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FERMATEAN FUZZY TOPOLOGICAL SPACES

HARIWAN Z. IBRAHIM

Abstract. The purpose of this paper is to introduce the notion of Fermatean fuzzy topological space by motivating from the notion of intuitionistic fuzzy topological space, and define Fermatean fuzzy continuity of a function defined between Fermatean fuzzy topological spaces. For this purpose, we define the notions of image and the pre-image of a Fermatean fuzzy subset with respect to a function and we investigate some basic properties of these notions. We also construct the coarsest Fermatean fuzzy topology on a non-empty set X which makes a given function f from X into Y a Fermatean fuzzy continuous where Y is a Fermatean fuzzy topological space. Finally, we introduce the concept of Fermatean fuzzy points and study some types of separation axioms in Fermatean fuzzy topological space.

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1. Introduction

Fuzzy sets were introduced by L.A. Zadeh [9] in 1965. The fuzzy set concept was the basis of mathematical testing of the fuzzy concept that exists in our real world and the formation of new branches in mathematics. The fuzzy set concept corresponding to unexplained physical situations gives useful applications on many topics such as statistics, data processing and linguistics. A lot of research has been done on this subject since 1965. In 1968, C.L. Chang [4] defined the concept of fuzzy topological space and generalized some basic notions of topology such as open set, closed set, continuity and compactness to fuzzy topological spaces. The idea of intuitionistic fuzzy set was first published by K. Atanassov [1] and many works by the same author and his colleagues appeared in the literature [2, 3]. D. Coker [5] subsequently initiated a study of intuitionistic fuzzy topological spaces. Later, R.R. Yager [8] launched a nonstandard fuzzy set referred to as Pythagorean fuzzy set. Recently, M. Olgun, M. Ünver and S.

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Yardımc [6] defined a Pythagorean fuzzy topological spaces. Fermatean fuzzy sets proposed by T. Senapati and R.R. Yager in 2020 [7], can handle uncertain information more easily in the process of decision making. They defined basic operations over the Fermatean fuzzy sets. The aim of this paper is to present the concept of Fermatean fuzzy topological space and study the continuity of a function defined among Fermatean fuzzy topological spaces. In the future works, categorical properties of Fermatean fuzzy topological spaces, applications of Fermatean fuzzy topological spaces, Fermatean fuzzy nano topological spaces and Fermatean fuzzy soft topological spaces may be studied.

2. Fermatean fuzzy sets

Definition 2.1. [2] The intuitionistic fuzzy sets are defined on a non-empty set X as objects having the form $I = \{\langle x, \alpha_I(x), \beta_I(x)\rangle : x \in X\}$, where $\alpha_I(x) : X \to [0,1]$ and $\beta_I(x) : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set I, respectively, and $0 \leq \alpha_I(x) + \beta_I(x) \leq 1$, for all $x \in X$.

Definition 2.2. [8] The Pythagorean fuzzy sets are defined on a non-empty set X as objects having the form $P = \{\langle x, \alpha_P(x), \beta_P(x)\rangle : x \in X\}$, where $\alpha_P(x) : X \to [0,1]$ and $\beta_P(x) : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set P, respectively, and $0 \leq (\alpha_P(x))^2 + (\beta_P(x))^2 \leq 1$, for all $x \in X$.

Definition 2.3. [7] Let X be a universe of discourse. A Fermatean fuzzy set (FFS) F in X is an object having the form $F = \{\langle x, \alpha_F(x), \beta_F(x)\rangle : x \in X\}$, where $\alpha_F(x) : X \to [0,1]$ and $\beta_F(x) : X \to [0,1]$, including the condition $0 \leq (\alpha_F(x))^3 + (\beta_F(x))^3 \leq 1$, for all $x \in X$. The numbers $\alpha_F(x)$ and $\beta_F(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set F.
For any FFS F and $x \in X$, $\pi_F(x) = \sqrt[3]{1 - (\alpha_F(x))^3 - (\beta_F(x))^3}$ is identified as the degree of indeterminacy of x to F. In the interest of simplicity, we shall mention the symbol $F = (\alpha_F, \beta_F)$ for the FFS $F = \{\langle x, \alpha_F(x), \beta_F(x)\rangle : x \in X\}$.

Definition 2.4. [7] Let $F = (\alpha_F, \beta_F)$, $F_1 = (\alpha_{F_1}, \beta_{F_1})$ and $F_2 = (\alpha_{F_2}, \beta_{F_2})$ be three Fermatean fuzzy sets (FFSs), then their operations are defined as follows:

(1) $F_1 \cap F_2 = (min\{\alpha_{F_1}, \alpha_{F_2}\}, max\{\beta_{F_1}, \beta_{F_2}\})$.
(2) $F_1 \cup F_2 = (max\{\alpha_{F_1}, \alpha_{F_2}\}, min\{\beta_{F_1}, \beta_{F_2}\})$.
(3) $F^c = (\beta_F, \alpha_F)$.

We say F_1 is a subset of F_2 or F_2 contains F_1 and we write $F_1 \subset F_2$ or $F_2 \supset F_1$ if $\alpha_{F_1} \leq \alpha_{F_2}$ and $\beta_{F_1} \geq \beta_{F_2}$.

Remark 2.1. If $\alpha_{F_1} = \alpha_{F_2}$ and $\beta_{F_1} = \beta_{F_2}$, then $F_1 = F_2$.

