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RESEARCH ARTICLE

Development of Mathematical Task Analytic Framework: Proactive and Reactive Features

Sheunghyun Yeo[1], Jung Colen[2], Na Young Kwon[3], Hoyun Cho[4], Jinho Kim[1], Woong Lim[5]

[1] Professor, Department of Mathematics Education, Daegu National University of Education
[2] Professor, Department of Mathematical & Natural Sciences, Chadron State College
[3] Professor, Department of Mathematics Education, Inha University
[4] Professor, School of Education, Capital University
[5] Professor, Graduate School of Education, Yonsei University

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Abstract

A large body of previous studies investigated mathematical tasks by analyzing the design process prior to lessons or textbooks. While researchers have revealed the significant roles of mathematical tasks within written curricular, there has been a call for studies about how mathematical tasks are implemented or what is experienced and learned by students as enacted curriculum. This article proposes a mathematical task analytic framework based on a holistic definition of tasks encompassing both written tasks and the process of task enactment. We synthesized the features of the mathematical tasks and developed a task analytic framework with multiple dimensions: breadth, depth, bridging, openness, and interaction. We also applied the scoring rubric to analyze three multiplication tasks to illustrate the framework by its five dimensions. We illustrate how a series of tasks are analyzed through the framework when students are engaged in multiplicative thinking. The framework can provide important information about the qualities of planned tasks for mathematics instruction (proactive) and the qualities of implemented tasks during instruction (reactive). This framework will be beneficial for curriculum designers to design rich tasks with more careful consideration of how each feature of the tasks would be attained and for teachers to transform mathematical tasks with the provision of meaningful learning activities into implementation.

Keywords cognitive complexity, mathematical tasks, task analytic framework, task features

• Corresponding Author: Sheunghyun Yeo email: shyeo@dnue.ac.kr

I. INTRODUCTION

As the standpoint of teaching and learning mathematics has been changed from static and passive to dynamic and active, research has focused more on the use of mathematical tasks that engage students in a process of meaning-making when solving them through discussion (Jäder et al., 2017; Romberg, 1994; Ruthven et al., 2009; Schoenfeld, 1992). However, tasks are often considered as only static and written problems in textbooks or instruction materials. Although this approach is beneficial of preservice or novice teachers in developing their expertise to identify and develop effective tasks (e.g., König et al., 2020), it is unclear to what extent students have actually the opportunity to learn mathematics with relation to mathematical tasks and how much the tasks can achieve their targeted (even beyond) learning goals (Francisco & Maher, 2011; Schmidt et al., 1997; Wijaya et al., 2015). In order for students to develop their conceptual understanding and mathematical thinking as active learners, researchers argue that written mathematical tasks should be combined with meaningful and worthwhile mathematical activities through the whole processes of task enactment (Horoks & Robert, 2007; Simon et al., 2016).

A large body of previous studies investigated mathematical tasks by analyzing the design process prior to lessons (e.g., Liljedahl et al., 2007) and printed textbooks (e.g., Son, 2012). While research revealed the significant roles of mathematical tasks within intended curriculum (e.g., the national or state level standards or guidelines) or written curriculum (e.g., adopted textbooks by districts or schools), there has been a call for studies about how mathematical tasks are implemented or what is experienced and learned by students as enacted curriculum (Remillard, 2005; Tarr et al., 2006; Watson & Ohtani, 2015). For instance, Boston (2012) suggested a toolkit for analyzing instructional quality of mathematics to assess the nature and characteristics of classroom instruction. The toolkit provided a rubric to assess cognitive demands of mathematical tasks in consideration of the potential and actual engagement of students in the tasks.

Researchers have examined the importance of mathematical tasks related to multiple aspects of the development of student's mathematical thinking and dispositions including high-ordered thinking (Hiebert & Wearne, 1993), mathematical justifications and reasoning (Arbaugh & Brown, 2005), and motivation (Clarke & Roche, 2018). In the same vein, the National Council of Teachers of Mathematics (NCTM) has consistently emphasized the importance of rich tasks, as stated that "effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem-solving and allow multiple entry points and varied solution strategies" (2014, p. 17). For example, tasks with a high level of cognitive demand can impact the way students interact with mathematical content to build their understanding through the whole problem-solving process (Stein & Smith, 1998).

Although the cognitive complexity in mathematical tasks plays a critical role in providing students the opportunity to learn, the engagement of high-leverage mathematical thinking depends on other multiple features of tasks and other classroom-based factors (Stein et al., 1996). Previous studies have shown the fundamental task features such as multiple representations (Fan & Zhu, 2007), multiple solution strategies (Jitendra et al., 2007), and mathematical discourse (Stein et al., 2009). Other studies have also explored various factors related to the enactment of tasks: students' prior knowledge (Stillman, 2000), teacher knowledge (Charalambous, 2010), belief (Raymond, 1997), curricular materials (Tarr et al., 2008), and professional development (Polly, 2015). Although research examined and revealed the rich relationship between mathematical tasks and various features or factors, there are little studies on the various features of tasks that emerge through the enactment of tasks and how those tasks features are attained by students. While most studies emphasize the features of static tasks prior to the enactment, few studies pay attention to how these various features of mathematical tasks are launched and managed during lessons. Furthermore, much less studies have explored how the features influence what is learned or performed by students.

Taken together, this study is situated in the process of task enactment from planning to implementation and proposes a task centric framework to analyze the assigned and enacted mathematical tasks. Therefore, in the present study, mathematical tasks include not only the potentials of assigned tasks but also mathematical activities that provide opportunities to develop students' mathematical thinking from holistic perspectives (Boston, 2012; Boston & Candela, 2018). Acknowledging the importance of both static and dynamic features of tasks, it is paramount to examine the nature of tasks when implemented. In this study, we synthesize the multi-dimensional characteristics of mathematical tasks by considering how the characteristics of tasks are unfolded in the instructions and focusing on students' engagement in learning through the tasks. In this paper, we aim (1) to propose a task analytic framework that addresses how the features of tasks are intended and attained in mathematical activities and (2) to illustrate classroom examples of using the framework to analyze a series of multiplication tasks.

