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CONSTRUCTIONS OF SEGAL ALGEBRAS IN $L^1(G)$ OF LCA GROUPS G IN WHICH A GENERALIZED POISSON SUMMATION FORMULA HOLDS

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Abstract. Let G be a non-discrete locally compact abelian group, and μ be a transformable and translation bounded Radon measure on G. In this paper, we construct a Segal algebra $S_\mu(G)$ in $L^1(G)$ such that the generalized Poisson summation formula for μ holds for all $f \in S_\mu(G)$, for all $x \in G$. For the definitions of transformable and translation bounded Radon measures and the generalized Poisson summation formula, we refer to L. Argabright and J. Gil de Lamadrid's monograph in 1974.

1. Preliminaries

In this paper, G denotes a non-discrete LCA group with the dual group Γ, and $L^1(G)$ and $M(G)$ the group algebra and the usual measure algebra with convolution "$*$" as multiplication, respectively. Haar measures dx on G, and $d\gamma$ on Γ are chosen so that $d\gamma$ is the Plancherel measure corresponding to dx. $\mathcal{K}(G)$ denotes the family of all compact subsets of $G, C_c(G)$ the space consisting of all continuous functions on G with compact supports. $C_{c,2}(G)$ is the linear subspace of $C_c(G)$ generated by $\{f * g : f, g \in C_c(G)\}$, which forms a dense subspace of $L^1(G)$. $\mathfrak{M}(G)$ is the space of all Radon measures on G. The symbols $\hat{f}$ and $\hat{\mu}$ for $f \in L^1(G)$ and $\mu \in M(G)$ express the Fourier transform and the Fourier-Stieltjes transform of f and μ, respectively:

$$\hat{f}(\gamma) = \int_G (-x, \gamma) f(x) dx \quad \text{and} \quad \hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \ (\gamma \in \Gamma).$$

We also use symbol: $\check{f}(\gamma) = \int_G (x, \gamma) f(x) dx = \hat{f}(-\gamma)$. For a function f on G and $y \in G$, f_y denotes the translation of f by y, that is, $f_y(x) = f(x - y)$ $(x \in G)$, and also $f^*(x) = \overline{f(-x)}$, $\mu'(x) = \mu(-x)$ $(x \in G)$.

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Definition 1 ([1, p. 8, (2.1)])**.** A measure $\mu \in \mathfrak{M}(G)$ is said to be transformable if there exists $\hat{\mu} \in \mathfrak{M}(\Gamma)$ which satisfies

$$\text{(1)} \qquad \check{f} \in L^2(\hat{\mu}) \text{ and } \int_G f * f^*(x) d\mu(x) = \int_\Gamma |\check{f}(\gamma)|^2 d\hat{\mu}(\gamma) \ (f \in C_c(G)).$$

(1) implies ([1, p. 8, (2.2)])

$$\text{(2)} \qquad \check{g} \in L^1(\hat{\mu}) \text{ and } \int_G g(x) d\mu(x) = \int_\Gamma \check{g}(\gamma) d\hat{\mu}(\gamma) \ (g \in C_{c,2}(G)).$$

Remark 1. It is easy to see that (2) implies (1). Therefore $\mu \in \mathfrak{M}(G)$ is transformable if and only if there exists a measure $\hat{\mu} \in \mathfrak{M}(\Gamma)$ which satisfies (2).

The measure $\hat{\mu}$ in (1) is called the Fourier transform of μ. The set of all transformable measures in $\mathfrak{M}(G)$ is denoted by $\mathfrak{M}_T(G)$ ([1, p. 8]). For $\mu \in \mathfrak{M}_T(G)$, it may happen that $\hat{\mu} \in \mathfrak{M}_T(\Gamma)$. In this case, the inversion formula $\check{\hat{\mu}} = \mu$ holds ([1, Theorem 3.4]). We put $\mathscr{I}(G) = \{\mu \in \mathfrak{M}_T(G) : \hat{\mu} \in \mathfrak{M}_T(\Gamma)\}$ ([1, p. 21]).