3. Fermatean fuzzy topological space

For understanding the Fermatean fuzzy set better, we give an instance to illuminate the understandability of the Fermatean fuzzy set. The point when someone needs will plan as much craving for the level for an alternative s_i on a criterion C_j, he might provide for the degree on which that alternative s_i fulfils those criteria C_j likewise 0.85, what is more correspondingly the elective s_i dissatisfies the criterion C_j similarly as 0.65. We can definitely get $0.85+0.65 = 1.5 > 1$, and, therefore, it does not follow the condition of intuitionistic fuzzy sets. Also, we can get $(0.85)^2 + (0.65)^2 = 0.7225 + 0.4225 = 1.145 > 1$, which does not obey the constraint condition of Pythagorean fuzzy set. However, we can get $(0.85)^3 + (0.65)^3 = 0.614125 + 0.274625 = 0.88875 \leq 1$, which is good enough to apply the Fermatean fuzzy set to control it [7].

Note here that, if the union and the intersection are infinite, then we use supremum "sup" and infimum "inf" instead of maximum "max" and minimum "min", respectively. Throughout this paper, we use the notation 1_X for the Fermatean fuzzy subset $(1, 0)$ and we use the notation 0_X for the Fermatean fuzzy subset $(0, 1)$, that is, $\alpha_{1_X} = 1$, $\beta_{1_X} = 0$, $\alpha_{0_X} = 0$ and $\beta_{0_X} = 1$. A Fermatean fuzzy subset F of a non-empty set X is a pair (α_F, β_F) of a membership function $\alpha_F(x) : X \to [0, 1]$ and a non-membership function $\beta_F(x) : X \to [0, 1]$ with $(\alpha_F(x))^3 + (\beta_F(x))^3 = (\gamma_F(x))^3$ for any $x \in X$ where $\gamma_F(x) : X \to [0, 1]$ is a function which is called the strength of commitment at point x.

Definition 3.1. Let X be a non-empty set and τ be a family of Fermatean fuzzy subsets of X. If

(1) $1_X, 0_X \in \tau$,
(2) for any $F_1, F_2 \in \tau$, we have $F_1 \cap F_2 \in \tau$,
(3) for any $\{F_i\}_{i \in I} \subset \tau$, we have $\bigcup_{i \in I} F_i \in \tau$ where I is an arbitrary index set then τ is called a Fermatean fuzzy topology on X.

The pair (X, τ) is said to be a Fermatean fuzzy topological space. Each member of τ is called an open Fermatean fuzzy subset. The complement of an open Fermatean fuzzy subset is called a closed Fermatean fuzzy subset.

Following is an example of a Fermatean fuzzy topological space.

Example 3.2. Let $X = \{c_1, c_2\}$. Consider the following family of Fermatean fuzzy subsets $\tau = \{1_X, 0_X, F_1, F_2, F_3, F_4, F_5\}$ where

$$F_1 = \{\langle c_1, \alpha_{F_1}(c_1) = 0.4, \beta_{F_1}(c_1) = 0.6\rangle, \langle c_2, \alpha_{F_1}(c_2) = 0.1, \beta_{F_1}(c_2) = 0.3\rangle\},$$

$$F_2 = \{\langle c_1, \alpha_{F_2}(c_1) = 0.5, \beta_{F_2}(c_1) = 0.4\rangle, \langle c_2, \alpha_{F_2}(c_2) = 0.2, \beta_{F_2}(c_2) = 0.8\rangle\},$$

$$F_3 = \{\langle c_1, \alpha_{F_3}(c_1) = 0.3, \beta_{F_3}(c_1) = 0.7\rangle, \langle c_2, \alpha_{F_3}(c_2) = 0, \beta_{F_3}(c_2) = 0.85\rangle\},$$

$F_4 = \{\langle c_1, \alpha_{F_4}(c_1) = 0.5, \beta_{F_4}(c_1) = 0.4\rangle, \langle c_2, \alpha_{F_4}(c_2) = 0.2, \beta_{F_4}(c_2) = 0.3\rangle\}$,
and

$F_5 = \{\langle c_1, \alpha_{F_5}(c_1) = 0.4, \beta_{F_5}(c_1) = 0.6\rangle, \langle c_2, \alpha_{F_5}(c_2) = 0.1, \beta_{F_5}(c_2) = 0.8\rangle\}$.
Observe that (X, τ) is a Fermatean fuzzy topological space.

Remark 3.1. Since every fuzzy set F on a non-empty set X is obviously a Fermatean fuzzy set having the form $F = \{\langle x, \alpha_F(x), 1 - \alpha_F(x)\rangle : x \in X\}$, so any fuzzy topological space (X, τ_1) in the sense of Chang is obviously a Fermatean fuzzy topological space in the form $\tau = \{F : \alpha_F \in \tau_1\}$ whenever we identify a fuzzy set in X whose membership function is α_F with its counter part $F = \{\langle x, \alpha_F(x), 1 - \alpha_F(x)\rangle : x \in X\}$ as in the following example.

Example 3.3. Let $X = \{c\}$. Consider the following family of fuzzy subsets $\tau = \{1_X, 0_X, F_1, F_2\}$ where

$1_X = \{\langle c, \alpha_{1_X}(c) = 1, 1 - \alpha_{1_X}(c) = \beta_{1_X}(c) = 0\rangle\}$,

$0_X = \{\langle c, \alpha_{0_X}(c) = 0, 1 - \alpha_{0_X}(c) = \beta_{0_X}(c) = 1\rangle\}$,

$F_1 = \{\langle c, \alpha_{F_1}(c) = 0.7, 1 - \alpha_{F_1}(c) = \beta_{F_1}(c) = 0.3\rangle\}$ and

$F_2 = \{\langle c, \alpha_{F_2}(c) = 0.2, 1 - \alpha_{F_2}(c) = \beta_{F_2}(c) = 0.8\rangle\}$.
Observe that (X, τ) is a fuzzy topological space and hence it is a Fermatean fuzzy topological space.