II. RELATED LITERATURE

Mathematical Tasks

Tasks potentially influence and structure the way students think and broaden (or limit) their views of the subject matter with which they are engaged (Carpenter et al., 1997). Mathematical tasks provide students opportunities for conceptual thinking and encourage them to make connections between specific mathematical ideas through deeper understanding about mathematical concepts, processes, and relationships (Stein et al., 2009). However, it is questionable whether every mathematical task gives the same level of opportunity for students to learn high-quality mathematics (Hiebert et al., 1996; Stein et al., 2009).

Doyle (1988) argues that tasks are assigned by teachers and devoted to developing students' understanding and practices. He defines task with four aspects of work in a classroom: "(a) a goal state or end product to be achieved; (b) a problem space or set of conditions and resources available to accomplish the tasks; (c) the operations involved in assembling and using resources to reach the goal state or generate the product; (d) the importance of the task in the overall work system of the class" (p. 169). The four aspects are interconnected and have reciprocal relationships with each other. For example, if students are provided with more resources, the way to attain their goals would be easier.

Above definitions of tasks are limited to the written tasks. However, mathematical tasks could be defined beyond such potentials. In fact, studies have used extensive definitions of mathematical tasks including students' actual learning through the activities. For example, Horoks and Robert (2007) extended the definition of tasks that are considers not only as mathematical concepts to be learned at each moment of the class but also as the work done by students. Suppose a teacher facilitates classroom discussions and some new mathematical idea emerges from students' dialogues. The teacher can develop this idea as a task even though it was not originally intended (Kim, 2014).

As researchers have shown that students' mathematical ideas through tasks implementation play a significant role in measuring the quality of mathematical tasks (Arbaugh & Brown, 2005; Boston, 2012; Crespo, 2003; Norton & Kastberg, 2012), the value of tasks can be measured by the extent which students are engaged in solving the given tasks. In this study, we do not limit mathematical tasks to assigned or written mathematical problems. Assigned mathematical tasks themselves can exert the potential when enacted by teachers and engaged by students. Therefore, we encompass the potentials of assigned tasks and the attainments of mathematical activities that provide opportunities to develop students' mathematical thinking in various situations in this study. Based on such a holistic definition of tasks, it is needed to re-identify characteristics of mathematical tasks considering how such characteristics develop students' mathematical thinking.

Change of Perspectives on Mathematical Task

This holistic perspective of mathematical tasks offers a dynamic view on tasks. Because mathematical tasks affect what students learn and how they think (Doyle, 1983), we consider the activities enacted by teachers and learners related to the tasks as a part of task implementation. As mentioned earlier, the original intentions of a task are difficult to be apart from implemented activities that comprise students' responses to the given task (Christiansen & Walther, 1986; Watson & Mason, 2007). At the same time, the tasks are not always implemented as intended. Even though teachers design a rich task to be able to elicit students' mathematical thinking, it often ends up with a rote practice of skills. From the task centric perspective, this task can be interpreted as a worthwhile but unsuccessfully implemented task. Therefore, teachers have to be ready to use tasks pedagogically and keep in mind the various features for the implementation process. Although the shift of cognitive demand was investigated during the multiple phases of lesson implementation through a task centered framework (e.g., Boston & Smith, 2009), there is still the remaining question of how other features of mathematical tasks such as multiple strategies and communication can be enacted dynamically.

As we take the definition of task in a holistic perspective, it is required to re-consider the features of tasks considering both planned and implemented task characteristics. We identify two distinct features of mathematical tasks: proactive and reactive. The proactive features of task refer to "addressing specifically the initial formulation of the [task] design" (Watson & Ohtani, 2015, p. 28). When teachers design (or select) a mathematical task, they should consider underlying mathematical content knowledge and students' cognitive engagement, mathematical understanding and reasoning, and problem-solving (NCTM, 1991). For example, Yeh and her colleagues (2016) proposed the Juicy Tasks emphasizing how meaningful and worthwhile a mathematical task can be, by connecting it to important mathematical learning goals, multiple entry points, and relevant situations in selecting and adapting the task. On the other hand, the reactive features focus on "attention on the process by which a designed sequence is integrated into the classroom environment, subsequently refined, and then theorized about" (Watson & Ohtani, 2015, p. 28). Teachers should consider the tasks that draw students' different prior knowledge and the classroom ecology, then develop a wide spectrum of their mathematical knowledge and skills. This is because the value of tasks can vary depending on current students' abilities and interactions during the implementation under the unique classroom situations. For example, Giménez and his colleagues (2013) suggested Task Suitability Criteria with important mathematical concepts, cognitive suitability, interactions, adequate materials, mathematical dispositions, and even school political environments.

The distinct views of task features might highlight only a partial nature of each perspective of tasks. Therefore, in this study, we focus on not only proactive features as potential of tasks (Baumert et al., 2010) but also the reactive features in the task enactment from a situated perspective that learners have the opportunity to acquire knowledge due to the interaction between them and the environment (Lave & Wenger, 1991). From this viewpoint, we can show to what extent the potential of tasks can be actually exerted through implementations.

Development of the Task Analytic Framework

The motivation for developing a task analytic framework stemmed from the observations of classroom instructions in elementary schools. From our observations, we noticed teachers' striving to improve the quality of their teaching. Due to the uniqueness of each classroom, there should be no absolute solution to improve a teacher's instruction. However, how mathematical tasks are designed and how they are implemented by considering students' knowledge and background are crucial for student learning across every classroom. Once we define mathematical tasks including assigned tasks and enacted tasks from the holistic perspective, we further need a modified version of the task framework. We first redefined the process of task enactment (Henningsen & Stein, 1997). Then, we developed a comprehensive analytic framework for mathematical instructional tasks (a.k.a., Mathematical Task Analytic Framework [MTAF]) that focuses on characterizing various features of mathematical tasks. To identify dimensions of the analytic framework, we synthesized previous literature about distinct features of tasks. In the following section, we describe the task enactment process and identify cross-cut dimensions and components for analyzing mathematical tasks from the holistic perspective.