A remarkable property of $\mu \in \mathfrak{M}(G)$, which is essentially important in this paper, is the translation boundedness.

Definition 2 ([1, p. 5])**.** A measure $\mu \in \mathfrak{M}(G)$ is said to be translation bounded if for every $K \in \mathcal{K}(G)$, $m_\mu(K) := \sup_{y \in G} |\mu|(K + y) < \infty$ holds, where $|\mu|$ is the total variation measure of μ. The set of all translation bounded measures on G will be denoted by $\mathfrak{M}_B(G)$.

Definition 3 ([2, p. 5])**.** $\mu \in \mathfrak{M}(G)$ is said to be shift-bounded if $f * \mu \in C_b(G)$ holds for every $f \in C_c(G)$.

Definition 4 ([3, p. 16])**.** A subspace S of $L^1(G)$ is called a Segal algebra if the following conditions are satisfied:

(i) The space S is dense in $L^1(G)$ in the norm topology of $L^1(G)$;
(ii) S is a Banach space under some norm $\| \ \|_S$ which dominates $\| \ \|_1$;
(iii) For each $f \in S$ and $x \in G$, $f_x \in S$ with $\|f_x\|_S = \|f\|_S$;
(iv) For each $f \in S$ and $\varepsilon > 0$, there is a neighborhood U of $0 \in G$ such that

$$\|f - f_x\|_S < \varepsilon \ (x \in U).$$

For the basic facts and notations, we refer to [5] and for Segal algebras we refer to [3, 4].

Definition 5. Let $\mu \in \mathfrak{M}_T(G) \cap \mathfrak{M}_B(G)$. Define

$$S_\mu(G) = \left\{ f \in L^1(G) : \||f| * |\mu|\|_\infty < \infty, \ \lim_{y \to 0} \||f - f_y| * |\mu|\|_\infty = 0, \hat{f} \in L^1(\hat{\mu}) \right\},$$

$$\|f\|_\mu = \|f\|_1 + \||f| * |\mu|\|_\infty + \|\hat{f}\|_{L^1(\hat{\mu})} \quad (f \in S_\mu(G)),$$

where $\||f| * |\mu|\|_\infty = \sup_{y \in G} \|f\|_{L^1(\mu'_y)}$. It is easy to see that $(S_\mu(G), \| \ \|_\mu)$ is a normed linear space.

In 1974, L. Argabright and J. Gil de Lamadrid introduced in [1] the notion of transformable Radon measures, and proved a generalized Poisson summation formula: Let $\mu \in \mathfrak{M}_T(G)$, and suppose that $f \in L^1(G)$ is convolvable with μ and $\hat{f} \in L^1(\hat{\mu})$. Then, for locally almost all $x \in G$, the following equality holds:

$$\int_G f(x-y)d\mu(y) = \int_\Gamma (x,\gamma)\hat{f}(\gamma)d\hat{\mu}(\gamma).$$

Furthermore, for any $u \in G$ such that the first integral in the above represents a continuous function of x in a neighborhood of u, the formula is valid for $x = u$. In the case where $G = \mathbb{R}^d$ and $\mu = m_{\mathbb{Z}^d}$, the counting measure on $\mathbb{Z}^d$, the formula reduces to the Poisson summation formula.

The contents of this paper: In §2, lemmas for the theorems in §3 are given. In §3, we construct, for each $\mu \in \mathfrak{M}_T(G) \cap \mathfrak{M}_B(G)$, a Segal algebra $S_\mu(G)$ such that for all $f \in S_\mu(G)$ and for all $x \in G$, the above generalized Poisson summation formula holds (Theorem 1). Then a characterization theorem for elements in $\mathscr{I}(G)$, and its corollaries are given. In §4, we exhibit some concrete examples for Theorem 1.