Remark 3.2. As any intuitionistic fuzzy subset or Pythagorean fuzzy subset of a set can be considered as a Fermatean fuzzy subset, we observe that any intuitionistic fuzzy topological space or Pythagorean fuzzy topological space is a Fermatean fuzzy topological space as well. On the other hand, it is obvious that a Fermatean fuzzy topological space needs not to be a intuitionistic fuzzy topological space and Pythagorean fuzzy topological space. Even an open Fermatean fuzzy subset maybe neither an intuitionistic fuzzy subset nor a Pythagorean fuzzy (see Example 3.4).

Example 3.4. Let $X = \{c_1, c_2\}$. Consider the following family of Fermatean fuzzy subsets $\tau = \{1_X, 0_X, F_1, F_2\}$ where
$F_1 = \{\langle c_1, \alpha_{F_1}(c_1) = 0.4, \beta_{F_1}(c_1) = 0.6\rangle, \langle c_2, \alpha_{F_1}(c_2) = 0.1, \beta_{F_1}(c_2) = 0.3\rangle\}$
and
$F_2 = \{\langle c_1, \alpha_{F_2}(c_1) = 0.9, \beta_{F_2}(c_1) = 0.6\rangle, \langle c_2, \alpha_{F_2}(c_2) = 0.2, \beta_{F_2}(c_2) = 0.3\rangle\}$.
Observe that (X, τ) is a Fermatean fuzzy topological space but (X, τ) is neither intuitionistic fuzzy topological space nor Pythagorean fuzzy topological space.

Remark 3.3. The family $\{1_X, 0_X\}$ is called the indiscreet Fermatean fuzzy topological space and the topology that contains all Fermatean fuzzy subsets is called the discrete Fermatean fuzzy topological space. Besides, a Fermatean fuzzy topology τ_1 on a set is said to be coarser than a Fermatean fuzzy topology τ_2 on the same set if $\tau_1 \subset \tau_2$.

Definition 3.5. Let (X, τ) be a Fermatean fuzzy topological space and $F = \{\langle x, \alpha_F(x), \beta_F(x)\rangle : x \in X\}$ be Fermatean fuzzy set in X. Then, the Fermatean fuzzy interior and Fermatean fuzzy closure of F are defined by

(1) $cl(F) = \bigcap\{H : H$ is closed Fermatean fuzzy set in X and $F \subset H\}$.
(2) $int(F) = \bigcup\{G : G$ is open Fermatean fuzzy set in X and $G \subset F\}$.

Remark 3.4. Let (X, τ) be a Fermatean fuzzy topological space and F be any Fermatean fuzzy set in X. Then,

(1) $int(F)$ is an open Fermatean fuzzy set.
(2) $cl(F)$ is a closed Fermatean fuzzy set.
(3) $int(1_X) = 1_X$ and $int(0_X) = 0_X$.
(4) $cl(1_X) = 1_X$ and $cl(0_X) = 0_X$.

Example 3.6. Consider the Fermatean fuzzy topological space (X, τ) in Example 3.2. If $F = \{\langle c_1, 0.62, 0.82\rangle, \langle c_2, 0.73, 0.69\rangle\}$, then
$int(F) = 0_X$ and $cl(F) = 1_X \cap F_3^c = F_3^c = \{\langle c_1, 0.7, 0.3\rangle, \langle c_2, 0.85, 0\rangle\}$.

Theorem 3.7. Let (X, τ) be a Fermatean fuzzy topological space and F_1, F_2 be Fermatean fuzzy sets in X. Then, the following properties hold:

(1) $int(F_1) \subset F_1$ and $F_1 \subset cl(F_1)$.
(2) If $F_1 \subset F_2$, then $int(F_1) \subset int(F_2)$ and $cl(F_1) \subset cl(F_2)$.
(3) F_1 is open Fermatean fuzzy if and only if $F_1 = int(F_1)$.
(4) F_1 is closed Fermatean fuzzy if and only if $F_1 = cl(F_1)$.
(5) $int(F_1) \cup int(F_2) \subset int(F_1 \cup F_2)$.
(6) $cl(F_1 \cap F_2) \subset cl(F_1) \cap cl(F_2)$.
(7) $int(F_1 \cap F_2) = int(F_1) \cap int(F_2)$.
(8) $cl(F_1) \cup cl(F_2) = cl(F_1 \cup F_2)$.

Proof. (1) and (2) are obvious. (3) and (4) follow from (1) and Definition 3.5.
(7) From $int(F_1 \cap F_2) \subset int(F_1)$ and $int(F_1 \cap F_2) \subset int(F_2)$ we obtain $int(F_1 \cap F_2) \subset int(F_1) \cap int(F_2)$. On the other hand, from the facts $int(F_1) \subset F_1$ and $int(F_2) \subset F_2$ we have $int(F_1) \cap int(F_2) \subset F_1 \cap F_2$ and $int(F_1) \cap int(F_2) \in \tau$ we see that $int(F_1) \cap int(F_2) \subset int(F_1 \cap F_2)$, and hence $int(F_1 \cap F_2) = int(F_1) \cap int(F_2)$.
(8) Can be easily deduced from (7).

□

Theorem 3.8. Let (X, τ) be a Fermatean fuzzy topological space and F be Fermatean fuzzy set in X. Then, the following properties hold:

(1) $cl(F^c) = int(F)^c$.
(2) $int(F^c) = cl(F)^c$.
(3) $cl(F^c)^c = int(F)$.
(4) $int(F^c)^c = cl(F)$.

