^2+3648p-600}{1}. 
\end{align*}
$$

Lemma 7. Let $a^{[p]}(x)$ be as in (2).

1. If $1 < p \leq \frac{15+\sqrt{85}}{7} (\approx 3.4599)$, then $\theta_{5,3} \geq 0 \text{ if and only if } 0 < x \leq \xi_0$.
2. If $p > \frac{15+\sqrt{85}}{7}$, then $\theta_{5,3} \geq 0 \text{ if and only if } 0 < x \leq \xi_2$.

Proof. In fact

$$
\theta_{5,3} = \sigma_5 \theta_{4,3} + \delta_5 \theta_{4,2} - \phi_4 \theta_{3,2} = x(\xi_2 - x) \frac{(p-1)^6 (151p^4 - 478p^3 + 576p^2 - 312p + 64)}{(7p-6)(6p-5)(5p-4)^2(4p-3)^2(3p-2)^2(2p-1)^2}.
$$

And $\xi_2 < \xi_0$ if and only if $p > \frac{15+\sqrt{85}}{7}$. Thus we have our conclusions. \hfill \Box

Note that $d_n(t) \geq 0$ for $n = 0, 1, 2, 3$. Observe by Lemma 3 that if $n \geq 6$, then

$$
\theta_{n,n-2} t^{n-2} + \theta_{n,n-1} t^{n-1} + \theta_{n,n} t^n = \delta_n \cdots \delta_6 t^{n-5}(\theta_{5,3} t^3 + \theta_{5,4} t^4 + \theta_{5,5} t^5).
$$
Thus if \( \theta_{5,3} t^3 + \theta_{5,4} t^4 + \theta_{5,5} t^5 \geq 0 \) for all \( t \geq 0 \), then \( d_n(t) \geq 0 \) for all \( n \geq 6 \) and \( t \geq 0 \) because other Maclaurin coefficients are nonnegative. So we will verify \( \theta_{n,n-2} t^{n-2} + \theta_{n,n-1} t^{n-1} + \theta_{n,n} t^n \geq 0 \) for \( n = 4, 5 \). That is \( (3) \),

\[
\theta_{4,2} t^2 + \theta_{4,3} t^3 + \theta_{4,4} t^4 \geq 0, \quad \text{and} \quad \theta_{5,3} t^3 + \theta_{5,4} t^4 + \theta_{5,5} t^5 \geq 0,
\]

for all \( t \geq 0 \).

**Theorem 8.** Let \( \alpha[p](x) \) be as in \( (2) \).

(a) If \( 1 < p \leq p_1 \), then \( W_{\alpha[p]}(x) \) is quadratically hyponormal if and only if \( 0 < x \leq \xi_0 \).

(b) If \( p > p_1 \) and \( 0 < x \leq \xi_3 \), then \( W_{\alpha[p]}(x) \) is quadratically hyponormal, where

\[
p_1 = \frac{494 + 2\sqrt{62743} \cos \omega}{291} \approx 3.4188, \quad \text{with} \quad \omega = \frac{1}{3} \arccos \left( \frac{15684659}{3936684049} \sqrt{62743} \right).
\]

**Proof.** From Proposition 5, we need to discuss the case of \( p > \frac{25+\sqrt{241}}{12} \approx 3.377 \).

By Lemma 4 and Lemma 7, we know that \( c(n, 3) \geq 0 \) for all \( n \geq 5 \), in one of the following two cases,

- **Case 1.** \( p > \frac{15+\sqrt{55}}{7} \) and \( 0 < x \leq \xi_2 \);
- **Case 2.** \( \frac{25+\sqrt{241}}{12} < p \leq \frac{15+\sqrt{55}}{7} \) and \( 0 < x \leq \xi_0 \).

Under **Case 1.** We have the following results.

**Claim I.** If \( p > \frac{15+\sqrt{55}}{7} \) and \( \xi_1 < x \leq \xi_3 \), then \( \theta_{4,3} < 0 \) and \( \theta_{4,2} t^2 + \theta_{4,3} t^3 + \theta_{4,4} t^4 \geq 0 \).