2. Lemmas

Lemma 1. *For $\mu \in \mathfrak{M}(G)$, μ is translation bounded if and only if μ is shift-bounded.*

Proof. Suppose that μ is translation bounded, and let $f \in C_c(G)$ be arbitrary. Put $K := \mathrm{supp}(f)$. Then

$$\begin{aligned}\|f * \mu\|_\infty &= \sup_{x\in G}\left|\int_G f(x-y)d\mu(y)\right| \le \sup_{x\in G}\int_G |f(x-y)|d|\mu|(y)\\ &= \sup_{x\in G}\int_{x-K} |f(x-y)|d|\mu|(y) \le \sup_{x\in G}\|f\|_\infty|\mu|(x-K)\\ &= \|f\|_\infty m_\mu(-K) < \infty.\end{aligned}$$

Hence μ is shift-bounded.

Conversely, suppose that μ is shift-bounded. Then $|\mu|$ is shift-bounded ([2, Proposition 1.12]), and let $K \in \mathcal{K}(G)$ be arbitrary. Choose $f \in C_c(G)$ such that $0 \le f \le 1$ with $f(x) = 1$ $(x \in -K)$. Then

$$\begin{aligned}m_\mu(K) &= \sup_{x\in G}|\mu|(K+x) \le \sup_{x\in G}\int_G f(x-y)d|\mu|(y)\\ &= \sup_{x\in G} f * |\mu|(x) = \|f * |\mu|\|_\infty < \infty.\end{aligned}$$

Hence μ is translation bounded. □

Lemma 2. *$C_{c,2}(G) \subset S_\mu(G)$.*

Proof. Suppose $f \in C_{c,2}(G)$. Then $\int_\Gamma |\hat{f}(\gamma)|d|\hat{\mu}|(\gamma) < \infty$ follows from (2) and $\||f| * |\mu|\|_\infty < \infty$ follows from Lemma 1. Let $\varepsilon > 0$ be given. Let $K =$

$\mathrm{supp}(f)$ and let U be a compact neighborhood of $0 \in G$ such that $\|f - f_y\|_\infty < \frac{\varepsilon}{2m_\mu(-K)}$ $(y \in U)$. Then

$$\begin{aligned}\||f - f_y| * |\mu|\|_\infty &\le \|f - f_y\|_\infty m_\mu(-\mathrm{supp}(f - f_y)) \\ &\le \frac{\varepsilon}{2m_\mu(-K)} m_\mu(-(K \cup (K + y))) \le \varepsilon \quad (y \in U).\end{aligned}$$

□

Lemma 3. *Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(S_\mu(G), \|\ \|_\mu)$. Then there exists $f \in S_\mu(G)$ such that $\|f - f_n\|_\mu \to 0$ $(n \to \infty)$.*

Proof. We can suppose without loss of generality that $\|f_n - f_{n-1}\|_\mu \le \frac{1}{n^2}$, $1 \le n$, $f_0 = 0$. Let $F_n := f_n - f_{n-1}$, $n = 1, 2, 3, \ldots$. We readily know that there exists $f \in L^1(G)$ such that

$$f(x) = \lim_{n\to\infty} f_n(x)(dx - a.e.), \text{ and } \|f - f_n\|_1 \to 0 \ (n \to \infty). \tag{3}$$

Further we have

$$\begin{aligned}\||f| * |\mu|\|_\infty &\le \|\Big|\sum_{n=1}^\infty F_n\Big| * |\mu|\|_\infty \\ &\le \sum_{n=1}^\infty \||F_n| * |\mu|\|_\infty \le \sum_{n=1}^\infty \frac{1}{n^2} < \infty,\end{aligned} \tag{4}$$

$$\begin{aligned}\||f - f_n| * |\mu|\|_\infty &\le \|\Big|\sum_{k=n+1}^\infty F_k\Big| * |\mu|\|_\infty \\ &\le \sum_{k=n+1}^\infty \||F_k| * |\mu|\|_\infty \le \sum_{k=n+1}^\infty \frac{1}{k^2} \to 0 \ (n \to \infty),\end{aligned} \tag{5}$$

$$\begin{aligned}\int_\Gamma |\hat{f}(\gamma)| d|\hat{\mu}|(\gamma) d\gamma &= \int_\Gamma \Big|\sum_{n=1}^\infty \widehat{F_n}(\gamma)\Big| d|\hat{\mu}|(\gamma) \\ &\le \sum_{n=1}^\infty \int_\Gamma |\widehat{F_n}(\gamma)| d|\hat{\mu}|(\gamma) \le \sum_{n=1}^\infty \frac{1}{n^2} < \infty,\end{aligned} \tag{6}$$