Proof. We only prove (1), the other parts can be proved similarly.
Let $F = \{\langle x, \alpha_F(x), \beta_F(x)\rangle : x \in X\}$ and suppose that the family of open Fermatean fuzzy sets contained in F are indexed by the family $\{\langle x, \alpha_{U_i}(x), \beta_{U_i}(x)\rangle : i \in J\}$. Then, $int(F) = \{\langle x, \bigvee \alpha_{U_i}(x), \bigwedge \beta_{U_i}(x)\rangle\}$ and hence
$int(F)^c = \{\langle x, \bigwedge \beta_{U_i}(x), \bigvee \alpha_{U_i}(x)\rangle\}$. Since $F^c = \{\langle x, \beta_F(x), \alpha_F(x)\rangle\}$ and $\alpha_{U_i} \leq \alpha_F$, $\beta_{U_i} \geq \beta_F$ for each $i \in J$, so we obtain that $\{\langle x, \beta_{U_i}(x), \alpha_{U_i}(x)\rangle : i \in J\}$ is the family of closed Fermatean fuzzy sets containing F^c, that is, $cl(F^c) = \{\langle x, \bigwedge \beta_{U_i}(x), \bigvee \alpha_{U_i}(x)\rangle\}$. Thus, $cl(F^c) = int(F)^c$. □

Definition 3.9. Let X and Y be two non-empty sets and $f : X \to Y$ be a function. Let A and B be Fermatean fuzzy subsets of X and Y, respectively. Then, the membership and non-membership functions of image of A with respect to f that is denoted by $f[A]$ are defined by

$$\alpha_{f[A]}(y) := \begin{cases} sup_{z\in f^{-1}(y)}\alpha_A(z) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\beta_{f[A]}(y) := \begin{cases} inf_{z\in f^{-1}(y)}\beta_A(z) & \text{if } f^{-1}(y) \neq \phi \\ 1 & \text{otherwise,} \end{cases}$$

respectively. The membership and non-membership functions of pre-image of B with respect to f that is denoted by $f^{-1}[B]$ are defined by
$\alpha_{f^{-1}[B]}(x) := \alpha_B(f(x))$, and $\beta_{f^{-1}[B]}(x) := \beta_B(f(x))$, respectively.

Remark 3.5. Note that $f[A]$ and $f^{-1}[B]$ are Fermatean fuzzy subsets. In fact, since α_A and β_A are non-negative functions, then one can obtain,
$(\alpha_{f[A]}(y))^3 + (\beta_{f[A]}(y))^3$
$= (sup_{z\in f^{-1}(y)}\alpha_A(z))^3 + (inf_{z\in f^{-1}(y)}\beta_A(z))^3$
$= sup_{z\in f^{-1}(y)}(\alpha_A(z))^3 + inf_{z\in f^{-1}(y)}(\beta_A(z))^3$
$= sup_{z\in f^{-1}(y)}((\gamma_A(z))^3 - (\beta_A(z))^3) + inf_{z\in f^{-1}(y)}(\beta_A(z))^3$
$\leq sup_{z\in f^{-1}(y)}(1 - (\beta_A(z))^3) + inf_{z\in f^{-1}(y)}(\beta_A(z))^3 = 1,$
whenever $f^{-1}(y)$ is non-empty. On the other hand if $f^{-1}(y) = \phi$, then we have $(\alpha_{f[A]}(y))^3 + (\beta_{f[A]}(y))^3 = 1$.
The proof is trivial for $f^{-1}[B]$.

Theorem 3.10. Let X and Y be two non-empty sets and let $f : X \to Y$ be a function. Then, we have

(1) $f^{-1}[B^c] = f^{-1}[B]^c$ for any Fermatean fuzzy subset B of Y.
(2) $f[A]^c \subset f[A^c]$ for any Fermatean fuzzy subset A of X.
(3) if $B_1 \subset B_2$, then $f^{-1}[B_1] \subset f^{-1}[B_2]$ where B_1 and B_2 are Fermatean fuzzy subsets of Y.
(4) if $A_1 \subset A_2$, then $f[A_1] \subset f[A_2]$ where A_1 and A_2 are Fermatean fuzzy subsets of X.
(5) $f[f^{-1}[B]] \subset B$ for any Fermatean fuzzy subset B of Y.
(6) $A \subset f^{-1}[f[A]]$ for any Fermatean fuzzy subset A of X.

Proof. (1) For any $x \in X$ and for any Fermatean fuzzy subset B of Y we get from the definition of the complement that
$\alpha_{f^{-1}[B^c]}(x) = \alpha_{B^c}(f(x))$
$= \beta_B(f(x))$
$= \beta_{f^{-1}[B]}(x)$
$= \alpha_{f^{-1}[B]^c}(x)$.
Similarly one can have $\beta_{f^{-1}[B^c]}(x) = \beta_{f^{-1}[B]^c}(x)$. Therefore, we have $f^{-1}[B^c] = f^{-1}[B]^c$.

(2) For any $y \in Y$ such that $f(y) \neq \phi$ and for any Fermatean fuzzy subset A of X, we can write $(\gamma_{f[A]}(y))^3 = (\alpha_{f[A]}(y))^3 + (\beta_{f[A]}(y))^3$
$= sup_{z\in f^{-1}(y)}(\alpha_A(z))^3 + inf_{z\in f^{-1}(y)}(\beta_A(z))^3$
$= sup_{z\in f^{-1}(y)}((\gamma_A(z))^3 - (\beta_A(z))^3) + inf_{z\in f^{-1}(y)}(\beta_A(z))^3$
$\leq sup_{z\in f^{-1}(y)}((\gamma_A(z))^3) - inf_{z\in f^{-1}(y)}(\beta_A(z))^3 + inf_{z\in f^{-1}(y)}(\beta_A(z))^3$
$= sup_{z\in f^{-1}(y)}((\gamma_A(z))^3)$ ————(*)
Now from (*), we have $\alpha_{f[A^c]}(y) = sup_{z\in f^{-1}(y)}\alpha_{A^c}(z)$
$= sup_{z\in f^{-1}(y)}\beta_A(z)$
$= sup_{z\in f^{-1}(y)}\sqrt[3]{(\gamma_A(z))^3 - (\alpha_A(z))^3}$
$\geq \sqrt[3]{sup_{z\in f^{-1}(y)}(\gamma_A(z))^3 - sup_{z\in f^{-1}(y)}(\alpha_A(z))^3}$
$\geq \sqrt[3]{(\gamma_{f[A]}(y))^3 - (\alpha_{f[A]}(y))^3}$
$= \beta_{f[A]}(y)$
$= \alpha_{f[A]^c}(y)$.
The proof is trivial for each $y \in Y$ such that $f(y) = \phi$. On the other hand, we have $\beta_{f[A^c]}(y) \leq \beta_{f[A]^c}(y)$ using the same idea. Hence, we obtain $f[A]^c \subset f[A^c]$.