**Proof of Claim I.** Under the condition of the Claim, we can get

\[
\sigma_5 \theta_{4,3} + \delta_5 \theta_{4,2} = \frac{(p - 1)^6 x \Phi_1}{(7p - 6)(6p - 5)(5p - 4)(4p - 3)(3p - 2)(2p - 1)^2} \geq 0,
\]

where

\[
\Phi_1 = (740p^5 - 2438p^4 + 2985p^3 - 1602p^2 + 316p) - (1522p^5 - 6055p^4 + 9662p^3 - 7744p^2 + 3128p - 512) x.
\]

Since \( \theta_{4,2} \geq 0 \) and \( \theta_{4,3} < 0 \), it follows that if \( 0 < t \leq \frac{7p-6}{4(6p-5)} \), where \( \delta_5 = \frac{7p-6}{4(6p-5)} \),

then \( \theta_{4,2} + \theta_{4,3} t \geq 0 \). Since \( \theta_{4,4} \geq 0 \), we have \( \theta_{4,2} t^2 + \theta_{4,3} t^3 + \theta_{4,4} t^4 \geq 0 \).

We also get that

\[
\sigma_5 \theta_{4,4} + \delta_5 \theta_{4,3} = \frac{4(p - 1)^6 x \Phi_2}{(7p - 6)(6p - 5)(5p - 4)(4p - 3)(3p - 2)(2p - 1)^2} \geq 0,
\]

where

\[
\Phi_2 = (376p^5 - 1121p^4 + 1242p^3 - 599p^2 + 104p) - (711p^5 - 2566p^4 + 3745p^3 - 2768p^2 + 1040p - 160) x.
\]
So if \( t > \frac{7p-6}{4(6p-5)} \), then \( t\theta_{4,4} + \theta_{4,3} \geq 0 \). Since \( \theta_{4,2} \geq 0 \), we have that \( \theta_{4,2}t^2 + \theta_{4,3}t^3 + \theta_{4,4}t^4 \geq 0 \). ▲

**Claim II.** If \( p > \frac{15+\sqrt{85}}{12} \) and \( \xi_1 < x \leq \xi_3 \), then \( \theta_{5,3}t^3 + \theta_{5,4}t^4 + \theta_{5,5}t^5 \geq 0 \).

**Proof of Claim II.** By the same argument as Claim I, it suffices to prove that if \( \xi_1 < x \leq \xi_3 \), then \( \sigma_6\theta_{5,4} + \delta_0\theta_{5,3} \geq 0 \) and \( \sigma_6\theta_{5,5} + \delta_6\theta_{5,4} \geq 0 \).

Indeed, a straightforward calculation shows that

\[
\sigma_6\theta_{5,4} + \delta_0\theta_{5,3} = \frac{4x (p - 1)^8 \Phi_3}{(8p - 7)(7p - 6)^2(6p - 5)^2(5p - 4)^2(4p - 3)^2(3p - 2)^2(2p - 1)^2} \geq 0,
\]

where

\[
\Phi_3 = (856p^5 - 2791p^4 + 3418p^3 - 1857p^2 + 376p) - (1809p^5 - 7126p^4 + 11335p^3 - 9104p^2 + 3696p - 608) x,
\]

and

\[
\sigma_6\theta_{5,5} + \delta_6\theta_{5,4} = \frac{32x (p - 1)^8 \Phi_4}{(8p - 7)(7p - 6)^2(6p - 5)^2(5p - 4)^2(4p - 3)^2(3p - 2)^2(2p - 1)^2} \geq 0.
\]

where

\[
\Phi_4 = (218p^5 - 655p^4 + 731p^3 - 355p^2 + 62p) - (413p^5 - 1501p^4 + 2205p^3 - 1640p^2 + 620p - 96) x.
\]

So \( \theta_{5,3}t^3 + \theta_{5,4}t^4 + \theta_{5,5}t^5 \geq 0 \). ▲

By Claim I and Claim II, we have proved that if \( p > \frac{15+\sqrt{85}}{12} \) and \( 0 < x \leq \xi_3 \), then \( W_{\alpha_{\nu}(x)} \) is quadratically hyponormal.