$$\begin{aligned}\int_\Gamma \Big|\widehat{f - f_n}(\gamma)\Big| d|\hat{\mu}|(\gamma) &\le \sum_{k=n+1}^\infty \int_\Gamma |\hat{F}_k(\gamma)| d|\hat{\mu}|(\gamma) \\ &\le \sum_{k=n+1}^\infty \frac{1}{k^2} \to 0 \ (n \to \infty).\end{aligned} \tag{7}$$

To show $\lim_{y\to 0} \||f - f_y| * |\mu|\|_\infty = 0$, let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\||f - f_N| * |\mu|\|_\infty \le \varepsilon/4$. Let U be a neighborhood of $0 \in G$ such that $\||f_N - (f_N)_y| * |\mu|\|_\infty < \varepsilon/2$ $(y \in U)$. Then

$$\begin{aligned}\||f - f_y| * |\mu|\|_\infty &\le \||f - f_N| * |\mu|\|_\infty + \||f_N - (f_N)_y| * |\mu|\|_\infty \\ &\quad + \|f_y - (f_N)_y| * |\mu|\|_\infty \le \varepsilon \quad (y \in U).\end{aligned} \tag{8}$$

Therefore $f \in S_\mu(G)$ by (4), (6) and (8), and $\|f - f_n\|_\mu \to 0 (n \to \infty)$ by (3), (5) and (7). □

3. Main results

Theorem 1. *Let $\mu \in \mathfrak{M}_T(G) \cap \mathfrak{M}_B(G)$. Then we have*

(i) $(S_\mu(G), \| \ \|_\mu)$ *is a Segal algebra.*

(ii) $C_{c,2}(G)$ *is a dense subspace of* $S_\mu(G)$.

(iii) *For all $f \in S_\mu(G)$ and all $x \in G$, the generalized Poisson summation formula holds* (*cf.* [1, Theorem 3.3]):

$$\int_G f(x-y)d\mu(y) = \int_\Gamma (x,\gamma)\hat{f}(\gamma)d\hat{\mu}(\gamma). \tag{9}$$

Proof. (i) By Lemma 3, $S_\mu(G)$ is a Banach space, and $\| \ \|_1 \le \| \ \|_\mu$ is clear. For $f \in S_\mu(G)$ and $y \in G$, we readily know $f_y \in S_\mu(G)$ and

$$\begin{aligned}\|f_y\|_\mu &= \|f_y\|_1 + \||f_y| * |\mu|\|_\infty + \|(-y,\gamma)\hat{f}\|_{L^1(\hat{\mu})}\\ &= \|f\|_1 + \||f| * |\mu|\|_\infty + \|\hat{f}\|_{L^1(\hat{\mu})} = \|f\|_\mu.\end{aligned}$$

Further, let $f \in S_\mu(G)$ and $\varepsilon > 0$ be given arbitrarily. We can choose a neighborhood U of $0 \in G$ such that

$$\begin{aligned}\|f - f_y\|_1 &\le \varepsilon/3 \quad (y \in U),\\ \||f - f_y| * |\mu|\|_\infty &\le \varepsilon/3 \quad (y \in U),\\ \|\hat{f} - \widehat{f_y}\|_{L^1(\hat{\mu})} = \|\hat{f} - (-y,\gamma)\hat{f}\|_{L^1(\hat{\mu})} &\le \varepsilon/3 \quad (y \in U).\end{aligned}$$

Therefore we have

$$\|f - f_y\|_\mu = \|f - f_y\|_1 + \||f - f_y| * |\mu|\|_\infty + \|\hat{f} - \widehat{f_y}\|_{L^1(\hat{\mu})} \le \varepsilon \quad (y \in U).$$

Since $S_\mu(G)$ contains $C_{c,2}(G)$ by Lemma 2 which is a dense subspace of $L^1(G)$, it follows that $S_\mu(G)$ is a dense subspace of $L^1(G)$. Hence $(S_\mu(G), \| \ \|_\mu)$ is a Segal algebra.

