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MOMENT ESTIMATE AND EXISTENCE FOR THE SOLUTION OF NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper, the existence and uniqueness for the global solution of neutral stochastic functional differential equation is investigated under the locally Lipschitz condition and the contractive condition. The implicit iterative methodology and the Lyapunov-Razumikhin theorem are used. The stability analysis for such equations is also applied. One numerical example is provided to illustrate the effectiveness of the theoretical results obtained.

1. Introduction

Many dynamical systems depend on both present and past states, and involve derivative with delays as well as the functional of the past history [13, 17, 23]. Neutral functional differential equation is often used to describe such systems [1, 14, 22], which is written as

$$\frac{d[x(t) - D(x_t)]}{dt} = f(t, x_t), \quad t \in [t_0, T]. \tag{1}$$

When the equation (1) is perturbed by random external perturbation [14], it can be presented as the following neutral stochastic functional differential equation (NSFDE):

$$d[x(t) - D(x_t)] = f(t, x_t)dt + g(t, x_t)d\mathcal{B}(t), \quad t \in [t_0, T]. \tag{2}$$

NSFDE (2) has some fundamental applications into many important fields such as mechanical, electrical, biological, medical and physical sciences [2, 10, 11, 19, 20] and the references therein. For some other dynamical properties of NSFDE (2), we can refer to [3,4,13,14,23,27]. Recently, the existence and uniqueness of the solution for NSFDE (2) has been developed, many useful results have been

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given in [1, 3, 5–7, 9, 11, 12, 14, 16, 17, 21, 23, 25, 26] and the references therein. For example, in [14], by using the Picard iterative method and the stochastic analysis theory, under the globally Lipschitz condition and the contractive condition, Mao discussed the existence and uniqueness of the solution for NSFDE (2). In [17, 18], when the globally Lipschitz condition is satisfied for the drift term and the diffusion term, and the contractive condition is fulfilled for the neutral term, the existence and uniqueness for NSFDE (2) with infinite delay was analyzed by using the Picard iterative method and the stochastic analysis method. In [1], under the globally Lipschitz condition and the contractive condition, the existence and uniqueness of the solution of NSFDE (2) was investigated by employing the implicit iterative methodology and the stochastic analysis theory.

Note that the results proposed above are applicable for the case, in which the drift term and the diffusion term satisfy the globally Lipschitz condition. However, in many realistic models, highly nonlinear stochastic differential equations usually exist in drift term and the diffusion term, see [3, 7, 9, 13, 18] and the references therein. Compared with stochastic highly nonlinear delay differential equation, highly nonlinear NSFDE is much more complex, due to the simultaneous existence of the neutral term and the stochastic perturbation. Thus, the existence and uniqueness, and stability of highly nonlinear NSFDE is very attracting. For example, in [9], with the drift term and the diffusion term only satisfy the local Lipschtiz condition, the linear growth condition is replaced by the monotonicity condition, and the contractive condition is satisfied for the neutral term, then the existence and uniqueness of the global solution of NSFDE has been investigated. In [13], under the locally Lipschitz condition and the contractive condition, the existence and uniqueness, the exponential stability in moment and the almost surely exponential stability for the global solution of neutral stochastic delay differential equations have been considered by using the Lyapunov function and stochastic analysis theory.

In [14], the stochastic version of Lyapunov-Razumikhin methodology has been firstly established to discuss the exponential stability in moment for NSFDE (2). Based on this excellent work, such useful methodology has been developed to analyze the stability of NSFDE (2), and some good results have been presented in [2, 8, 15] and the references therein. The results in [8, 15] are obtained under the globally Lipschitz condition for the drift term and the diffusion term. In [24], when the drift term and the diffusion term satisfy the locally Lipschitz condition with the locally Lipschitz coefficients being time-varying, the existence and uniqueness, moment estimate for the global solution of stochastic functional differential equations have been investigated by using the Lyapunov-Razumikhin theorem and stochastic analysis theory. To our knowledge, there is few work on the existence and uniqueness, moment estimate for the global solution of NSFDE under the locally Lipschitz condition for the drift term and the diffusion term.

In this paper, we will mainly discuss the existence and uniqueness, moment estimate for the global solution of NSFDE, when the drift term and the diffusion term satisfy the locally Lipschitz condition with the locally Lipschitz coefficients being time-varying. The implicit iterative methodology, the stochastic analysis theory and the Lyapunov-Razumikhin theorem are used. The exponential stability for the global solution of NSFDE will also be investigated. One numerical example is given to show the effectiveness of the theoretical results derived.

2. Preliminaries

In this section, let us recall some notations and basic definitions, and introduce some lemmas. $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq t_0}, \mathcal{P})$ represents a completed probability space with the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual condition (i.e., it is right continuous and increasing while $\mathcal{F}_{t_0}$ contains all $\mathcal{P}$-null sets). Let $\mathcal{B}(t)$ be an m-dimensional Brownian motion defined on this probability space, i.e., $\mathcal{B}(t) = (\mathcal{B}_1(t), \mathcal{B}_2(t), \ldots, \mathcal{B}_m(t))^T$. '$E$' denotes the mathematical expectation. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$ or the trace norm of a matrix in $\mathbb{R}^{n\times m}$. If A is a matrix or vector, then its transpose is denoted by A^T. Here let $\mathcal{L}^1([a,b];\mathbb{R}^n)$ be the family of $\mathbb{R}^n$-valued and $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b |f(t)|dt \leq \infty$(a.s.). $\mathcal{L}^p([a,b];\mathbb{R}^{n\times m})$ $(p \geq 2)$ means the family of $\mathbb{R}^n$-valued and $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b |f(t)|^p dt \leq \infty$(a.s.). To avoid confusion, if A is a matrix, its Frobenius norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\mathcal{C}([t_0-\tau, t_0];\mathbb{R}^n)$ be the family of all bounded continuous functions φ from $[t_0-\tau, t_0]$ to $\mathbb{R}^n$ equipped with the norm $\|\varphi\| = \sup_{t_0-\tau\leq\theta\leq t_0} |\varphi(\theta)|$. Let $\mathcal{C}^p_{\mathcal{F}_{t_0}}([t_0-\tau, t_0];\mathbb{R}^n)$ denotes the family of all $\mathcal{F}_{t_0}$-measurable, bounded and $\mathcal{C}([t_0-\tau, t_0];\mathbb{R}^n)$-valued random processes $\phi = \{\phi(\theta) : t_0-\tau \leq \theta \leq t_0\}$ with $E\|\phi\|^p < \infty$ $(p \geq 2)$. For $a, b \in \mathbb{R}$, $L(\cdot) \in \mathcal{H}^1([a,b];X)$ means that $\int_a^b L(u)du < \infty$, where $X = \mathbb{R}$ or $X = [0,+\infty)$, and $a \vee b$ denotes the maximum of a and b. It is assumed that the function $L(t) : [a,b] \to \mathbb{R}_+$ to be finite if $L(t) < \infty$ for any $t \in [a,b]$.

Now, we consider one n-dimension NSFDE:

$$d[x(t) - D(x_t)] = f(t, x_t)dt + g(t, x_t)d\mathcal{B}(t), \quad t \geq t_0, \tag{3}$$

with the initial value $x_{t_0} = \xi \in \mathcal{C}([t_0-\tau, t_0];\mathbb{R}^n)$. The neutral term $D(\cdot) : \mathcal{C}([t_0-\tau, t_0];\mathbb{R}^n) \to \mathbb{R}^n$, the drift term $f(\cdot,\cdot) : [0,T] \times \mathcal{C}([t_0-\tau, t_0];\mathbb{R}^n) \to \mathbb{R}^n$, and the diffusion term $g(\cdot,\cdot) : [0,T] \times \mathcal{C}([t_0-\tau, t_0];\mathbb{R}^n) \to \mathbb{R}^{n\times m}$ are all Borel measurable.