Under **Case 2.** If \( \frac{25+\sqrt{241}}{12} < p \leq \frac{15+\sqrt{85}}{12} \) and \( \xi_1 < x \leq \xi_3 (\leq \xi_2) \), then \( \theta_{4,3} < 0, \theta_{5,3} \geq 0 \). By Lemma 2, \( \theta_{n,n-1} < 0 \) for all \( n \geq 4 \). Note that if \( \frac{25+\sqrt{241}}{12} < p \leq p_1 \), then \( \xi_3 \geq \xi_0 \), and if \( p_1 < p \leq \frac{15+\sqrt{85}}{7} \), then \( \xi_3 < \xi_0 \). By the same way as Claim I and Claim II, we can easily prove that if \( \frac{25+\sqrt{241}}{12} < p \leq p_1 \) and \( \xi_1 < x \leq \xi_0 \), or if \( p_1 < p \leq \frac{15+\sqrt{85}}{7} \) and \( \xi_1 < x \leq \xi_3 \), then \( \theta_{n,n-2}t^{n-2} + \theta_{n,n-1}t^{n-1} + \theta_{n,n}t^n \geq 0 \) for \( n = 4, 5 \).

Therefore, if \( 1 < p \leq p_1 \), then \( W_{\alpha_{\nu}(x)} \) is quadratically hyponormal if and only if \( 0 < x \leq \xi_0 \). If \( p > p_1 \) and \( 0 < x \leq \xi_3 \), then \( W_{\alpha_{\nu}(x)} \) is quadratically hyponormal.

**Remark.** Let \( \xi_0, \xi_1, \xi_2, \xi_3 \) as in (3).

1. When \( \frac{25+\sqrt{241}}{12} < p < p_1 \), we get \( \xi_1 < \xi_0 < \xi_3 < \xi_2 \).
2. When \( p_1 < p < \frac{15+\sqrt{85}}{7} \), we get \( \xi_1 < \xi_3 < \xi_0 < \xi_2 \).
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When \( p > \frac{15 + \sqrt{85}}{7} \), we get \( \xi_1 < \xi_3 < \xi_2 < \xi_0 \).

**Example 9.** If \( p = 4 \), then \( \alpha^{[4]}(x) : \sqrt{x}, \sqrt{\frac{x}{7}}, \sqrt{\frac{x}{17}}, \sqrt{\frac{x}{31}}, \ldots \). By the results as above, we know that

- If \( 0 < x \leq \frac{22037}{48882} \approx 0.46677 \), then \( W_{\alpha^{[4]}(x)} \) is quadratically hyponormal. (By Theorem 8)
- \( W_{\alpha^{[4]}(x)} \) is positively quadratically hyponormal if and only if \( 0 < x \leq \frac{379}{670} \approx 0.56567 \). (By Proposition 5)
- If \( \frac{379}{670} < x \leq \frac{22037}{48882} \), then \( W_{\alpha^{[4]}(x)} \) is quadratically hyponormal but not positively quadratically hyponormal. In particular, \( W_{\alpha^{[4]}(x_0)} \) is quadratically hyponormal but not positively quadratically hyponormal, here \( x_0 = \frac{366}{500} = \frac{283}{350} \).
- \( W_{\alpha^{[4]}(x)} \) is 2-hyponormal if and only if \( 0 < x \leq \frac{49}{115} \approx 0.42609 \).
- \( W_{\alpha^{[4]}(x)} \) is 3-hyponormal if and only if \( 0 < x \leq \frac{4900}{13039} \approx 0.37580 \).
- \( W_{\alpha^{[4]}(x)} \) is 4-hyponormal if and only if \( 0 < x \leq \frac{207025}{591904} \approx 0.34976 \).
- \( W_{\alpha^{[4]}(x)} \) is \( n \)-hyponormal if and only if \( 0 < x \leq \frac{1}{4} \left( \frac{3n(n)}{3n+1} \right) \).
- \( W_{\alpha^{[4]}(x)} \) is subnormal if and only if \( 0 < x \leq \frac{1}{4} \).

5. Conclusion

After the subnormality, \( n \)-hyponormalty, and positively quadratic hyponormality [10], this paper considered the quadratic hyponormality of \( W^{[p]}_{\alpha} (x) \). The cubic hyponormality, semi-weakly hyponormality and other topics, also in particular, new techniques for solving these problems can be considered for further research. We leave them to interested readers.

References


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