Intended to Enacted Mathematical Tasks

To understand the transformative process of intended to enacted tasks, this study modified a task enactment model suggested by Henningsen and Stein (1997). They proposed three distinct phases: (a) represented in instructional materials, (b) set-up by teachers, and (c) implemented by students. We revised the process with four distinct phases to expand the meaning of the task as the entire mathematical process of instructions: planned by teachers or developers, set-up by teachers and students, performed by teachers and students, and discussed by teachers and students (Figure 1). Prior to instructions, tasks are designed by teachers or textbook developers in written forms (Tasks as planned). At this phase, tasks are considered as mathematical interventions which have potential features and cognitive complexity. During the enactment process, instead of only teachers having an authority to set up mathematical tasks, we believe students might also contribute to create and launch tasks collaboratively with teachers or they could volunteer to create their own problem situations (Tasks as set-up). In the performing tasks phase, students carry out tasks using their own strategies with support from the teacher (Tasks as performed). In the final phase of the enactment of tasks, the teacher facilitates a discussion about students' various solution strategies to develop their mathematical understanding (Tasks as discussed). Each phase of task enactment is a collection of interactive activities among students and teachers within tasks and activities. After the engagement in solving the tasks, individual students achieve mathematical learning outcomes (Achieved tasks). In this study, the proactive features of tasks are evaluated in the planning phase, while the reactive features are evaluated in the enactment phase (set-up, performed, and discussed).

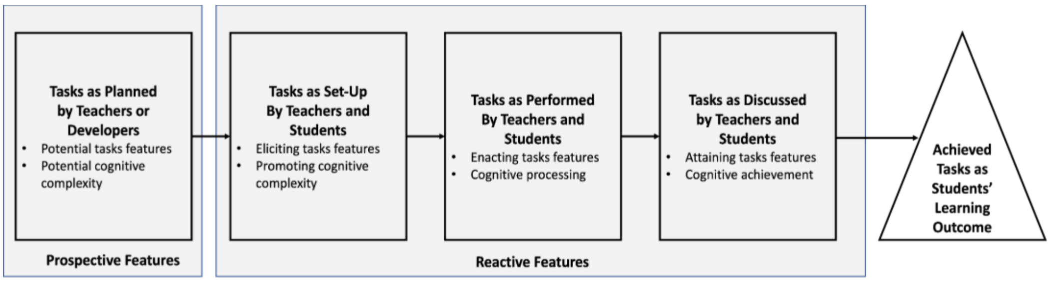


Figure 1. The Enactment Process of Mathematical Tasks (adapted from Henningsen and Stein,1997)
Note. The shaded box represents the enactment process of mathematical tasks.

We specified major dimensions of enactment process of mathematical tasks: the tasks features and cognitive complexity. The tasks features refer to "aspects of tasks that mathematics educators have identified as important considerations for the development of mathematical understanding, reasoning, and sense making" (Henningsen & Stein, 1997, pp. 528-529). This dimension includes multiple solution strategies, multiple representations, and mathematical communication. Cognitive complexity refers to "the kind of thinking processes entailed in solving the task" (Stein et al., 1996, p. 461). These features will be detailed and exemplified in the later sections.

Mathematical Task Analytic Framework

As explained in the previous section, we extended our definition of mathematical tasks incorporating assigned tasks (Tasks as set-up) and enacted tasks (Tasks as performed and discussed). This requires revision of the characteristics of mathematical tasks. Here, we propose MTAF (Table 1). Building on two dimensions of tasks (tasks features and cognitive complexity) by Henningsen and Stein (1997), we specified the dimensions of task enactment into five domains. Specifically, the cognitive domain was specified with breadth and depth of mathematical ideas and the task features were specified as bridging, openness, and interaction. This comprehensive framework is to guide, analyze, and reflect upon from the assignment to enactment of mathematical tasks. Compared to the previous task frameworks, The MATF is more comprehensive to analyze multi-facet characteristics of mathematical tasks and would provide a lens to analyze planned tasks for mathematics instruction (proactive) implemented tasks during instruction (reactive) at the same time.

Table 1. Mathematical Task Analytic Framework

Dimensions	Components	Description
Breadth (National Research Council, 2001; Li, 2000; Son & Senk, 2010)	Knowledge of Concepts	Articulating mathematical concepts and underlying conceptual understanding
	Knowledge of Procedures	Explaining routine procedures or invented strategies
Depth (Stein et al., 1996; Stein et al., 2000)	Memorization	Recalling facts or formulas
	Procedure without Connection	Using procedures without connecting to underlying meanings
	Procedure with Connection	Using procedure for deeper understanding of concepts
	Doing Mathematics	Focusing on non-algorithmic thinking and exploring mathematics
Bridging (Clarke & Roche, 2018; Kisker et al., 2012)	Mathematical Connection	Connecting other mathematical knowledge including students' prior knowledge
	Contextualization	Connecting students' interests and or contextualized experiences
Openness (Watson & Ohtani, 2015; Yeo, 2017)	Multiple Entries	Providing open choices for essential information
	Multiple Strategies	Providing open choice for solution strategies
	Multiple Solutions	Providing a chance to have different answers depending on problem conditions
Interaction (Cohen et al., 2003; Herbst & Chazan, 2012; Lampert, 2001)	Teacher-Student	Communicating between a teacher and students to support and advanced mathematical thinking
	Student-Student	Communicating between students to develop shared understanding

Breadth of Mathematical Ideas

The breadth of the mathematical ideas indicates the types of mathematical knowledge students use when carrying out mathematical tasks. We synthesize cognitive expectation and mathematical proficiency (National Research Council, 2001; Li, 2000; Son & Senk, 2010) and categorize them into two components: knowledge of concepts and knowledge of procedures.