Let $C^{1,2}([t_0,+\infty)\times\mathbb{R}^n;(0,+\infty))$ denote the family of all functions $V(t,x)$ on $\mathbb{R}\times\mathbb{R}^n$, which are once continuously differentiable in t and twice in x. For each $V(t, x-D(\xi)) \in C^{1,2}([t_0,+\infty)\times\mathbb{R}^n;(0,+\infty))$, an operator $\mathcal{L}V(t, x-D(\xi))$ of NSDFE (3) is given by

$$\mathcal{L}V(t, x-D(\xi)) = V_t(t, x-D(\xi)) + V_x(t, x-D(\xi))f(t,\xi)$$

$$+\frac{1}{2}\text{trace}[g^T(t,\xi)V_{xx}(t,x-D(\xi))g(t,\xi)],$$

where

$$V_t(t,x)=\frac{V(t,x-D(\xi))}{\partial t},$$
$$V_x(t,x)=\left[\frac{\partial V(t,x-D(\xi))}{\partial x_1},\frac{\partial V(t,x-D(\xi))}{\partial x_2},\ldots,\frac{\partial V(t,x-D(\xi))}{\partial x_n}\right],\text{ and}$$
$$V_{xx}(t,x-D(\xi))=\left[\frac{\partial^2 V(t,x-D(\xi))}{\partial x_i\partial x_j}\right]_{n\times n}.$$

Let B_T denote the family of all stochastic processes $\xi(t,\omega):[t_0-\tau,T]\times\Omega\to\mathbb{R}^n$, in which $\xi(t,\omega)$ is measurable for each fixed $t\in[t_0-\tau,T]$, and is bounded continuous in t for a.e. fixed $\omega\in\Omega$. For any $\varphi\in B_T$,

$$\|\varphi(t)\|_{B_T}=\{E(\sup_{t\in[t_0,T]}|\varphi(t,\omega)|^p)\}^{\frac{1}{p}},\quad p\geq 2.$$

It can be proved that B_T is a Banach space with its norm $\|\cdot\|_{B_T}$ (see [5]).

For NSFDE (3), the solution is written as the integral form: for any $t\geq t_0$,

$$x(t)-D(x_t)=x(t_0)-D(x_{t_0})+\int_{t_0}^t f(s,x_s)\,ds+\int_{t_0}^t g(s,x_s)\,d\mathcal{B}(s),\tag{4}$$

and $x_{t_0}=\xi\in\mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$.

Definition ([14])**.** An $\mathbb{R}^n$-valued stochastic process $x(t)$ on $t_0-\tau\leq t\leq T$ $(T>t_0)$ is called a solution of NSFDE (3) with initial data $x_{t_0}=\xi=\{\xi(\theta):-\tau\leq\theta\leq 0\}\in\mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$ if it has the following properties:

(i) it is continuous and $\{x_t\}_{t\in[t_0,T]}$ is $\mathcal{F}_t$-adapted;

(ii) $\{f(t,x_t)\}\in\mathcal{L}^1([t_0,T];\mathbb{R}^n)$ and $\{g(t,x_t)\}\in\mathcal{L}^p([t_0,T];\mathbb{R}^{n\times m})$ $(p\geq 2)$;

(iii) $x_{t_0}=\xi$ and (4) holds for all $t\in[t_0,T]$.

A solution $x(t)$ is said to be unique if other solution $\overline{x}(t)$ is indistinguishable from it, that is

$$P\{x(t)=\overline{x}(t)\text{ for all }t_0-\tau\leq t\leq T\}=1.$$

Definition ([14])**.** Let $x(t)$, $t\in[t_0-\tau,\sigma_\infty)$ be a continuous $\mathcal{F}_t$-adapted $\mathbb{R}^n$-valued local process, where σ_∞ is a stopping time. It is called a local solution of equation (3) with initial data $x_{t_0}=\xi=\{\xi(\theta):-\tau\leq\theta\leq 0\}\in\mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$, if $x_{t_0}=\xi$ and

$$x(t\wedge\sigma_k)-D(x_{t\wedge\sigma_k})=\xi(0)+\int_{t_0}^{t\wedge\sigma_k}f(s,x_s)ds+\int_{t_0}^{t\wedge\sigma_k}g(s,x_s)d\mathcal{B}(s),\quad\forall t\geq t_0,$$

holds for any $k\geq 1$, where $\{\sigma_k\}_{k\geq 1}$ is a nondecreasing sequence of finite stopping time such that $\sigma_k\uparrow\sigma_\infty(a.s.)$. Furthermore, if $\limsup_{t\to\sigma_\infty}|x(t)|\geq\limsup_{k\to\infty}|x_k(\sigma_k)|=\infty$ is satisfied whenever $\sigma_\infty<\infty$, it is called a maximal local solution and σ_∞ is called the explosion time. A maximal local solution

$x(t),\ t \in [t_0 - \tau, \sigma_\infty)$ is said to be unique if for any other maximal local solution $\hat{x}(t),\ t \in [t_0 - \tau, \hat{\sigma}_\infty)$, we have $\sigma_\infty = \hat{\sigma}_\infty$ (a.s.) and $x(t) = \hat{x}(t)$ for all $t \in [t_0 - \tau, \hat{\sigma}_\infty)$ (a.s.).

Lemma 2.1 ([14]). *For any $a, b \geq 0$, $p \geq 2$ and $\varepsilon \in (0, 1)$, we have*

$$|a + b|^p = [1 + \varepsilon^{\frac{1}{p-1}}]^{p-1}\left(|a|^p + \frac{|b|^p}{\varepsilon}\right).$$

Lemma 2.2 ([14]). *Let $p \geq 2$. Then there exist universal positive constant c_p (depending only on p) and $g \in \mathcal{L}^p([t_0, T; \mathbb{R}^{n\times m})$ such that $E\int_{t_0}^T |g(s)|^p ds < \infty$. Then*

$$E\left(\sup_{t_0 \leq t \leq T} |\int_{t_0}^t g(s)d\mathcal{B}(s)|^p\right) \leq c_p E\int_{t_0}^T |g(s)|^p ds,$$

where $c_p = \left(\frac{p^3}{2(p-1)}\right)^{p/2}(T - t_0)^{\frac{p-2}{2}}$.

3. Main result

In this section, the existence and uniqueness for the global solution of NSFDE (3) is firstly investigated. Before conducting, we define an operator:

$$(5) \qquad (\Pi x)(t) := \begin{cases} D(x_t) + x(t_0) - D(x_{t_0}) + \int_0^t f(s)ds \\ + \int_0^t g(s)d\mathcal{B}(s), & t \in [t_0, T], \\ \xi \in \mathcal{C}([t_0 - \tau, t_0]; \mathbb{R}^n), & t \in [t_0 - \tau, t_0]. \end{cases}$$

Under the condition (7) (see below), it can be proved that the operator $\Pi : B_T \to B_T$ has a unique fixed point. The detailed proof can refer to Lemma 3.4 in [1].