(3) Assume that $B_1 \subset B_2$. Then, we have for any $x \in X$ that
$\alpha_{f^{-1}[B_1]}(x) = \alpha_{B_1}(f(x)) \leq \alpha_{B_2}(f(x)) = \alpha_{f^{-1}[B_2]}(x)$.
Therefore, one can get $\alpha_{f^{-1}[B_1]}(x) \leq \alpha_{f^{-1}[B_2]}(x)$. Similarly, it is not difficult to show that $\beta_{f^{-1}[B_1]}(x) \geq \beta_{f^{-1}[B_2]}(x)$.

(4) Assume that $A_1 \subset A_2$ and $y \in Y$. If $f(y) = \phi$, then the proof is trivial. Assume that $f(y) \neq \phi$. Then, we have
$\alpha_{f[A_1]}(y) = sup_{z\in f^{-1}(y)}\alpha_{A_1}(z)$
$\leq sup_{z\in f^{-1}(y)}\alpha_{A_2}(z)$
$= \alpha_{f[A_2]}(y)$.
Thus, $\alpha_{f[A_1]} \leq \alpha_{f[A_2]}$ follows. Similarly, we have $\beta_{f[A_1]} \geq \beta_{f[A_2]}$.

(5) For any $y \in Y$ such that $f(y) \neq \phi$, we can write
$\alpha_{f[f^{-1}[B]]}(y) = sup_{z\in f^{-1}(y)}\alpha_{f^{-1}[B]}(z)$
$= sup_{z\in f^{-1}(y)}\alpha_B(f(z))$
$\leq \alpha_B(y)$.
On the other hand if $f(y) = \phi$, then we have $\alpha_{f[f^{-1}[B]]}(y) = 0 \leq \alpha_B(y)$. Similarly, we have $\beta_{f[f^{-1}[B]]}(y) = 0 \geq \beta_B(y)$.

(6) For any $x \in X$, we have
$\alpha_{f^{-1}[f[A]]}(x) = \alpha_{f[A]}(f(x))$

$= sup_{z\in f^{-1}(f(x))}\alpha_A(z)$
$\geq \alpha_A(x)$.
Similarly, we have $\beta_{f^{-1}[f[A]]} \leq \beta_A$.

□

The proof of the following result is easy and hence it is omitted.

Theorem 3.11. Let X and Y be two non-empty sets and $f : X \to Y$ be a function. Then, the following statements are true:

(1) $f[\cup_{i\in I}A_i] = \cup_{i\in I}f[A_i]$ for any Fermatean fuzzy subset A_i of X.
(2) $f^{-1}[\cup_{i\in I}B_i] = \cup_{i\in I}f^{-1}[B_i]$ for any Fermatean fuzzy subset B_i of Y.
(3) $f[A_1 \cap A_2] \subset f[A_1] \cap f[A_2]$ for any two Fermatean fuzzy subsets A_1 and A_2 of X.
(4) $f^{-1}[\cap_{i\in I}B_i] = \cap_{i\in I}f^{-1}(B_i)$ for any Fermatean fuzzy subset B_i of Y.

Definition 3.12. Let A and U be two Fermatean fuzzy subsets in a Fermatean fuzzy topological space. Then, U is said to be a neighbourhood of A if there exists an open Fermatean fuzzy subset E such that $A \subset E \subset U$.

Theorem 3.13. *A Fermatean fuzzy subset A is open in a Fermatean fuzzy topological space if and only if it contains a neighbourhood of its each subset.*

Proof. The proof is easy. □

Definition 3.14. Let (X, τ_1) and (Y, τ_2) be two Fermatean fuzzy topological spaces and $f : X \to Y$ be a function. Then, f is said to be Fermatean fuzzy continuous if for any Fermatean fuzzy subset A of X and for any neighbourhood V of $f[A]$ there exists a neighbourhood U of A such that $f[U] \subset V$.

Theorem 3.15. Let (X, τ_1) and (Y, τ_2) be two Fermatean fuzzy topological spaces and $f : X \to Y$ be a function. Then, the following statements are equivalent:

(1) f is Fermatean fuzzy continuous.
(2) For any Fermatean fuzzy subset A of X and for any neighbourhood V of $f[A]$, there exists a neighbourhood U of A such that for any $B \subset U$ we have $f[B] \subset V$.
(3) For any Fermatean fuzzy subset A of X and for any neighbourhood V of $f[A]$, there exists a neighbourhood U of A such that $U \subset f^{-1}[V]$.
(4) For any Fermatean fuzzy subset A of X and for any neighbourhood V of $f[A]$, $f^{-1}[V]$ is a neighbourhood of A.