(ii) Let I be the closure of $C_{c,2}(G)$ in $S_\mu(G)$. Then I is an ideal of $S_\mu(G)$. Indeed, for any $f \in I$ and $g \in S_\mu(G)$, we can choose sequences $\{f_n\}, \{g_n\}$ in $C_{c,2}(G)$ such that $\|f - f_n\|_\mu \to 0$ $(n \to \infty)$, $\|g - g_n\|_1 \to 0$ $(n \to \infty)$. Then we have (see [3, §4, Proposition 1])

$$\|f * g - f_n * g_n\|_\mu \le \|f - f_n\|_\mu \|g\|_1 + \|f_n\|_\mu \|g - g_n\|_1 \to 0 \ (n \to \infty),$$

so $f * g \in I$. By an ideal theorem for Segal algebras (cf. [4, Theorem 6.2.9]), $I = \bar{I} \cap S_\mu(G)$. Since $\bar{I} = L^1(G)$, we have $I = S_\mu(G)$, that is, $C_{c,2}(G)$ is a dense subspace of $S_\mu(G)$.

(iii) Let $f \in S_\mu(G)$ and $y \in G$ be given. By (ii), we can choose a sequence $\{f_n\}_{n=1}^\infty \subset C_{c,2}(G)$ such that $\|f - f_n\|_\mu \to 0$ $(n \to \infty)$. It follows that

$$\left| \int_G f(y-x)d\mu(x) - \int_G f_n(y-x)d\mu(x) \right| \le \|f - f_n\|_\mu \to 0 \ (n \to \infty) \tag{10}$$

and

$$\left| \int_{\Gamma} (y, \gamma) \hat{f}(\gamma) d\hat{\mu}(\gamma) - \int_{\Gamma} (y, \gamma) \widehat{f_n}(\gamma) d\hat{\mu}(\gamma) \right| \leq \|f - f_n\|_{\mu} \to 0 \ (n \to \infty). \tag{11}$$

Since $f_n(x)$ (hence $(f_n(x - y), y \in G) \in C_{c,2}(G)$, $n = 1, 2, 3, \ldots$, we have from (2)

$$\int_G f_n(y - x) d\mu(x) = \int_{\Gamma} (y, \gamma) \widehat{f_n}(\gamma) d\hat{\mu}(\gamma). \tag{12}$$

By (10), (11) and (12), the desired equality (9) follows. □

Theorem 2. $\mathfrak{M}_T(G) \cap \mathfrak{M}_B(G) = \mathscr{I}(G)$.

Proof. $\supseteq$: Suppose $\mu \in \mathcal{I}(G)$. Then $\mu \in \mathfrak{M}_T(G)$ and $\hat{\mu} \in \mathfrak{M}_T(\Gamma)$ with $\mu' = \hat{\hat{\mu}}$ by [1, Theorem 3.4], and we have $\mu' \in \mathfrak{M}_B(G)$ by [1, Theorem 2.5], and hence $\mu \in \mathfrak{M}_B(G)$.

$\subseteq$: Suppose $\mu \in \mathfrak{M}_T(G) \cap \mathfrak{M}_B(G)$, and let $h \in C_c(\Gamma)$ be given arbitrarily. Since $\|\hat{h}\|_2 = \|h\|_2 < \infty$, we have

$$\widehat{h * h^*} = |\hat{h}|^2 \in L^1(G). \tag{13}$$

From (13) and the inversion theorem, we have

$$h * h^*(\gamma) = \int_G (x, \gamma) |\hat{h}|^2(x) dx = \int_G (-x, \gamma) |\check{h}(x)|^2 dx \quad (\gamma \in \Gamma). \tag{14}$$