Lemma 3.1. *Assume that there exist a finite function $L(t) \in \mathcal{H}^1([t_0, T]; [0, +\infty))$ and constant $\kappa \in (0, 1)$ for $p \geq 2$ such that*

$$(6) \qquad |f(t, \varphi) - f(t, \phi)| \leq L(t)\|\varphi - \phi\|,\ |g(t, \varphi) - g(t, \phi)|^p \leq L(t)\|\varphi - \phi\|^p,$$

$$(7) \qquad |D(\varphi) - D(\phi)| \leq \kappa\|\varphi - \phi\|,$$

for all $t \in [t_0, T]$, $\varphi, \phi \in \mathcal{C}([t_0 - \tau, t_0]; \mathbb{R}^n)$ with

$$(8) \qquad D(0) = 0, \quad f(t, 0) \in \mathcal{L}^1([t_0, b]; \mathbb{R}), \quad g(t, 0) \in \mathcal{L}^p([t_0, T]; \mathbb{R}^{n\times m}).$$

Then the equation (3) has a unique solution $x(t)$ with the initial value $x_{t_0} = \xi \in \mathcal{C}([t_0 - \tau, T]; \mathbb{R}^n)$.

Proof. Based on the operator (5), we can define the following implicit iterative scheme:

$$
\begin{aligned}
&x^n(t) = \xi(t), \quad t \in [t_0 - \tau, t_0], \quad n = 0, 1, 2, \ldots, \\
&x^0(t) = \xi(0), \quad t \in [t_0, T], \\
&x^n(t) = D(x_t^n) + x(t_0) - D(x_{t_0}) + \int_{t_0}^{t} f(s, x_s^{n-1}) ds \\
&\qquad + \int_{t_0}^{t} g(s, x_s^{n-1}) d\mathcal{B}(s), \quad t \in [t_0, T], \quad n = 1, 2, \ldots.
\end{aligned} \tag{9}
$$

Denote

$$
J_{n-1}(t) = x(t_0) + \int_{t_0}^{t} f(s, x_s^{n-1}) ds + \int_{t_0}^{t} g(s, x_s^{n-1}) d\mathcal{B}(s), \;\; t \in [t_0, T].
$$

Applying (6), we have

$$
\begin{aligned}
|f(t,\varphi)|^p &= |f(t,0) + f(t,\varphi) - f(t,0)|^p \\
&\le 2^{p-1}|f(t,0)|^p + 2^{p-1}|f(t,\varphi) - f(t,0)|^p \\
&\le 2^{p-1}|f(t,0)|^p + 2^{p-1}(L(t))^p\|\varphi\|^p,
\end{aligned} \tag{10}
$$

and

$$
|g(t,\varphi)|^p \le 2^{p-1}|g(t,0)|^p + 2^{p-1}L(t)\|\varphi\|^p. \tag{11}
$$

<u>Existence</u>: Apparently, we have $x^0(t) \in B_T$, $t \in [t_0, T]$. Moreover, we can prove that $x^n(t) \in B_T$ $(n = 1, 2, \ldots)$ for $t \in [t_0, T]$. In fact, from (10) and (11), it follows that

$$
\begin{aligned}
|x^n(t)|^p &\le \left(1 + \varepsilon_1^{\frac{1}{p-1}}\right)^{p-1} \left(|D(x_t^n) - D(x_{t_0})|^p + \frac{|J_{n-1}(t)|^p}{\varepsilon_1}\right) \\
&\le \left(\frac{1}{\kappa}\right)^{p-1} (|D(x_t^n) - D(x_{t_0})|^p) + \frac{1}{(1-\kappa)^{p-1}}|J_{n-1}(t)|^p \\
&\le \kappa \sup_{-\tau \le \theta \le 0} \left[\left(1 + \varepsilon_2^{\frac{1}{p-1}}\right)^{p-1} \left(|x^n(t+\theta)|^p + \frac{|x(t_0+\theta)|^p}{\varepsilon_2}\right)\right] \\
&\quad + \frac{1}{(1-\kappa)^{p-1}}|J_{n-1}(t)|^p \\
&\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}\|x_t^n\|^p + \frac{\kappa}{(1-\sqrt{\kappa})^{p-1}}\|x_{t_0}\|^p + \frac{1}{(1-\kappa)^{p-1}}|J_{n-1}(t)|^p,
\end{aligned}
$$

where $\varepsilon_1 = \left(\frac{1-\kappa}{\kappa}\right)^{p-1}$ and $\varepsilon_2 = \left(\frac{1-\sqrt{\kappa}}{\sqrt{\kappa}}\right)^{p-1}$, then applying $J_{n-1}(t)$ yields that

$$
\begin{aligned}
|x^n(t)|^p &\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}\|x_t^n\|^p + \frac{\kappa}{(1-\sqrt{\kappa})^{p-1}}\|x_{t_0}\|^p + \left(\frac{3}{1-\kappa}\right)^{p-1}|x(t_0)|^p \\
&\quad + \left(\frac{3}{1-\kappa}\right)^{p-1}\left|\int_{t_0}^{t} f(s, x_s^{n-1}) ds\right|^p
\end{aligned} \tag{12}
$$

$$+\left(\frac{3}{1-\kappa}\right)^{p-1}\left|\int_{t_0}^{t} g(s,x_s^{n-1})d\mathcal{B}(s)\right|^p.$$

Using the Hölder inequality, Lemma 2.1 and Lemma 2.2, one derives from (12) that

$$\begin{aligned}
&E\sup_{t_0\le s\le t}|x^n(s)|^p\\
&\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)+\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}}{(1-\kappa)^{p-1}}E\|\xi\|^p\\
&\quad+\left(\frac{3}{1-\kappa}\right)^{p-1}E\left(\sup_{t_0\le\ \theta\le t}\int_{t_0}^{\theta}|f(s,x_s^{n-1})|ds\right)^p\\
&\quad+\left(\frac{3}{1-\kappa}\right)^{p-1}c_pE\int_{t_0}^{t}|g(s,x_s^{n-1})|^pds.
\end{aligned}$$

From (10) and (11), it implies that

$$\begin{aligned}
&E(\sup_{t_0\le s\le t}|x^n(s)|^p)\\
&\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)+\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}}{(1-\kappa)^{p-1}}E\|\xi\|^p\\
&\quad+\left(\frac{6}{1-\kappa}\right)^{p-1}\int_{t_0}^{t}[(t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p]ds\\
&\quad+\left(\frac{6}{1-\kappa}\right)^{p-1}\left\{\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right\}\int_{t_0}^{t}E(\sup_{t_0\le s\le t}\|x_s^{n-1}\|^p)ds.
\end{aligned}$$

Due to the fact that $E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)\le E\|\xi\|^p+E(\sup_{t_0\le s\le t}|x^n(s)|^p)$, it implies

$$\begin{aligned}
&E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)\\
&\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)+\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}+(1-\kappa)^{p-1}}{(1-\kappa)^{p-1}}E\|\xi\|^p\\
&\quad+\left(\frac{6}{1-\kappa}\right)^{p-1}\int_{t_0}^{t}[(t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p]ds\\
&\quad+\left(\frac{6}{1-\kappa}\right)^{p-1}\left\{\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right\}\int_{t_0}^{t}E(\sup_{t_0\le s\le t}\|x_s^{n-1}\|^p)ds,
\end{aligned}$$

which follows that

$$\begin{aligned}
&E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)\\
&\le \frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}+(1-\kappa)^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}E\|\xi\|^p
\end{aligned}\tag{13}$$

$$+\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\int_{t_0}^{t}[(t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p]ds$$
$$+\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\left\{\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right\}$$
$$\times\int_{t_0}^{t}E(\sup_{t_0-\tau\le s\le t}|x_s^{n-1}|^p)ds.$$