Proof. (1) ⇒ (2): Assume that f is Fermatean fuzzy continuous. Let A be a Fermatean fuzzy subset of X and V be a neighbourhood of $f[A]$. Then, there exists a neighbourhood U of A such that $f[U] \subset V$. Now, if $B \subset U$, then we get $f[B] \subset f[U] \subset V$.
(2) ⇒ (3): Let A be a Fermatean fuzzy set of X and V be a neighbourhood of $f[A]$. From (2), there exists a neighbourhood U of A such that for any $B \subset U$ we have $f[B] \subset V$. Then, we can write $B \subset f^{-1}[f[B]] \subset f^{-1}[V]$. As B is an

arbitrary subset of U, we have $U \subset f^{-1}[V]$.
(3) $\Rightarrow$ (4): Let A be a Fermatean fuzzy subset of X and V be a neighbourhood of $f[A]$. Then from (3), there exists a neighbourhood U of A such that $U \subset f^{-1}[V]$. Since U is a neighbourhood of A there exists an open Fermatean fuzzy subset K of X such that $A \subset K \subset U$. On the other hand as $U \subset f^{-1}[V]$, one can get $A \subset K \subset f^{-1}[V]$ which implies $f^{-1}[V]$ is a neighbourhood of A.
(4) $\Rightarrow$ (1): Let A be a Fermatean fuzzy subset of X and V be a neighbourhood of $f[A]$. From the hypothesis, we have $f^{-1}[V]$ is a neighbourhood of A. Therefore, there exists an open Fermatean fuzzy subset K of X such that $A \subset K \subset f^{-1}[V]$ which implies $f[K] \subset f[f^{-1}[V]] \subset V$. Moreover, as K is open it is a neighbourhood of A. Hence, f is Fermatean fuzzy continuous.

□

Theorem 3.16. Let (X, τ_1) and (Y, τ_2) be two Fermatean fuzzy topological spaces. A function $f : X \to Y$ is Fermatean fuzzy continuous if and only if for each open Fermatean fuzzy subset B of Y we have $f^{-1}[B]$ is an open Fermatean fuzzy subset of X.

Proof. Assume that f is continuous. Let B be an open Fermatean fuzzy subset of Y and $A \subset f^{-1}[B]$. Then, we get $f[A] \subset B$. Since B is open, then by Theorem 3.13, there exists a neighbourhood V of $f[A]$ such that $V \subset B$. Thus, Fermatean fuzzy continuity of f and (4) of Theorem 3.15 imply that $f^{-1}[V]$ is a neighbourhood of A. On the other hand from (3) of Theorem 3.10 we have $f^{-1}[V] \subset f^{-1}[B]$. Therefore, $f^{-1}[B]$ is a neighbourhood of A as well. As A is an arbitrary subset of $f^{-1}[B]$, then by Theorem 3.13, the Fermatean fuzzy subset $f^{-1}[B]$ is open.
Conversely, let A be a Fermatean fuzzy subset of X and V be a neighbourhood of $f[A]$. Then, there exists an open Fermatean fuzzy subset L of Y such that $f[A] \subset L \subset V$. Now, from the hypothesis $f^{-1}[L]$ is open. On the other hand, we can write $A \subset f^{-1}[f[A]] \subset f^{-1}[L] \subset f^{-1}[V]$. Hence, $f^{-1}[V]$ is a neighbourhood of A which proves the Fermatean fuzzy continuity of f. □

Example 3.17. Consider $X = \{c_1, c_2\}$ with the Fermatean fuzzy topology $\tau_1 = \{1_X, 0_X, A_1\}$ and $Y = \{n_1, n_2\}$ with the Fermatean fuzzy topology $\tau_2 = \{1_Y, 0_Y, B_1\}$, where $A_1 = \{\langle c_1, 0.7, 0.8\rangle, \langle c_2, 0.9, 0.6\rangle\}$ and
$B_1 = \{\langle n_1, 0.9, 0.6\rangle, \langle n_2, 0.7, 0.8\rangle\}$.

Let $f : X \to Y$ defined as follows:

$$f(x) = \begin{cases} n_2 & \text{if } x = c_1, \\ n_1 & \text{if } x = c_2. \end{cases}$$

Since $1_Y, 0_Y$ and B_1 are open Fermatean fuzzy subsets of Y, then

$$f^{-1}[1_Y] = \{\langle c_1, 1, 0\rangle, \langle c_2, 1, 0\rangle\},$$

$f^{-1}[0_Y] = \{\langle c_1, 0, 1\rangle, \langle c_2, 0, 1\rangle\}$ and

$f^{-1}[B_1] = \{\langle c_1, 0.7, 0.8\rangle, \langle c_2, 0.9, 0.6\rangle\}$ are open Fermatean fuzzy subsets of X. Thus, f is Fermatean fuzzy continuous.

Example 3.18. Consider $X = \{c_1, c_2\}$ with the Fermatean fuzzy topology $\tau_1 = \{1_X, 0_X\}$ and $Y = \{n_1, n_2\}$ with the Fermatean fuzzy topology $\tau_2 = \{1_Y, 0_Y, B_1\}$, where $B_1 = \{\langle n_1, 0.81, 0.59\rangle, \langle n_2, 0.51, 0.92\rangle\}$.

Let $f : X \to Y$ defined as follows:

$$f(x) = \begin{cases} n_1 & \text{if } x = c_1, \\ n_2 & \text{if } x = c_2. \end{cases}$$

Since B_1 is open Fermatean fuzzy subset of Y, but

$f^{-1}[B_1] = \{\langle c_1, 0.81, 0.59\rangle, \langle c_2, 0.51, 0.92\rangle\}$ is not open Fermatean fuzzy subset of X. Thus, f is not Fermatean fuzzy continuous.

Remark 3.6. A function $f : X \to Y$ is Fermatean fuzzy continuous if and only if for each closed Fermatean fuzzy subset B of Y we have $f^{-1}[B]$ is a closed Fermatean fuzzy subset of X.

Corollary 3.19. The following are equivalent to each other:

(1) $f : (X, \tau_1) \to (Y, \tau_2)$ is Fermatean fuzzy continuous.
(2) $cl(f^{-1}[B]) \subset f^{-1}[cl(B)]$ for each Fermatean fuzzy set in Y.
(3) $f^{-1}[int(B)] \subset int(f^{-1}[B])$ for each Fermatean fuzzy set in Y.