Knowledge of concepts is about what and how students articulate underlying mathematical concepts and explain the meaning of a mathematical concept or operation without any computation. For example, suppose students are asked to explain the meaning of addition or multiplication. Note that this knowledge includes any conceptual understanding which makes relational understanding by connecting previous knowledge (Hiebert & Wearne, 1993). Knowledge of procedures regards students' use of routine procedures or invented algorithms without justifying each step. For example, students might recall the addition algorithm of two-digit numbers by lining up two numbers and adding numbers that share the same place value.

Depth of Mathematical Ideas

Regarding the depth of mathematical ideas, we draw on the cognitive demand (Stein et al., 1996; Stein et al., 2000). Stein and her colleagues defined four levels of cognitive demand: memorization, procedures without connections, procedures with connections, and doing mathematics. Tasks that require a lower level of thinking and reasoning are described as memorization (reproduce previously learned facts and have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definition being learned) or procedures without connections (use a procedure or algorithms without connecting mathematical concepts and have no connection to the concepts or meaning that underlie the procedure being used). In the meantime, a higher level of thinking imposes procedures with connections (meaningfully use a procedure connected with mathematical understandings or concepts and make connections among multiple representations helps to develop meaning) or doing mathematics (explore complicated and non-algorithmic pathways as solutions and require students to explore and understand the nature of mathematical concepts, processes, or relationships). Students would gain both conceptual and procedural knowledge through the higher-level tasks (Stein et al., 1996; Stein et al., 2000). The depth with the level of cognitive demand is an important dimension when considering assigned and enacted tasks because it affects students' exploration of mathematical concepts.

Bridging

Bridging is about connecting various mathematical knowledge or students' contexts. In this dimension, we consider mathematical connection and contextualization. Even though students work with the same task in the same classroom, their learning varies due to their different background such as prior knowledge and experience. Students' prior mathematical knowledge, mathematical learning experiences, and connected other mathematical topics are important considerations in designing a rich task for students' deep understanding (NCTM, 2000). In addition, contextualized and authentic mathematical tasks engage students in more meaningful learning (Clarke & Roche, 2018). This contextual connection helps students make sense of the tasks authentically by connecting individual cultural backgrounds (in or out of classrooms) and access to targeted mathematical content (Civil & Andrade, 2002; Kisker et al., 2012).

Openness

The openness of a task is a key factor for successful problem-solving that students have their agency to develop various solution strategies and to discuss them, rather than drilling a particular solution (Watson & Ohtani, 2015). To provide multiple open choices to students, there are three subcategories of openness (Yeo, 2017): multiple entries, multiple strategies, and multiple solutions. Rich tasks make students look for multiple entry points, to use multiple pathways to solve them, and often to have more than one possible answer. (Drake et al., 2015). Multiple entry tasks provide open access to the same mathematics to all learners by adjusting the levels of cognitive difficulty. Mathematical tasks that make students use multiple strategies, enhance their ability to solve problems, to justify their statement, and to think mathematically by comparing various representations and strategies. Tasks that require students to create a situation to meet certain conditions entail multiple solutions. This type of task provokes students' deeper understanding.

Interaction

In mathematics classrooms, discursive interactions occur between the teacher and his/her students and amongst students (Cohen et al., 2003; Herbst & Chazan, 2012; Lampert, 2001). In other words, it should be noted the interactions that focused on the current study are not only teacher-student but also student-student conversations that concern the content of the subject domain. From task centric perspective, instructional tasks being enacted become the context of such interaction (Ni et al., 2014).

Teacher-Student interaction refers to teachers' dialogic efforts including questioning for in-the-moment responses, asking students mathematical meaning and justification to develop mathematical thinking (Kazemi & Stipek, 2009), and asking them to orient towards other students' mathematical ideas (Shaughnessy et al., 2021). This interaction can be initiated by a student's posed questions as well. Student-Student interaction can be used to support what has been called as knowledge building by Scardamalia and Bereiter (2006) or a process that involves creative and sustained work with ideas. When building knowledge, students work collaboratively to improve shared ideas and to extend the frontiers of public knowledge.

III. METHODS

Context

We used MTAF to assess a sequence of lessons that was implemented by a teacher-researcher (one of the authors) in an elementary school in South Korea. The teacher taught two days per a week in third-grade mathematics and the duration of each lesson was two consecutive class periods. Since the teacher had continuously demonstrated his effort in local schools to bridge between practices and theory (Kim, 2020; 2021), it was expected to draw fully the potentials of mathematical tasks through interactions with students. The third-grade elementary students who participated in this study consisted of 12 male and 12 female students. The students were expected to have informal and formal knowledge about the multiplication based on previous learning opportunities or daily life experience.

Mathematical Tasks

In this paper, we examined the whole number multiplication tasks, "The Number of Students in Our School". The goal of the tasks was to elicit students' multiplicative reasoning when figuring out "a total number of students". The tasks consisted of three phases (Table 2): (a) Task 1 in which students were to find the total number of grade 3 students in their school (the number of students in each of three classes is fixed as 24), (b) Task 2 in which students were to find the total number of students from grades 1 to 3 in their school (the classroom size is different from class to class: either 23 or 24 students per class), and (c) Task 3 in which students were to find the number of total students in their own made-up schools (the classroom sizes are selected freely by the students).

Table 2. The Number of Students in Our School Tasks

Task 1	Task 2	Task 3
How many students are in Grade 3? Please solve as many as possible strategies and explain how you solve the problem. *Given information* Grade 3: 24, 24, 24	How many students are in Grade 1 to 3? Please solve as many as possible strategies and explain how you solve the problem. *Given information* Grade 1: 25, 24, 24 Grade 2: 25, 24, 25 Grade 3: 24, 24, 24	How many students are in your imaginary school? Please explain how you solve the problem and exchange the problem with your peers. *Given information* The number of students varies.