From (13), one yields that for any $N\ge 1$,

$$\max_{1\le n\le N}E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)$$
$$\le\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}+(1-\kappa)^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}E\|\xi\|^p$$
$$+\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\int_{t_0}^{t}[(t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p]ds$$
$$+\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\left\{\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right\}$$
$$\times\int_{t_0}^{t}\max_{1\le n\le N}E(\sup_{t_0-\tau\le u\le s}|x^n(u)|^p)ds.$$

Then, by using the Gronwall inequality, we have

$$\max_{1\le n\le N}E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)$$
$$\le\left\{\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}+(1-\kappa)^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}E\|\xi\|^p\right.$$
$$\left.+\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\int_{t_0}^{t}[(t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p]ds\right\}\tag{14}$$
$$\times\exp\left\{\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\left[\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right]\right\}.$$

Since N is arbitrary, it implies from (14) that

$$E(\sup_{t_0-\tau\le s\le t}|x^n(s)|^p)$$
$$\le\left\{\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}+(1-\kappa)^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}E\|\xi\|^p\right.$$

$$+\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\int_{t_0}^{t}[(t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p]ds\Bigg\}$$
$$\times\exp\Bigg\{\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\left[\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right]\Bigg\}$$

for all $t\in[0,T]$ $(n=1,2,\ldots)$. Therefore, $x^n(t)\in B_T$ $(n=0,1,2,\ldots)$ for all $t\in[t_0-\tau,T]$. Similar to the previous reasoning process, it follows

$$\begin{aligned}|x^1(t)|^p &\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}\sup_{t_0-\tau\le s\le t}|x^1(s)|^p+\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}}{(1-\kappa)^{p-1}}\|\xi\|^p\\ &\quad+\left(\frac{3}{1-\kappa}\right)^{p-1}\left(\int_{t_0}^{t}|f(s,x_s^0)|ds\right)^p\\ &\quad+\left(\frac{3}{1-\kappa}\right)^{p-1}\left|\int_{t_0}^{t}g(s,x_s^0)d\mathcal{B}(s)\right|^p.\end{aligned}$$

Hence, it is also shown that

$$\begin{aligned}&E(\sup_{t_o-\tau\le s\le t}|x^1(t)|^p)\\ &\le\frac{\kappa(1+\sqrt{\kappa})^{p-1}+3^{p-1}+(1-\kappa)^{p-1}+6^{p-1}\left[\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right]}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\\ &\quad\times E\|\xi\|^p\\ &\quad+\frac{6^{p-1}}{(1-\kappa)^{p-1}\left(1-\kappa^{\frac{p+1}{2}}\right)}\left\{\int_{t_0}^{t}[(t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p]ds\right\}\\ &:=C_1.\end{aligned}\tag{15}$$

Similarly,

$$\begin{aligned}&|x^1(t)-x^0(t)|^p\\ &\le\frac{1}{\kappa^{p-1}}|D(x_t^0)-D(x_0)|^p\\ &\quad+\left(\frac{2}{1-\kappa}\right)^{p-1}\left(\left|\int_{t_0}^{t}f(s,x_s^0)ds\right|^p+\left|\int_{t_0}^{t}g(s,x_s^0)d\mathcal{B}(s)\right|^p\right)\\ &\le\frac{\kappa}{(\sqrt{\kappa})^{p-1}}\sup_{t_0-\tau\le s\le t}|x^1(s)|^p+\frac{\kappa(1+\sqrt{\kappa})^{p-1}}{(1-\kappa)^{p-1}}\|\xi\|^p\\ &\quad+\left(\frac{2}{1-\kappa}\right)^{p-1}\left(\left|\int_{t_0}^{t}f(s,x_s^0)ds\right|^p+\left|\int_{t_0}^{t}|g(s,x_s^0)|d\mathcal{B}(s)\right|^p\right),\end{aligned}$$

and

$$
\begin{aligned}
&E(\sup_{t_0\le s\le t}|x^1(t)-x^0(t)|)^p\\
&\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}\sup_{t_0-\tau\le s\le t}|x^1(s)|^p+\frac{\kappa(1+\sqrt{\kappa})^{p-1}}{(1-\kappa)^{p-1}}\|\xi\|^2\\
&\quad+\left(\frac{4}{1-\kappa}\right)^{p-1}\left[\int_{t_0}^t (t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p ds\right]\\
&\quad+\left(\frac{4}{1-\kappa}\right)^{p-1}\left[\left(\int_{t_0}^t L(s)ds\right)^p+c_p\int_{t_0}^t L(s)ds\right]\int_{t_0}^t E[\sup_{t_0-\tau\le s\le t}|x_s|^p]ds\\
&\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}\sup_{t_0-\tau\le s\le t}|x^1(s)|^p+\frac{\kappa(1+\sqrt{\kappa})^{p-1}}{(1-\kappa)^{p-1}}\|\xi\|^2\\
&\quad+\left(\frac{4}{1-\kappa}\right)^{p-1}\left[\int_{t_0}^t (t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p ds\right]\\
&\quad+\left(\frac{4}{1-\kappa}\right)^{p-1}\left[\left(\int_{t_0}^t L(s)ds\right)^p+c_p\int_{t_0}^t L(s)ds\right](t-t_0)\|\xi\|^p.
\end{aligned}
$$

Therefore, applying (15), it gives

$$
\begin{aligned}
&E(\sup_{t_0-\tau\le s\le t}|x^1(t)-x^0(t)|)^p\\
&\le E(\sup_{t_0\le s\le t}|x^1(t)-x^0(t)|)^p \qquad (16)\\
&\le \frac{\kappa}{(\sqrt{\kappa})^{p-1}}C_1+\left(\frac{4}{1-\kappa}\right)^{p-1}\left[\int_{t_0}^t (t-t_0)^{p-1}|f(s,0)|^p+c_p|g(s,0)|^p ds\right]\\
&\quad+\frac{4^{p-1}+\kappa(1+\sqrt{\kappa})^{p-1}}{(1-\kappa)^{p-1}}\left[\left(\int_{t_0}^t L(s)ds\right)^p+c_p\int_{t_0}^t L(s)ds\right]E\|\xi\|^p\\
&\equiv C
\end{aligned}
$$

for all $t\in[t_0,T]$.

Now, we will prove that for any $n\ge 0$,

$$
\begin{aligned}
&E(\sup_{t_0-\tau\le s\le t}|x^{n+1}(t)-x^n(t)|)^p \qquad (17)\\
&\le C\left\{\frac{2^{p-1}}{(1-\kappa)^p}\left[(t-t_0)^{p-1}\left(\int_{t_0}^t L(s)ds\right)^p+c_p\int_{t_0}^t L(s)ds\right]\right\}^n\frac{(t-t_0)^n}{n!},
\end{aligned}
$$

where C is defined in (16).