(14) shows that the Fourier transform of $|\check{h}|^2$ has compact support, and by [4, Proposition 6.2.5] (see also [3, §5, Examples (vii)]), $|\check{h}|^2$ belongs to $S_{\mu}(G)$. By (9) and (14), it follows that

$$\int_G |\check{h}(x)|^2 d\mu'(x) = \int_G |\check{h}(-x)|^2 d\mu(x) = \int_{\Gamma} h * h^*(\gamma) d\hat{\mu}(\gamma). \tag{15}$$

The definition of the transformable measures and (15) imply $\hat{\mu} \in \mathfrak{M}_T(\Gamma)$ with its Fourier transform μ', that is, $\mu \in \mathscr{I}(G)$ with $\check{\hat{\mu}} = \mu$. □

Definition 6 ([1, p. 39])**.** Let H be a closed subgroup of G, and let $\mu \in \mathfrak{M}(H)$. We can consider μ as a measure in $\mathfrak{M}(G)$ whose support is contained in H. In this case we express it by $\iota\mu \in \mathfrak{M}(G)$. A measure $\nu \in \mathfrak{M}(G)$ is called H-invariant if $\nu * \delta_h = \nu$ for every element $h \in H$. We denote by $\mathfrak{M}_H(G)$ the set of all H-invariant measures in $\mathfrak{M}(G)$.

Obviously, in the case where $\nu(\neq 0)$ is concentrated in H, ν is H-invariant if and only if $\nu|_H$ is a Haar measure of H.

A measure $\mu \in \mathfrak{M}(G)$ is called periodic if G/I is compact, where I is the closed subgroup consisting of all $x \in G$ satisfying $\delta_x * \mu = \mu$.

Corollary 1. *Every periodic measure is contained in $\mathscr{I}(G)$.*

Proof. Suppose that $\mu \in \mathfrak{M}(G)$ is a periodic measure. By [1, Corollary 6.1], μ belongs to $\mathfrak{M}_T(G)$. Since μ is periodic, there exists a closed subgroup I of G such that the quotient group G/I is compact and $\delta_x * \mu = \mu$ for all $x \in I$. There exists a compact subset H of G such that $H + I = G$. Hence for any $x \in G$, there exist $h \in H$ and $y \in I$ such that $x = h + y$. Then for any $K \in \mathcal{K}(G)$, we have

$$\begin{aligned} |\mu|(K + x) &= |\mu|(K + h + y) \leq |\mu|(K + H + y) \\ &= |\mu| * \delta_{-y}(K + H) = |\mu|(K + H) < \infty, \end{aligned}$$

that is, $\mu \in \mathfrak{M}_B(G)$. Hence we have $\mu \in \mathscr{I}(G)$ from Theorem 2. □

A measure $\mu \in \mathfrak{M}(G)$ is called positive definite if $\int_G f * f^*(x) d\mu(x) \geq 0$ for all $f \in C_c(G)$ ([1, p. 23)]). It is known that every positive definite measure is transformable, but not necessarily translation bounded ([1, Theorem 4.1 and Proposition 7.1]). Therefore the next corollary is an immediate consequence of Theorem 2.

Corollary 2. *A positive definite measure $\mu \in \mathfrak{M}(G)$ belongs to $\mathscr{I}(G)$ if and only if it is translation bounded.*

Theorem 3. *Let H be a closed subgroup of G, and let $\mu \in \mathfrak{M}(H)$. Then, μ belongs to $\mathscr{I}(H)$ if and only if $\iota\mu$ belongs to $\mathscr{I}(G)$.*

Proof. Suppose $\mu \in \mathscr{I}(H)$. Then $\iota\mu \in \mathfrak{M}_T(G)$ by [1, Theorem 6.2]. On the other hand, since $\mu \in \mathfrak{M}_B(H)$, μ is shift-bounded by Lemma 1, and by [2, Proposition 1.16] $\iota\mu$ is shift-bounded. By Lemma 1 again, $\iota\mu \in \mathfrak{M}_B(G)$. Hence $\iota\mu \in \mathscr{I}(G)$ by Theorem 2.