In Task 1, the teacher elicited necessary information to find the total number of 3rd grade students from their prior grade levels' experience. The elementary school being presented was a lab school which had 24 students (12 boys and 12 girls) in most classrooms and three classes at each grade level. The students were expected to find the total number of third grade students by using various ways such as addition (e.g., 24+24+24, 20+4+20+4+20+4) or multiplication (e.g., 3 × 24). In Task 2, the instructor expanded the number of students to three grade levels (three classes per grade). Unlike 3rd grade, first grade and second grade did not have the unified number of students: In grade 1, there are 25, 24, and 24 students; in grade 2, there are 25, 24, and 25 students, and in grade 3, there are 24, 24, and 24 students in each class. Due to two equal size of groups (24 and 25), the students were expected to use a wider spectrum of strategies such as repeated additions (e.g., 25 + 24 + 24 + 25 + 24 + 25 + 24 + 24 + 24), grouping numbers with multiplication (e.g., 3 × 25 + 6 × 24), and decomposing and recomposing numbers (e.g., 9 × 24 + 3). Lastly, in Task 3, the students were asked to use only two numbers (e.g., 24 and 26) to make up their own schools with various classroom sizes of grades 1 to 3. This last phase gave students flexible opportunities to consider various mathematical strategies as well as solutions. It is notable that students were able to build their own problems in which the solutions could be different from everyone else. We found this series of tasks was effective to illustrate our developed framework because the features of each phase of the tasks are clearly diverse despite the same content orientation of multiplicative reasoning and the instructor used the same types of teaching practices across the lessons that might minimize the effect of the instructor.

Data Source

As the data sources, this study collected video recording of the lessons with two video cameras. One camera recorded the whole classroom from the front of the classroom and the other captured from the teacher's perspective. The duration of each lesson was about 80 minutes and all video recordings were transcribed. In addition, we collected students' written responses during the lessons as artifacts.

Data Analysis

Based on MTAF, we develop a scoring rubric to evaluate the quality of task (Table 3). Raters, four authors except the instructor and the last author, assigned a score for each component of MTAF on a scale of 1-3. We set up intentional hierarchy for each dimension. To aim effective mathematical learning (NCTM, 2014), we stratified the quality of components with various combinations. Across rubrics, 3 refers to high-quality features, 2 refers to medium-quality features of the component, and 1 refers to low-quality features or the absence of the component. To evaluate proactive features, the raters initially evaluate the potential of each written task prior to watching the recorded videos. Then, the four authors evaluated reactive features, by watching the recorded videos and finding evidence for each component of the rubric. During this iterative coding process, each rater coded individually first then consolidated any disparity to make consensus scores through discussions.

Three lessons are not enough for quantitative analysis, but the goal of this study is not to generalize the use of the MTAF rubric. Instead, we qualitatively analyzed scores and provided specific examples from the lessons to describe how the MTAF can support effective implementation of mathematical tasks.

Table 3. Scoring Rubric for Task Enactment

Dimensions	Score	Descriptions
Breadth	3	Focusing on knowledge of various concepts and procedures.
	2	Focusing on either knowledge of concepts or knowledge of procedures.
	1	Focusing on a single concept or procedure.
Depth	3	Focusing on non-algorithmic thinking and exploring mathematics.
	2	Using procedure for deeper understanding of concepts.
	1	Recalling facts or requiring algorithms.
Bridging	3	Extending mathematical ideas and connecting individual cultural backgrounds.
	2	Extending mathematical ideas or connecting individual cultural backgrounds.
	1	Providing routine problems with superficial mathematical structures.
Openness	3	Providing multiple entry points, strategies, and solutions altogether.
	2	Providing multiple entry points, strategies, or solutions.
	1	Providing none of entry points, strategies, and solutions.
Interaction	3	Providing co-building knowledge through engaged participation and discussions.
	2	Providing guided discursive interactions by teachers.
	1	Providing teacher-centered interactions.

IV. FINDINGS

Table 4 shows MTAF scores for the three tasks. Overall, the reactive scores were higher than the proactive score, which indicates the features of mathematical tasks implemented by the teacher and students with higher quality than the original potentials. In addition, the later task showed better quality in both proactive and reactive dimensions. In the following section, we explain how the tasks are analyzed for each dimension and component of the framework, evidenced by the potential of the tasks and the teacher's and students' actual responses from the recorded videos.

Table 4. MTAF Scores for Each Task Enactment

	Task 1		Task 2		Task 3	
Dimension	Proactive	Reactive	Proactive	Reactive	Proactive	Reactive
Breadth	2	3	2	3	2	3
Depth	2	1	2	2	3	3
Bridging	1	2	1	2	3	3
Openness	2	2	2	2	3	3
Interaction	2	3	2	2	2	3

Breadth

The three tasks were expected to elicit more procedure knowledge about repeated addition or multi-digit multiplication. For this reason, the scores for proactive features were 2 (see Table 5). On the other hand, during implementation, students shared not only their conceptual knowledge such as the meaning of multiplication and properties of addition and multiplication (the bold fonts in Table 5) but also additional procedural knowledge about one-digit by one-digit multiplication. Thus, all reactive features score 3.

Table 5. Breadth of Mathematical Idea

	Breadth	Task 1	Task 2	Task 3
Proactive	Knowledge of Concepts	-	-	-
	Knowledge of Procedures	One-digit and two-digit addition Two-digit by one-digit multiplications	Two-digit and three-digit addition Two-digit by one-digit multiplications	Two-digit and three-digit addition Two-digit by one-digit, two-digit by two-digit multiplications
Reactive	Knowledge of Concepts	**Multiplication as equal groups** **Commutative property of multiplication**	**Multiplication as equal groups** **Commutative property of multiplication** **Commutative and associative property of addition**	**Multiplication as equal groups** **Commutative property of multiplication** **Commutative and associative property of addition**
	Knowledge of Procedures	One-digit and two-digit addition Two-digit by one-digit multiplications	Two-digit and three-digit addition **One-digit by one-digit multiplications** Two-digit by one-digit multiplications	Two-digit and three-digit addition **One-digit by one-digit multiplications** Two-digit by one-digit multiplications Two-digit by two-digit multiplications

Note that the bold font only appeared in the reactive analysis.