According to (16), it is easily seen that (17) holds when $n=0$. Under the inductive assumption that (17) holds for some $n\ge 0$. We shall show that inequality (17) still holds for $n+1$. Note that

$$
|x^{n+2}(t)-x^{n+1}(t)|^p=\Bigg|D(x_t^{n+2})-D(x_t^{n+1})+\int_{t_0}^t (f(s,x_s^{n+1})-f(s,x_s^n))ds
$$

$$
\begin{aligned}
&+\int_{t_0}^{t}(g(s,x_s^{n+1})-g(s,x_s^{n}))d\mathcal{B}(s)\Big|^p\\
&\le \kappa \sup_{t_0-\tau\le s\le t}|x^{n+2}(s)-x^{n+1}(s)|^p+\left(\frac{2}{1-\kappa}\right)^{p-1}\\
&\times\left|\int_{t_0}^{t}[f(s,x_s^{n+1})-f(s,x_s^{n})]ds\right|^p\\
&+\left(\frac{2}{1-\kappa}\right)^{p-1}\left|\int_{t_0}^{t}[g(s,x_s^{n+1})-g(s,x_s^{n})]d\mathcal{B}(s)\right|^p.
\end{aligned}
$$

Then, by using Lemma 2.2, it implies

$$
\begin{aligned}
&E(\sup_{t_0-\tau\le s\le t}|x^{n+2}(t)-x^{n+1}(t)|^p)\\
&\le E\sup_{t_0\le s\le t}|x^{n+2}(t)-x^{n+1}(t)|^p\\
&\le \kappa E(\sup_{t_0-\tau\le s\le t}|x^{n+2}(s)-x^{n+1}(s)|^p)\\
&+\left(\frac{2}{1-\kappa}\right)^{p-1}(t-t_0)^{p-1}\left[\left(\int_{t_0}^{t}L(s)ds\right)^p\right.\\
&\left.\times\int_{t_0}^{t}E(\sup_{t_0-\tau\le s\le t}|x^{n+1}(s)-x^{n}(s)|^p)ds\right]\\
&+\left(\frac{2}{1-\kappa}\right)^{p-1}\left[c_p\int_{t_0}^{t}L(s)ds\int_{t_0}^{t}E(\sup_{t_0-\tau\le s\le t}|x^{n+1}(t)-x^{n}(t)|^p)ds\right].
\end{aligned}
$$

It can also be claimed that

$$
\begin{aligned}
&E(\sup_{t_0-\tau\le s\le t}|x^{n+2}(t)-x^{n+1}(t)|^p)\\
&\le \frac{2^{p-1}}{(1-\kappa)^p}\left[(t-t_0)^{p-1}\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right]\\
&\quad\times\int_{t_0}^{t}E(\sup_{t_0-\tau\le s\le t}|x^{n+1}(s)-x^{n}(s)|^p)ds\\
&\le C\left\{\frac{2^{p-1}}{(1-\kappa)^p}\left[(t-t_0)^{p-1}\left(\int_{t_0}^{t}L(s)ds\right)^p+c_p\int_{t_0}^{t}L(s)ds\right]\right\}^{n+1}\frac{(t-t_0)^{n+1}}{(n+1)!},
\end{aligned}
$$

which implies that (17) holds for $n+1$. Hence, the inequality (17) holds for all $n>0$.

For any $m\ge n\ge 1$, we obtain

$$
\begin{aligned}
&\|x^m-x^n\|_{B_T}^p\\
&=E(\sup_{t_0-\tau\le s\le t}|x^m(t)-x^n(t)|^p)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{k=n}^{+\infty} E(\sup_{t_0-\tau\leq s\leq t}|x^{k+1}(t)-x^k(t)|^p)\\
&\leq \sum_{k=n}^{+\infty}\bigg\{C\bigg[\frac{2^{p-1}}{(1-k)^p}\bigg((t-t_0)^{p-1}\bigg(\int_{t_0}^t L(s)ds\bigg)^p+c_p\int_{t_0}^t L(s)ds\bigg)\bigg]^k\frac{(t-t_0)^k}{k!}\bigg\}\\
&\to 0 \ \text{ as } n\to+\infty.
\end{aligned}
$$

Thus, $\{x^n(t)\}_{n\geq 1}$ is a Cauchy sequence in Banach space B_T. Denote the limit by $x(t)\in B_T(t\in[t_0,T])$. When letting $n\to+\infty$ in (9), we can derive the solution of NSFDE (3) with the initial value $x_{t_0}=\xi=\{\xi(\theta):-\tau\leq\theta\leq 0\}\in\mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$. In other words, the existence of the solution have been shown.

Uniqueness. Let $x(t)$ and $\overline{x}(t)$ be two solutions of (3) with same initial value. Similar to the derivation process of inequality (17), we have

$$
\begin{aligned}
&E(\sup_{t_0-\tau\leq s\leq t}|x(s)-\overline{x}(s)|^p)\\
&\leq E(\sup_{t_0\leq s\leq t}|x(s)-\overline{x}(s)|^p)\\
&\leq \frac{2^{p-1}}{(1-\kappa)^p}\bigg[(t-t_0)^{p-1}\bigg(\int_{t_0}^t L(s)ds\bigg)^p+c_p\int_{t_0}^t L(s)ds\bigg]\\
&\quad\times\int_{t_0}^t E(\sup_{t_0-\tau\leq r\leq s}|x(r)-\overline{x}(r)|^p)ds
\end{aligned}
\tag{18}
$$

for $t\in[t_0,T]$. Then, by applying the Gronwall inequality to the inequality (18), it shows

$$E(\sup_{t_0-\tau\leq s\leq t}|x(s)-\overline{x}(s)|^p)=0 \ \text{ for } t\in[t_0,T].$$

Hence, it yields

$$\|x-\overline{x}\|_{B_T}^p=E(\sup_{t_0-\tau\leq s\leq t}|x(s)-\overline{x}(s)|^p)=0,$$

which implies that the uniqueness of the solution for NSFDE (3) with $x_{t_0}=\xi=\{\xi(\theta):-\tau\leq\theta\leq 0\}\in\mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$ is guaranteed. ☐

Lemma 3.2. *Suppose that conditions* (7)-(8) *hold, and*

$$
\begin{aligned}
&|f(t,\varphi)-f(t,\phi)|\leq K_m(t)\|\varphi-\phi\|,\\
&|g(t,\varphi)-g(t,\phi)|^p\leq K_m(t)\|\varphi-\phi\|^p,
\end{aligned}
\tag{19}
$$

are satisfied for all $t\in[t_0,T]$, $\varphi,\phi\in\mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$, *with* $\|\varphi\|\vee\|\phi\|\leq m$, *and* $p\geq 2$, *where* $K_m(t)\in\mathcal{H}^1([t_0,T];[0,+\infty))$. *Then, there exists a uniqueness maximal local solution* $x(t)$ *of NSFDE* (1).

Proof. Since (19) holds for $\varphi\in\mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$ with $\|\varphi\|\leq m$, we can define

$$f^m(t,\varphi)=f\bigg(t,\frac{m\wedge\|\varphi\|}{\|\varphi\|}\varphi\bigg),\qquad g^m(t,\varphi)=g\bigg(t,\frac{m\wedge\|\varphi\|}{\|\varphi\|}\varphi\bigg),$$

and $\frac{\|\varphi\|}{\|\varphi\|} = 1$, when $\varphi \equiv 0$. Then, it is seen that $f^m(\cdot,\cdot)$ and $g^m(\cdot,\cdot)$ satisfy the condition (6). Moreover, the following equation

$$d(x^m(t) - D(x_t^m)) = f^m(t, x^m(t))dt - g^m(t, x^m(t))d\mathcal{B}(t), \quad t \in [t_0, T], \tag{20}$$

$$x^m(t_0) = x^m(t_0 + s) = \begin{cases} \xi & \text{if} \|\xi\| \le m \\ 0 & \text{if} \|\xi\| > m \end{cases} \quad s \in [-\tau, 0],$$

has a unique continuous solution $x^m(t)$.