Proof. They can be easily proved using Theorems 3.8, 3.10 and 3.16 and Remark 3.6. □

Theorem 3.20. Let (Y, τ) be a Fermatean fuzzy topological spaces, X be a non-empty set and $f : X \to Y$ be a function. Then, there exists a coarsest Fermatean fuzzy topology τ_1 over X such that f is Fermatean fuzzy continuous.

Proof. Let us define a class of Fermatean fuzzy subsets τ_1 of X by
$\tau_1 := \{f^{-1}[V] : V \in \tau\}$.

We prove that τ_1 is the coarsest Fermatean fuzzy topology over X such that f is continuous.

(1) We can write for any $x \in X$ that
$\alpha_{f^{-1}[0_Y]}(x) = \alpha_{0_Y}(f(x)) = 0 = \alpha_{0_X}(x)$.
Similarly, we immediately have $\beta_{f^{-1}[0_Y]}(x) = \beta_{0_X}(x)$ for any $x \in X$ which implies $f^{-1}[0_Y] = 0_X$. Now, as $0_Y \in \tau$ we have $0_X = f^{-1}[0_Y] \in \tau_1$. In similar manner, it is easy to see that $1_X = f^{-1}[1_Y] \in \tau_1$.
(2) Assume that $F_1, F_2 \in \tau_1$. Then, for $i = 1, 2$ there exists $B_i \in \tau$ such that $f^{-1}[B_i] = F_i$ which implies $\alpha_{f^{-1}[B_i]} = \alpha_{F_i}$ and $\beta_{f^{-1}[B_i]} = \beta_{F_i}$. Thus, we obtain for any $x \in X$ that
$\alpha_{F_1 \cap F_2}(x) = min\{\alpha_{F_1}(x), \alpha_{F_2}(x)\}$

$= min\{\alpha_{f^{-1}[B_1]}(x), \alpha_{f^{-1}[B_2]}(x)\}$
$= min\{\alpha_{B_1}(f(x)), \alpha_{B_2}(f(x))\}$
$= \alpha_{B_1 \cap B_2}(f(x))$
$= \alpha_{f^{-1}[B_1 \cap B_2]}(x)$.
Similarly, it is not difficult to see that $\beta_{F_1 \cap F_2} = \beta_{f^{-1}[B_1 \cap B_2]}$. Hence, we get $F_1 \cap F_2 \in \tau_1$.

(3) Assume that $\{F_i\}_{i \in I}$ be an arbitrary sub-family of τ_1. Then for any $i \in I$, there exists $B_i \in \tau_1$ such that $f^{-1}[B_i] = F_i$ which implies $\alpha_{f^{-1}[B_i]} = \alpha_{F_i}$ and $\beta_{f^{-1}[B_i]} = \beta_{F_i}$. Therefore, one can get for any $x \in X$ that
$\alpha_{\bigcup_{i \in I} F_i}(x) = sup_{i \in I} \alpha_{F_i}(x)$
$= sup_{i \in I} \alpha_{f^{-1}[B_i]}(x)$
$= sup_{i \in I} \alpha_{B_i}(f(x))$
$= \alpha_{\bigcup_{i \in I} B_i}(f(x))$
$= \alpha_{f^{-1}[\bigcup_{i \in I} B_i]}(x)$.
On the other hand, it is easy to see that
$\beta_{\bigcup_{i \in I} F_i} = \beta_{f^{-1}[\bigcup_{i \in I} B_i]}$. Thus, we have $\bigcup_{i \in I} F_i \in \tau_1$.

From Theorem 3.16, the continuity of f is trivial. Now, we prove that τ_1 is the coarsest Fermatean fuzzy topology over X such that f is Fermatean fuzzy continuous. Let $\tau_2 \subset \tau_1$ be a Fermatean fuzzy topology over X such that f is Fermatean fuzzy continuous. If $B \in \tau_1$ then there exists $V \in \tau$ such that $f^{-1}[V] = B$. Since, f is Fermatean fuzzy continuous with respect to τ_2 we have $B = f^{-1}[V] \in \tau_2$. Hence, we have $\tau_2 = \tau_1$.

□

Definition 3.21. Let X be a non-empty set and $x \in X$ a fixed element in X. Suppose $r_1 \in (0,1]$ and $r_2 \in [0,1)$ are two fixed real numbers such that $r_1^3 + r_2^3 \leq 1$. Then, a Fermatean fuzzy point $p^x_{(r_1,r_2)} = \{\langle x, \alpha_p(x), \beta_p(x)\rangle\}$ is defined to be a Fermatean fuzzy set of X given by

$$p^x_{(r_1,r_2)}(y) := \begin{cases} (r_1, r_2) & \text{if } y = x, \\ (0,1) & \text{otherwise,} \end{cases}$$

for $y \in X$. In this case, x is called the support of $p^x_{(r_1,r_2)}$. A Fermatean fuzzy point $p^x_{(r_1,r_2)}$ is said to belong to a Fermatean fuzzy set $F = \{\langle x, \alpha_F(x), \beta_F(x)\rangle\}$ of X denoted by $p^x_{(r_1,r_2)} \in F$ if $r_1 \leq \alpha_F(x)$ and $r_2 \geq \beta_F(x)$. Two Fermatean fuzzy points are said to be distinct if their supports are distinct.

Remark 3.7. Let $F_1 = \{\langle x, \alpha_{F_1}(x), \beta_{F_1}(x)\rangle\}$ and $F_2 = \{\langle x, \alpha_{F_2}(x), \beta_{F_2}(x)\rangle\}$ be two Fermatean fuzzy sets of X. Then, $F_1 \subset F_2$ if and only if $p^x_{(r_1,r_2)} \in F_1$ implies $p^x_{(r_1,r_2)} \in F_2$ for any Fermatean fuzzy point $p^x_{(r_1,r_2)}$ in X.