In Task 1, many students employed standards algorithms for repeated addition of two-digit numbers or multiplication of one-digit by two-digit (*Knowledge of Procedures*) and some of them also explained the fundamental meaning of multiplication ("adding 24 three times is equal to 3 times 24 or 3 groups of 24."). In Task 2, the students were asked to figure out the total number of students in Grades 1 to 3. Recall that the number of students in each 3rd grade class is 24 equally in the previous task. In Task 2, however, the student numbers in the classes of Grades 1 and 2 were either 24 or 25. The students were able to conceptualize clearly the meaning of multiplication as equal groups and explore some properties of multiplication such as commutative property (*Knowledge of Concepts*). Some students combined addition and multiplication (*Knowledge of Procedure*). For example, Hojun re-arranged the numbers of all nine classes as 24 × 3 + 24 × 3 + 25 × 3 = 72 + 72 + 75 = 144 + 75 = 219 (Figure 2-left). Others used multi-step two-digit additions. For example, Siyoung decomposed 25 into 24 and 1 in her head. Then she calculated the sum of three 24s for each grade (24 + 24 + 24 =72 for each grade). Since there were three grades in each three classes, she added 72 three times (72 + 72 + 72 = 216). Because there were three classes of 25 students, she knew that there were three 1s left after using 24s from those three 25s. She added 3 to 216 and answered 219 as the total number of 1st to 3rd graders (Figure 2-right).

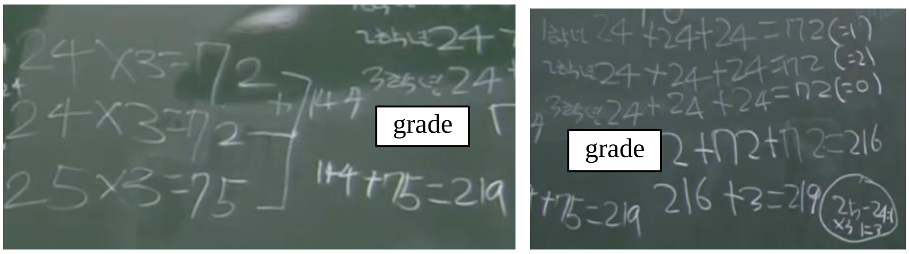


Figure 2. Hojun's Multiplication Strategy (left) and Siyoung's Addition Strategy (right)

In Task 3, the students calculated the number of students by making up their own schools. Students demonstrated the components of Breadth more dynamically. Jiwoo, for an instance, created a table to present the number of students in her school problem (Figure 3-left). When finding the total number of students, she decomposed each number by tens and ones (20 + 6 and 20 + 4). Because she made 5 classes in each 3 grades, there were total 15 classes—11 classes with 26 students and 4 classes with 24 students. She calculated 20 × 15 = 300, 6 × 11 = 66, and 4 × 4 = 16, then added the products to find the total number of students, 382 (Figure 3-right). Such strategy included various mathematical properties such as associative property of addition and commutative property of addition (*Knowledge of Concepts*). The task also elicited students' *Knowledge of Procedures*. Some students used additions of two-digit (e.g., 90 + 90 + 90) or three-digit (e.g., 270 + 297) as well as multiplications of two-digit by two-digit (e.g., 20 × 15), two-digit by one-digit (e.g., 6 × 11), and one-digit by one-digit (e.g., 4 × 4).

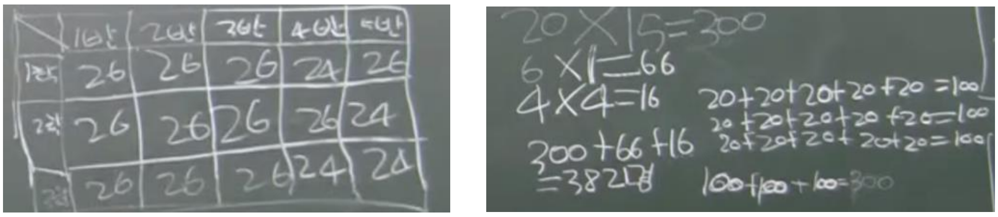


Figure 3. Jiwoo's Imaginary School (left) and Solution Strategy (Right)

Depth

The three tasks were expected to follow the same procedure without connection as assigned tasks, which scores 1 (Table 6). However, the three tasks actually showed different levels of the depth of mathematical ideas during implementation.

Table 6. Depth of Mathematical Idea

	Depth	Task 1	Task 2	Task 3
Proactive	Memorization	Multiplication facts	-	-
	Procedure without Connection	Repeated addition Multiplication algorithm	Repeated addition Multiplication algorithm	Repeated addition Multiplication algorithm
	Procedure with Connection	-	-	-
	Doing Mathematics	-	-	-
Reactive	Memorization	Multiplication facts	-	-
	Procedure without Connection	Repeated addition Multiplication algorithm		
	Procedure with Connection	-	**Understand the conception of multiplication with iterate #s to use**	-
	Doing Mathematics	-	-	**Non-algorithmic, complex with diverse numbers, decide to choose appropriate #s**

Note that the bold font is only appeared in the reactive analysis.

In Task 1, as an initial step for set-up, students began with recalling basic multiplication facts such as 2 × 3 and they were asked the meaning of this expressions, "Three groups of two[1]" (*Memorization*). Students often solved the launched task using standard algorithms of addition and multiplication. Shown in the other two phases, students had abilities to perform beyond standard algorithms such as decomposing two-digit numbers. However, the majority of the students simply used the algorithms in this phase. They had little opportunity to make connections between other representations or relevant

[1] 2 × 3 represents two groups of three in the US and other western countries, while it represents three groups of two in South Korea.

mathematical concepts (*Procedure without Connection*: score 1).

In Task 2, students' cognitive demand of the task increased, because the students reorganized two different numbers to make them easier to add or multiply. In doing this, the students focused on the concept of multiplication, identifying the number of classes with the same number of students as one factor (number of groups) and the number of students in the classrooms as the other factor (amount in each group) at the same time (*Procedure with Connection*: score 2).