Now, we define the sequence of stopping time:

$$\delta_m = T \wedge \inf\{t \in [t_0, T] : |x_m(t)| \ge m\}.$$

It is not difficult to show that

$$x^m(t) = x^{m+1}(t), \quad \text{if } t_0 \le t \le \delta_m. \tag{21}$$

This implies that δ_m is increasing, and has its limit $\delta_\infty = \lim_{m\to\infty} \delta_m$. Now we define $\{x(t) : t_0 \le t < \delta_m\}$ by

$$x(t) = x^m(t), \quad t \in [\delta_{m-1}, \delta_m),\ m \ge 1,$$

where $\delta_0 = t_0$. By (21), $x(t \wedge \delta_m) = x^m(t \wedge \delta_m)$. Therefore, it follows from (20) that

$$\begin{aligned} &x(t \wedge \delta_m) \\ &= \xi(0) + D(x_t^0) - D(\xi) + \int_{t_0}^{t\wedge\delta_m} f(s, x(s))ds + \int_{t_0}^{t\wedge\delta_m} g(s, x(s))d\mathcal{B}(s) \\ &= \xi(0) + D(x_t^0) - D(\xi) + \int_{t_0}^{t\wedge\delta_m} f^m(s, x(s))ds + \int_{t_0}^{t\wedge\delta_m} g^m(s, x(s))d\mathcal{B}(s) \end{aligned}$$

for any $t \in [t_0, T)$ and $m \ge 1$. It is seen that if $\delta_\infty < T$, then

$$\lim \sup_{t\to\delta_\infty} |x(t)| \ge \lim \sup_{m\to\infty} |x(\delta_m)| = \lim \sup_{m\to\infty} |x_m(\delta_m)| = \infty.$$

Hence, $\{x(t) : t_0 \le t < \delta_\infty\}$ is a maximal local solution. To prove the uniqueness, let $\{\bar{x}(t) : t_0 \le t < \bar{\delta}_\infty\}$ be another maximal local solution. Define

$$\bar{\delta}_m = \bar{\delta}_\infty \wedge \inf\{t \in [t_0, \bar{\delta}_\infty) : |\bar{x}(t)| \ge m\}.$$

It is shown that $\bar{\delta}_m \to \bar{\delta}_\infty$ a.s. and

$$P\{x(t) = \bar{x}(t),\ \forall t \in [t_0, \delta_m \wedge \bar{\delta}_m)\} = 1, \quad \forall m \ge 1.$$

As $m \to \infty$, it yields

$$P\{x(t) = \bar{x}(t),\ \forall t \in [t_0, \delta_\infty \wedge \bar{\delta}_\infty)\} = 1.$$

To complete the proof, we need to show that $\delta_\infty = \bar{\delta}_\infty$ a.s. In fact, for almost any $\omega \in \{\delta_\infty < \bar{\delta}_\infty\}$, we have

$$|\bar{x}(\delta_\infty, \omega)| = \lim_{m\to\infty} |\bar{x}(\delta_m, \omega)| = \lim_{m\to\infty} |x(\delta_m, \omega)| = \infty$$

which contradicts with the continuity of $\bar{x}(t,\omega)$ on $t \in [t_0, \bar{\delta}_\infty(\omega))$, then we must have $\delta_\infty \geq \bar{\delta}_\infty$ a.s. Similarly, we can show that $\delta_\infty \leq \bar{\delta}_\infty$ a.s. Therefore, we must have $\delta_\infty = \bar{\delta}_\infty$. □

Theorem 3.3. *Let conditions* (7)-(8) *and* (19) *hold. Suppose that there exist a Lyapunov function* $V \in C^{1,2}([t_0-\tau, T) \times \mathbb{R}^n; \mathbb{R}_+)$, $q \in \mathcal{H}^1([t_0-\tau, t]; [0, +\infty)])$, *and* $p \in \mathcal{H}^1([t_0-\tau, t]; \mathbb{R})$ *such that*

$$\lim_{|x|\to\infty} [\inf_{t_0-\tau\leq t<T} V(t, \xi(0) - D(\xi))] = \infty, \tag{22}$$

and

$$\mathcal{L}V(t, \xi(0) - D(\xi)) \leq -p(t)V(t, \xi(0) - D(\xi)) + q(t), \tag{23}$$

for any $(t, \xi) \in [t_0, +\infty) \times \mathcal{C}([t_0-\tau, t_0]; \mathbb{R}^n)$, *whenever* $V(s, x(s) - D(x_s)) \leq V(t, x(t) - D(x_t))e^{\int_s^t p(u)du}$ *for any* $s \in [t_0-\tau, t]$.

Then, NSFDE (3) *with* $x_{t_0} = \xi \in \mathcal{C}([t_0-\tau, t_0]; \mathbb{R}^n)$ *has a unique continuous solution* $x(t)$ *for* $t \in [t_0, T)$, *and the solution satisfies the following moment estimate*

$$\begin{aligned}
&E[V(t, x(t) - D(x_t))] \\
&\leq e^{-\int_{t_0}^t p(s)ds} E \sup_{t_0-\tau\leq s\leq t_0} [V(s, x(s) - D(x_s))e^{\int_{t_0}^s p(u)du}] \\
&\quad + \int_{t_o}^t q(u)e^{-\int_u^s p(s)ds} du \\
&\leq e^{\mathcal{K}} e^{-\int_{t_0}^t p(s)ds} E \sup_{t_0-\tau\leq s\leq t_0} [V(s, x(s) - D(x_s))] + \int_{t_o}^t q(u)e^{-\int_u^s p(s)ds} du
\end{aligned} \tag{24}$$

for $t \in [t_0, T)$, *where* $\mathcal{K} = \int_{t_0-\tau}^{t_0} |p(s)|ds$.

Furthermore, $EV(t, \xi(0) - D(\xi))$ *is continuous in* $[t_0, T)$, *and* $E|x(t) - D(x_t)|^p$ $(p \geq 2)$ *is continuous in* $[t_0, T)$ *if there exists a constant* $c > 0$ *such that*

$$c|\xi(0) - D(\xi)|^p \leq V(t, \xi(0) - D(\xi)), \quad \forall t \in [t_0, T). \tag{25}$$

Proof. Since $D(\cdot)$, $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ in NSFDE (3) satisfies conditions (8) and (19), Lemma 3.1 guarantees that there exists a unique continuous solution $x(t)$ with $x_{t_0} \in \mathcal{C}([t_0-\tau, t_0]; \mathbb{R}^n)$, for $t \in [t_0, \delta_\infty)$, where δ_∞ is the explosion time. Define $\overline{\omega}(t) = \sup_{t-\tau\leq s\leq t} \omega(s)$ for $t \in [t_0, \delta_\infty)$, where

$$\omega(t) = \begin{cases} V(t, x(t) - D(x_t))e^{\int_{t_0}^t p(s)ds} - \int_{t_0}^t q(u)e^{\int_{t_0}^u p(s)ds} du & \text{for } t \in [t_0, \delta_\infty), \\ V(t, x(t) - D(x_t))e^{\int_{t_0}^t p(s)ds} & \text{for } t \in [t_0-\tau, t_0]. \end{cases} \tag{26}$$

Then, it is seen that $\omega(t)$ is continuous for any $t \in [t_0, \delta_\infty)$. Meanwhile, there exists an $s_0 = s_0(\omega) \in [t_0-\tau, t]$ such that $\overline{\omega}(t) = \omega(s_0)$. Thus, $s_0 = t$ or $s_0 < t$ and $\omega(s) < \omega(s_0)$ for $s_0 < s \leq t$.

If $s_0 < t$, then for sufficiently small $h > 0$, $\overline{\omega}(t+h) = \overline{\omega}(t)$ and

$$\mathcal{L}\overline{\omega}(t) = 0,$$

If $s_0 = t$, then $\overline{\omega}(t) = \omega(t)$, i.e.,

$$\omega(t) \geq \omega(s)$$

for any $s \in [t_0 - \tau, t]$.