Definition 3.22. Let $r_1, r_3 \in (0,1]$, $r_2, r_4 \in [0,1)$ and $x, y \in X$. A Fermatean fuzzy topological space (X,τ) is said to be:

(1) T_0 if for each pair of distinct Fermatean fuzzy points $p^x_{(r_1,r_2)}, p^y_{(r_3,r_4)}$ in X, there exist two open Fermatean fuzzy sets L and K such that
$$L = \{\langle x, 1, 0\rangle, \langle y, 0, 1\rangle\}$$
or
$$K = \{\langle x, 0, 1\rangle, \langle y, 1, 0\rangle\}.$$

(2) T_1 if for each pair of distinct Fermatean fuzzy points $p^x_{(r_1,r_2)}, p^y_{(r_3,r_4)}$ in X, there exist two open Fermatean fuzzy sets L and K such that
$$L = \{\langle x, 1, 0\rangle, \langle y, 0, 1\rangle\}$$
and
$$K = \{\langle x, 0, 1\rangle, \langle y, 1, 0\rangle\}.$$

Example 3.23. Consider $X = \{c_1, c_2\}$ with the Fermatean fuzzy topology $\tau = \{1_X, 0_X, F_1, F_2\}$, where
$F_1 = \{\langle c_1, 1, 0\rangle, \langle c_2, 0, 1\rangle\}$ and $F_2 = \{\langle c_1, 0, 1\rangle, \langle c_2, 1, 0\rangle\}$. Then, (X,τ) is T_0 and T_1.

Corollary 3.24. Let (X,τ) be a Fermatean fuzzy topological space. If (X,τ) is T_1, then (X,τ) is T_0.

Proof. The proof is straightforward from the Definition 3.22. □

Here is an example which shows that the converse of above corollary is not true in general.

Example 3.25. Consider $X = \{c_1, c_2\}$ with the Fermatean fuzzy topology $\tau = \{1_X, 0_X, F\}$, where $F = \{\langle c_1, 1, 0\rangle, \langle c_2, 0, 1\rangle\}$. Then, (X,τ) is T_0 but not T_1 because there exists no open Fermatean fuzzy set K such that $K = \{\langle x, 0, 1\rangle, \langle y, 1, 0\rangle\}$.

Theorem 3.26. Let (X,τ) be a Fermatean fuzzy topological space, $r_1, r_3 \in (0,1]$ and $r_2, r_4 \in [0,1)$. If (X,τ) is T_0, then for each pair of distinct Fermatean fuzzy points $p^x_{(r_1,r_2)}$, $p^y_{(r_3,r_4)}$ of X, $cl(p^x_{(r_1,r_2)}) \neq cl(p^y_{(r_3,r_4)})$.

Proof. Let (X,τ) be T_0 and $p^x_{(r_1,r_2)}$, $p^y_{(r_3,r_4)}$ be any two distinct Fermatean fuzzy points of X. Then, there exist two open Fermatean fuzzy sets L and K such that
$$L = \{\langle x, 1, 0\rangle, \langle y, 0, 1\rangle\}$$
or
$$K = \{\langle x, 0, 1\rangle, \langle y, 1, 0\rangle\}.$$

Let $L = \{\langle x, 1, 0\rangle, \langle y, 0, 1\rangle\}$ be exists. Then, $L^c = \{\langle x, 0, 1\rangle, \langle y, 1, 0\rangle\}$ is a closed Fermatean fuzzy set which does not contain $p^x_{(r_1,r_2)}$ but contains $p^y_{(r_3,r_4)}$. Since $cl(p^y_{(r_3,r_4)})$ is the smallest closed Fermatean fuzzy set containing $p^y_{(r_3,r_4)}$, then $cl(p^y_{(r_3,r_4)}) \subset L^c$ and therefore $p^x_{(r_1,r_2)} \notin cl(p^y_{(r_3,r_4)})$. Consequently $cl(p^x_{(r_1,r_2)}) \neq cl(p^y_{(r_3,r_4)})$.

□

Theorem 3.27. Let (X, τ) be a Fermatean fuzzy topological space. Then, (X, τ) is T_1 if $p^x_{(1,0)}$ is closed Fermatean fuzzy set for every $x \in X$.

Proof. Suppose $p^x_{(1,0)}$ is a closed Fermatean fuzzy set for every $x \in X$. Let $p^x_{(r_1,r_2)}$, $p^y_{(r_3,r_4)}$ be any two distinct Fermatean fuzzy points of X, then $x \neq y$ implies ${p^x_{(1,0)}}^c$ and ${p^y_{(1,0)}}^c$ are two open Fermatean fuzzy sets such that

$${p^y_{(1,0)}}^c = \{\langle x, 1, 0\rangle, \langle y, 0, 1\rangle\}$$

and

$${p^x_{(1,0)}}^c = \{\langle x, 0, 1\rangle, \langle y, 1, 0\rangle\}.$$

Thus, (X, τ) is T_1. □

4. Conclusions

We defined a Fermatean fuzzy topology, Fermatean fuzzy neighborhood and Fermatean fuzzy continuous mapping, and obtained some of their properties. Also we introduced the concept of Fermatean fuzzy points and studied separation axioms in Fermatean fuzzy topological space. In the future, we will try to introduce the compactness, and connectedness in Fermatean fuzzy topological space.

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Hariwan Z. Ibrahim is an assistant professor at the University of Zakho. He received both his Ph.D. and M.Sc. from Zakho University. His research interests include fuzzy topology, topological algebra, ditopology, ideal topology and soft topology. He has published more than 70 research papers in international journals.

Department of Mathematics, Faculty of Education, University of Zakho, Zakho, Kurdistan Region-Iraq.
e-mail: hariwan_math@yahoo.com