When posing a problem situation in Task 3, students bore in mind a multiplicative structure with two repeated numbers. This task further required students' self-monitoring to manage structure with their own cognitive processes (Singer et al., 2017). The posed tasks were not predictable due to the nature of flexible number combinations and might be not easy to solve with only standardized ways. Therefore, students should understand how to use multiplication concepts and relationships between addition and multiplication in representing the context with mathematical expressions (*Doing Mathematics*: score 3).

Bridging

Table 7 below shows how the three tasks drew other mathematical topics including students' prior mathematical knowledge and their situated context. Evaluating assigned tasks, Tasks 1 and 2 were considered as traditional routine story problems (score 1) and Task 3 was considered to be available to extend students' prior knowledge (score 3). Since students had learned two-digit addition, one-digit multiplication, it was easy to anticipate they would use relevant arithmetic knowledge of procedures in the domain of Number and Operations. However, during implementation, the students utilized their informal knowledge of halving and decomposition.

Table 7. Bridging to Prior Knowledge and Context During the Enactment of the Tasks

	Bridging	Task 1	Task 2	Task 3
Proactive	Mathematical Connection	Addition Multiplication	Addition Multiplication	Addition Multiplication
	Contextualization	Routine problem	Routine problem	Imaginary school
Reactive	Mathematical Connection	Addition Multiplication **Halving**	Addition Multiplication **Decomposition**	Addition Multiplication **Decomposition**
	Contextualization	**Current grade level**	**Current school**	Imaginary school

Note that the bold font is only appeared in the reactive analysis.

To be specific, in Task 1, the students employed a halving strategy to decompose the given number 24 into 12 and 12. This strategy is shown only in this task. In the later tasks, students decomposed two-digit numbers into tens and ones (e.g., 20 + 4).

Regarding contextualization, the first two tasks were situated in their own school setting, which the raters did not anticipate. The elementary school was affiliated with a regional college of education and students in the elementary school employed lottery for entering and the total number of students should not exceed a certain amount of capacity. In addition, each classroom had a fixed number of students, 24 (12 boys and 12 girls) in Grade 3, but this fixed number was slightly varied in Grades 1 and 2. Because of the authenticity and relevance of the tasks, students were motivated and engaged in solving the problems. Students might not have considered asking this type of mathematical inquiry (i.e., the number of total students) within their context. Therefore, Bridging for the first two tasks scores 2. In Task 3, students are immersed in the task that they created. This posing new context stemmed from their own culture. When solving the problems posed by themselves, students were more actively engaged cognitively (score 3).

Openness

Interestingly, the dimension of openness showed the match scores for both proactive and reactive features. In the first two tasks, since the students used the designated numbers to carry out the tasks, they were expected to use specific information. There was not much room for the beginning and end of problem solving (see Table 8). In other words, the tasks during these first two tasks did not provide students opportunities for multiple entries or multiple solutions. Instead they were able to use multiple strategies including direct modeling, arithmetic operations (e.g., addition and multiplication), and regrouping. This is the reason why Tasks 1 and 2 scored 2 for both proactive and reactive features.

Table 8. Openness During the Enactment of the Tasks

	Openness	Task 1	Task 2	Task 3
Proactive	Entries	Same information	Same information	Different information
	Strategies	Multiple ways	Multiple ways	Multiple ways
	Solutions	Single answer	Single answer	Multiple answers
Reactive	Entries	Same information	Same information	Different information
	Strategies	Multiple ways	Multiple ways	Multiple ways
	Solutions	Single answer	Single answer	Multiple answers

Similarly, in Task 3, students generated the numbers of students for each classroom in several grades (different information). Some students had three classes, while others had five. Most students chose two-digit numbers (11 to 99) for the class sizes. In this phase, students developed diverse strategies even without the teacher's directions or support. Giving students the authorship of the task, this phase yielded multiple conditions, strategies, and answers (score 3).

Interaction

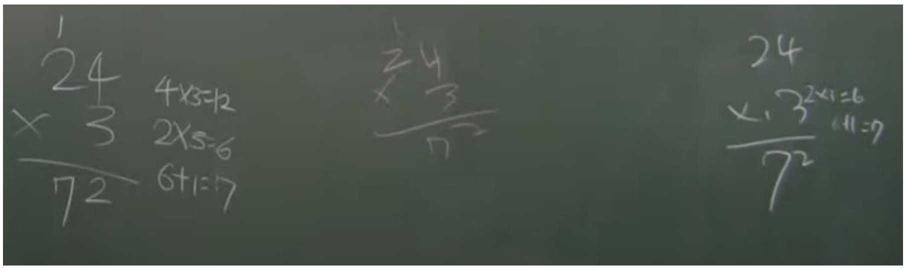
As potential of the three tasks, we expected the tasks could be guided by the teacher for effective classroom discussions (Table 9). The teacher might support elicit foundational multiplication concepts (Task 1) or multiplicative thinking (Tasks 2 and 3). In addition, in Task 3, students were expected to exchange their posed problems. Therefore, all three tasks scored 2 for proactive features.

Table 9. Interactions in Teacher-Student or Student-Student During the Enactment of the Tasks

	Interaction	Task 1	Task 2	Task 3
Proactive	Teacher-Student	Supporting to elicit the conception of multiplication Comparing strategies	Supporting and extending the multiplicative thinking Comparing strategies	Supporting and extending the multiplicative thinking Comparing strategies
	Student-Student	-	-	Solving other's posed problems
Reactive	Teacher-Student	Supporting to elicit the conception of multiplication Comparing strategies **Unpacking procedures**	Supporting and extending the multiplicative thinking Comparing strategies **Unpacking procedures**	Supporting and extending the multiplicative thinking Comparing strategies
	Student-Student	**Questioning other's strategies**	**Questioning other's strategies**	**Questioning other's strategies** Solving other's posed problems

Note that the bold font is only appeared in the reactive analysis.