Now, for any $s \in [t_0 - \tau, t]$,

$$e^{\int_{t_0}^{s} p(u)du} V(s, x(s) - D(x_s)) \leq e^{\int_{t_0}^{t} p(s)ds} V(t, x(t) - D(x_t)).$$

Since $t > t_0$, we put it into two cases to prove the assertion:

(i) for any $s \in [t_0 - \tau, t_0]$, from (23) and (25), it implies that

$$\begin{aligned} e^{\int_{t_0}^{s} p(u)du} V(s, x(s) - D(x_s)) &\leq \omega(s) \\ &\leq \omega(t) \\ &\leq \omega(t) + \int_{t_0}^{t} q(u) e^{\int_{t_0}^{u} p(s)ds} du \\ &= e^{\int_{t_0}^{t} p(s)ds} V(t, x(t) - D(x_t)). \end{aligned}$$

(ii) for any $s \in [t_0, t]$, from (23) and (25), it follows that

$$\begin{aligned} e^{\int_{t_0}^{s} p(u)du} V(s, x(s) - D(x_s)) &= \omega(s) + \int_{t_0}^{s} q(u) e^{\int_{t_0}^{u} p(z)dz} du \\ &\leq \omega(t) + \int_{t_0}^{t} q(u) e^{\int_{t_0}^{u} p(s)ds} du \\ &= e^{\int_{t_0}^{t} p(s)ds} V(t, x(t) - D(x_t)). \end{aligned}$$

Consequently, (26) holds, that is,

$$V(s, x(s) - D(x_s)) \leq e^{\int_{t_0}^{t} p(u)du} V(t, x(t) - D(x_t)) \tag{27}$$

is satisfied for any $s \in [t_0 - \tau, t]$.

Thus, from (22), (23) and the continuity of $\overline{\omega}(t)$, it gives that

$$\overline{\omega}(t) = \sup_{t_0 - \tau \leq s \leq t} [V(s, x(s) - D(x_s))] e^{\int_{t_0}^{t} p(s)ds} - \int_{t_0}^{t} q(u) e^{\int_{t_0}^{u} p(s)ds} du,$$

and when $s_0 = t$, $\overline{\omega}(t) = \omega(t)$. Hence, the Itô operator for $\overline{\omega}(t)$ is calculated as follows

$$\begin{aligned} \mathcal{L}^{+}\overline{\omega}(t) = {} & [V(t, x(t) - D(x_t))] D^{+}\left(e^{\int_{t_0}^{t} p(s)ds}\right) \\ & + e^{\int_{t_0}^{t} p(s)ds} \mathcal{L}V(t, x(t) - D(x_t)) - D^{+}\left[\int_{t_0}^{t} q(u) e^{\int_{t_0}^{u} p(s)ds} du\right] \\ \leq {} & e^{\int_{t_0}^{t} p(s)ds} [q(t) - p(t)V + p(t)V - q(t)] \\ \leq {} & 0, \end{aligned}$$

where $D^+(\cdot)$ denotes the upper Dini right-hand derivative. From (24), for $s_0 < t$ and $s_0 = t$, we have $\mathcal{L}\overline{\omega}(t) \geq 0$.

To prove that the solution $x(t)$ exists globally, we set δ_m as the stopping time, which means the time at which $x(t)$ firstly leaves $U_m = \{|x| < m\}$. Letting $x^m(t) = x(\delta_m(t))$ and $V^m[t, x(t) - D(x_t)] = V[\delta_m(t), x(\delta_m(t) - D(x_{\delta_m(t)}))]$, where $\delta_m(t) = t \wedge \delta_m$. Then, we have $x^m(t)$ and $V^m[t, x(t) - D(x_t)]$ are both continuous for all $t \in [t_0, \delta_\infty)$. For any $t \in [t_0, \delta_\infty)$ and $\delta_\infty \in (t_0, T)$, $x^m(t)$ satisfies the following equation

$$d(x^m(t) - D(x_t^m)) = I_{[t_0-\tau,\delta_m]}(t)[f(t, x_t^m)dt + g(t, x_t^m)dB(t)] \quad a.s.,$$

then we have $E[V^m(t, x(t) - D(x_t))] < \infty$. From $\mathcal{L}\overline{\omega}(t) \leq 0$, we have $\mathcal{L}\overline{\omega}(\delta_m(t)) \leq 0$ for any $t \in [t_0, \delta_\infty)$. Hence, since $\delta_m(t) = t \wedge \delta_m$, it gives

$$E\overline{\omega}(\delta_m(t)) \leq E\overline{\omega}(t_0)$$

for $t \in [t_0, \delta_\infty)$. From (24) and (27), we have $\omega(\delta_m(t)) \leq \overline{\omega}(\delta_m(t))$, and

$$\begin{aligned}
&E\left[V^m(t, x(t) - D(x_t))e^{\int_{t_0}^{\delta_m(t)} p(s)ds}\right] \\
&\leq E \sup_{t_0-\tau \leq s \leq t_0}\left[V(s, x(s) - D(x_s))e^{\int_{t_0}^{s} p(u)du}\right] + E\int_{t_0}^{\delta_m(t)} q(u)e^{-\int_{t_0}^{u} p(s)ds}du \\
&= u(t).
\end{aligned}$$

For any $\delta_\infty \in [t_0, T)$, there exists a constant $\mathcal{K} > 0$ such that $\int_{t_0}^{b} |p(s)|ds < \mathcal{K}$. When $|x^m(t)| \leq m$, by using the Chebyshev's inequality [3], it yields

$$\begin{aligned}
P\{\delta_m \in [t_0, \delta_\infty)\} &\leq \frac{E\left[V^m(t, x(t) - D(x_t))e^{\int_{t_0}^{\delta_m(t)} p(s)ds}\right]}{\inf\limits_{s\in[t_0,\delta_\infty)\times\{|x|>m\}}\left[V(s, x(s))e^{\int_{t_0}^{s} p(u)du}\right]} \\
&\leq \frac{\sup\limits_{s\in[t_0,\delta_\infty)} u(s)}{e^{\mathcal{K}}\inf\limits_{s\in[t_0,\delta_\infty)\times\{|x|>m\}} V(s, x(s))}.
\end{aligned} \tag{28}$$

Since $q \in \mathcal{H}^1([t_0, \delta_\infty]; [0, +\infty))$, $p \in \mathcal{H}^1([t_0, \delta_\infty]; \mathbb{R})$, $\int_{t_0}^{\delta_m(t)} q(u)e^{-\int_{t_0}^{u} p(s)ds}du \in \mathcal{H}^1([t_0-\tau, t]; [0, +\infty))$, $\sup\limits_{s\in[t_0,\delta_\infty)} u(s) < \infty$ for $\delta_\infty \in [t_0, T)$. From (22) and (28), it follows that $\lim\limits_{m\to\infty} P(\delta_m \in [t_0, \delta_\infty)) = 0$, and $x(t) = \lim\limits_{m\to\infty} x^m(t)$ exists on $[t_0, T)$, which show the existence and uniqueness of the solution for NSFDE (1) on $[t_0, T)$. As $m \to \infty$, we have

$$\begin{aligned}
&E\left[V^m(t, x(s) - D(x_s))e^{\int_{t_0}^{\delta_m(t)} p(s)ds}\right] \\
&\leq E \sup_{t_0-\tau \leq s \leq t_0}\left[V(s, x(s) - D(x_s))e^{\int_{t_0}^{s} p(u)du}\right] + E\int_{t_0}^{\delta_m(t)} q(u)e^{-\int_{t_0}^{u} p(s)ds}du.
\end{aligned}$$

Then, from (23) and (25), by using the dominated convergence theorem in [14], $EV(t,\xi(0)-D(\xi))$ and $E|x(t-D(x_t))|^p$ are continuous. ☐

Corollary 3.4. *If all conditions in Theorem 3.3 are satisfied except that* (23) *replaced by*

$$E\mathcal{L}V(t,\xi(0)-D(\xi)) \leq -p(t)EV(t,\xi(0)-D(\xi))+q(t), \tag{29}$$

where $\xi \in \mathcal{C}([t_0-\tau,t_0];\mathbb{R}^n)$, *whenever* $EV(s,x(s)-D(x_s)) \leq EV(t,x(t)-D(x_t))e^{\int_s^t p(u)du}$ *for* $s \in [t_0-\tau,t]$. *Then, the conclusions are still satisfied, except that* (24) *replace by*

$$\begin{aligned}
&E\Big[V(t,x(t)-D(x_t))\Big] \\
&\leq e^{-\int_{t_0}^t p(s)ds}E\sup_{t_0-\tau\leq s\leq t_0}\Big[V(s,x(s)-D(x_s))e^{\int_{t_0}^s p(u)du}\Big]+\int_{t_o}^t q(u)e^{-\int_u^s p(s)ds}du \\
&\leq e^{\mathcal{K}}e^{-\int_{t_0}^t p(s)ds}\sup_{t_0-\tau\leq s\leq t_0}\Big[EV(s,x(s)-D(x_s))\Big]+\int_{t_o}^t q(u)e^{-\int_u^t p(s)ds}du
\end{aligned}$$

for $t \in [t_0,T)$.

Proof. Since $E[V^m(t,\xi(0)-D(\xi))] < \infty$, and both $\xi(0)-D(\xi)$ and $V^m(t,\xi(0)-D(\xi))$ are continuous, by dominated convergence theorem in [14], $E[V^m(t,\xi(0)-D(\xi))]$ is continuous. Then, for small enough $h>0$, we have

$$\begin{aligned}
&\Delta E[V^m(t,x(t)-D(x_t))] \\
&= E[V^m(t,x(t+h)-D(x_{t+h}))]-E[V^m(t,x(t)-D(x_t))] \\
&= E\int_{t\wedge\delta_m}^{(t+h)\wedge\delta_m}\mathcal{L}V^m(s,x(s)-D(x_s))ds.
\end{aligned}$$

As $h\to 0^+$, it follows

$$D^+E[V^m(t,x(t)-D(x_t))] = E\mathcal{L}V^m(t,x(t)-D(x_t)) \quad \text{for } t\in[t_0,T).$$

Then, from condition (29), and as $n\to\infty$, we have

$$D^+E[V(t,x(t)-D(x_t))] \leq -p(t)EV(t,x(t)-D(x_t))+q(t) \quad \text{for } t\in[t_0,T),$$

whenever $EV(s,x(s)-D(x_s)) \leq EV(t,x(t)-D(x_t))e^{\int_s^t p(u)du}$ for $s\in[t_0-\tau,t]$. This leads to the fact that the condition (23) is satisfied. Applying Theorem 3.3, the conclusion of this corollary is derived. ☐

Corollary 3.5. *Let* $C^1(\mathbb{R},[0,+\infty))$ *denote the family of all functions* $\psi(t)$ *on R, which is once continuously differentiable in t with* $\psi'(t)\geq 0$. *Suppose that all conditions of Theorem 3.3 are satisfied except that condition* (23) *replaced by*

$$E\mathcal{L}V(t,x(t)-D(x_t)) \leq -\lambda\frac{\psi'(t)}{\psi(t)}EV(t,x(t)-D(x_t)), \quad \forall t\in[t_0,T),$$

for some constant $\lambda > 0$, whenever

$$EV(s, x(s) - D(x_s)) \le \frac{\psi^\lambda(t)}{\psi^\lambda(s)} EV(t, x(t) - D(x_t)), \quad \forall s \in [t_0 - \tau, t).$$

Then, NSFDE (3) *has a unique solution $x(t)$ for $t \in [t_0, T)$, and*

$$EV(t, x(t) - D(x_t)) \le \frac{\psi^\lambda(t)}{\psi^\lambda(s)} E \sup_{-\tau \le s \le 0} V(t_0 + s, x(t_0 + s) - D(x_{t_0+s})), \quad \forall t \ge t_0.$$

Proof. Taking $p(t) = \lambda \frac{\psi'(t)}{\psi(t)}$, we have $e^{\int_t^s p(u)du} = e^{\lambda \int_s^t \frac{\psi'(u)}{\psi(u)} du} = \frac{\psi^\lambda(t)}{\psi^\lambda(s)}$. □

4. An example

In this section, we consider the following NSFDE:

$$\begin{aligned} d[x(t) - D(x_t)] = & -\left[a(t)[x(t) - D(x_t)]^3 + \frac{1}{2}|\sin t|[x(t) - D(x_t)]\right]dt \\ & + \sin t \int_{-\tau}^0 h(s)[x(t+s) - D(x_{t+s})]ds d\mathcal{B}(t), \end{aligned} \tag{30}$$

on $t \ge 0$, with the initial value $x_0 = \xi \in \mathcal{C}([-\tau, 0]; \mathbb{R})$, where $a(t) \ge 0$ and $h(s)$ is a real-valued function with $\int_{-\tau}^0 e^{-\mu s}|h(s)|ds = \bar{h}(\mu) < \infty$ for some positive number μ. In (30), for neutral term $D(\xi)$, there exists a constant $\kappa \in (0, 1)$ such that $|D(\xi)| \le \kappa \|\xi\|$. Let $\zeta_\lambda = \int_{-\tau}^0 e^{-\frac{\lambda}{2}s}|h(s)|ds$ and $\lambda \in [0, \mu]$, such that

$$1 - \zeta_\lambda^2 \ge \lambda. \tag{31}$$

Letting $V(t, \xi(0) - D(\xi)) = |\xi(0) - D(\xi)|^2$, from (30) and (31), we have

$$\begin{aligned} & \mathcal{L}V(t, x(t) - D(x_t)) \\ = & -2[x(t) - D(x_t)]\left[a(t)[x(t) - D(x_t)]^3 + \frac{1}{2}|\sin t|[x(t) - D(x_t)]\right] \\ & + \left[\sin t \int_{-\tau}^0 h(s)[x(t+s) - D(x_{t+s})]ds\right]^2 \\ \le & -|\sin t|\left\{V(x(t)) - \left[\int_{-\tau}^0 h(s)\sqrt{x(t+s) - D(x_{t+s})}ds\right]^2\right\} \\ \le & -|\sin t|\left[1 - \left(\int_{-\tau}^0 e^{\frac{\lambda}{2}\int_{t+s}^t |\sin u|du} h(s)ds\right)^2\right] V(x(t)) \\ \le & -|\sin t|\left[1 - \left(\int_{-\tau}^0 e^{\frac{\lambda}{2}s} h(s)ds\right)^2\right] V(x(t)) \\ \le & -\lambda|\sin(t)|V(t, x(t) - D(x_t)), \end{aligned}$$

whenever $V(s, x(s) - D(x_s)) \le e^{\lambda \int_s^t |\sin(u)|du} V(t, x(t) - D(x_t))$ for all $s \in [t_0 - \tau, t]$. From Theorem 3.3, the existence and uniqueness for the global solution

$x(t)$ of NSFDE (30) with initial value $\xi \in \mathcal{C}([t_0-\tau, t_0];\mathbb{R})$ is guaranteed, and $x(t)$ satisfies

$$E|x(t)-D(x_t)|^2 = EV(t, x(t)-D(x_t)) \le E\|\xi\|^2 e^{-\lambda \int_{t_0}^{t} |\sin(s)| ds}, \quad t \ge t_0.$$

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