During implementation, encouraging students to explain the meaning of multiplication and justify their thinking processes, the students were supported to elicit their multiplication conceptions and to develop multiplicative thinking through teacher-student interaction. In Task 1, the students predominantly used either the standard algorithm of multiplication between two-digit and one-digit or repeated addition method. The first three students shared their solution strategies using the standard algorithm (Figure 4).



student 1 student 2 student 3

Figure 4. Using Standard Algorithm for the Fixed Number Phases

Since two students presented the same procedure, $2 \times 3 = 6$ instead of $20 \times 3 = 60$, the teacher asked students to consider the meaning of 2 in 24 by emphasizing the place value. We observed the following interactions between teacher and students to request justification of their solution strategies.

Student 1: 4 times 3 is 12 and 2 times 3 is 6. When you add 1 and 6, you get 7.

Teacher: If you add 12 [from 4×3] and 6 [from 2×3], it would be 18, but you have 72 not 17. Can you explain this?

Students were constantly asked to compare their own strategies and solutions with other students' and to understand them. Not only the teacher-student interaction, but student-student interaction was also prevalent throughout the lessons. The students often had opportunities to evaluate the validity of mathematical statements of their solution strategies through small or whole group discussion. For example, in Task 1, a student presented his halving strategy, $24 \times 3 = (12 + 12) \times 3 = 12 \times 3 + 12 \times 3$ and another student inquired why the distributive property did not work for an addition situation. He restated his question: why $(4 + 3) + 5 \neq (4 + 5) + (3 + 5)$? Without the teacher involved, other students jumped into this conversation and explained using some concrete examples. This was a significant moment that students could build and expand their mathematical knowledge through student-student interaction (score 3). We acknowledged that the nature of various interactions in mathematics instructions depends on facilitation by teachers, but the students could open up new entry points to access mathematical thinking during peer discussions. Furthermore, in Task 3, the students were engaged in other students' invented problems and challenged by solving the problems using their own strategies (score 3).

V. DISCUSSION

Researchers have documented how mathematical tasks can impact the way students interact with mathematical content to build their understanding (Ni et al., 2014; Stein & Smith, 1998; Tekkumru-Kisa et al., 2020). Nonetheless, the limited definition of mathematical tasks — written prompts in the textbooks or assigned by only teachers makes it difficult to track to what extent features of the tasks influence students' actual learning outcomes. In this study, we developed the MTAF, based on a holistic definition of mathematical tasks which includes both the written, intended tasks for the potential learning of mathematics and the process of learning and teaching through the enactment of the tasks. In this study, we applied the scoring rubric to analyze three multiplication tasks to illustrate the MTAF by its five dimensions: breadth of mathematical idea, depth of mathematical idea, bridging, openness, and interaction.

The MATF can provide important information about the qualities of planned tasks for mathematics instruction (proactive) and the qualities of implemented tasks during instruction (reactive). That is, the potentials and realization of tasks can be identified in terms of various task features through the analytic framework. In the case of Task 3, the planned tasks seem to have high level of depth, bridging, and openness (i.e., score 3) and medium level of breadth and interaction (i.e., score 2). However, during the implementation, the focus of classroom discussion oriented toward conceptual and procedural knowledge of multiplication, and students had opportunity to co-construct multiplicative reasoning by solving their peers' posed problems. With related to cognitive demands, research has shown that teachers can change the levels of cognitive demands during lessons through instructional practices (Boston & Smith, 2009). However, our transformative results in the breadth and interaction show various features of mathematics tasks can be also altered by interactions during the instructions. Therefore, teachers are required to begin to value the high-level tasks with multiple features and can be equipped with more professional learning experience of how to maintain or increase the quality of tasks during implementation.

During the implementation, the MATF can evaluate the quality of students' opportunity to learn and engagement in lessons. While the MATF can be used to assess the planned tasks with five components, students' learning practices and what they had learned can be a part of analysis as well when applying the developted framework. About the interaction component, for instance, we anticipated the quality of the written tasks as the medium level in the Tasks 1 (i.e., score 2), which is often guided and facilitated by teachers. However, during the lesson with Task 1, students were engaged in whole-group discussion to discuss whether the distribute property was working in addition. This conversation was initiated by one student and the students were able to build their knowledge of mathematical properties through the discussion (score 3). Therefore, the value of tasks can be interconnected with student learning by analyzing the reactive features of tasks.

The MTAF enables us to specify the ongoing process of how each component of the task features are achieved while developing students' mathematical proficiency. In addition, the framework proves its validity to indicate what elements of the task features correspond to the quality of student mathematical activity by considering both design and implementation of tasks (Henningsen & Stein, 1997). The MTAF also provides a unified lens to bridge a gap between what teachers do in classrooms and how or what students learn in the association with mathematical tasks (Tekkumru-Kia et al., 2020).

The static view of task features often restricts a way of interpreting and conducting mathematical tasks (Liljedahl, 2020). Suppose a teacher adopts a high-level cognitive demand task and provides it to his/her students While we acknowledge higher potential to leverage students' mathematical learning with this task, how can we guarantee that all teachers provide students the same quality of instruction as the task developers intended? Based on the fundamental design of a task, the ways how to maintain the various task features would also play an important role in developing students' understanding in the enactment of the tasks (Berg, 2012). In short, the MTAF supplements existing task frameworks (e.g., van de Walle et al., 2019) by extending the notion of tasks to the entire enactment processes.

Although this study illustrates the MTAF with only three mathematical tasks, future research opportunities exist in various forms. We suggest future research questions including but not limited to: How do teachers (or preservice teachers) use the MTAF to plan their instructions? How can this MTAF effectively be used to evaluate tasks given in the textbooks? How can the MTAF be improved? To what extent does the MTAF contribute for better implementations of a task in an authentic classroom setting? How do teachers' experiences influence their implementation of mathematical tasks in consideration of the tasks by each dimension? Along with other studies that incorporate the MTAF, such future studies will help teachers identify task characteristics in order to develop students' mathematical thinking.

A potential contribution of this study is to understand tasks in a direct association with students' learning throughout the process of task enactment. This can be helpful for curriculum designers to select or construct mathematical tasks with more careful consideration of how each task feature would be attained through teacher and student interactions. In addition, employing the MTAF enhances teachers' selection and implementation of optimal tasks.

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