Conversely, suppose $\iota\mu \in \mathscr{I}(G)$. Then $\widehat{\iota\mu} \in \mathscr{I}(\Gamma)$ which is $H^\perp$-invariant ([1, Proposition 6.1]). For each $\eta \in C_c(\Gamma)$ define $T\eta \in C_c(\Gamma/H^\perp)$ by

$$T\eta(\dot{\gamma}) = \int_{H^\perp} \eta(\gamma + \gamma') d\gamma' \quad (\dot{\gamma} = \gamma + H^\perp \in \Gamma/H^\perp).$$

There exists an isomorphism $\nu \to \dot{\nu}$ from $\mathfrak{M}_{H^\perp}(\Gamma)$ onto $\mathfrak{M}(\Gamma/H^\perp)$ defined by the so called generalized Weil formula (cf. [1, p. 40, (6.2)])

$$\int_\Gamma \eta(\gamma) d\nu(\gamma) = \int_{\Gamma/H^\perp} T\eta(\dot{\gamma}) d\dot{\nu}(\dot{\gamma}) \quad (\eta \in C_c(\Gamma)). \tag{16}$$

By [1, Theorem 6.1], $\dot{\widehat{\iota\mu}} \in \mathfrak{M}_T(\Gamma/H^\perp)$ which satisfies

$$\widehat{\widehat{\iota\mu}} = \iota\widehat{\dot{\widehat{\iota\mu}}}. \tag{17}$$

On the other hand, since $\iota\mu \in \mathscr{I}(G)$, we have from [1, Theorem 3.4]

$$(\iota\mu)' = \widehat{\widehat{\iota\mu}}. \tag{18}$$

From (17) and (18), we have

$$\iota\mu' = (\iota\mu)' = \iota\widehat{\dot{\widehat{\iota\mu}}}, \text{ that is, } \mu' = \widehat{\dot{\widehat{\iota\mu}}}. \tag{19}$$

Next, we show that $\dot{\widehat{\iota\mu}} \in \mathfrak{M}_B(\Gamma/H^\perp)$. Since $\widehat{\iota\mu} \in \mathfrak{M}_B(\Gamma)$, it is shift-bounded by Lemma 1. If $\eta \in C_c(\Gamma)$ and $\gamma_1 \in \Gamma$, we have from (16)

$$\begin{aligned}\int_\Gamma \eta(\gamma_1-\gamma)d\widehat{\iota\mu}(\gamma) &= \int_\Gamma \eta'_{\gamma_1}(\gamma)d\widehat{\iota\mu}(\gamma) = \int_{\Gamma/H^\perp} (T\eta'_{\gamma_1})(\dot\gamma)d\dot{\widehat{\iota\mu}}(\dot\gamma)\\ = \int_{\Gamma/H^\perp}\left(\int_{H^\perp}\eta(\gamma_1-\gamma+\gamma')d\gamma'\right)&d\dot{\widehat{\iota\mu}}(\dot\gamma) = \int_{\Gamma/H^\perp}(T\eta)(\dot\gamma_1-\dot\gamma)d\dot{\widehat{\iota\mu}}(\dot\gamma),\end{aligned}$$

and hence

$$\begin{aligned}\|(T\eta)*\dot{\widehat{\iota\mu}}\|_\infty &= \sup_{\dot\gamma_1\in\Gamma/H^\perp}\left|\int_{\Gamma/H^\perp}(T\eta)(\dot\gamma_1-\dot\gamma)d\dot{\widehat{\iota\mu}}(\dot\gamma)\right|\\ &= \sup_{\gamma_1\in\Gamma}\left|\int_\Gamma \eta(\gamma_1-\gamma)d\widehat{\iota\mu}(\gamma)\right| = \|\eta*\widehat{\iota\mu}\|_\infty<\infty \quad (\eta\in C_c(\Gamma)).\end{aligned} \tag{20}$$

Since T maps $C_c(\Gamma)$ onto $C_c(\Gamma/H^\perp)$ ([1, p. 40]), it follows from (20) that $\dot{\widehat{\iota\mu}}$ is shift-bounded, and by Lemma 1, $\dot{\widehat{\iota\mu}} \in \mathfrak{M}_B(\Gamma/H^\perp)$. Thus, by Theorem 2, we have $\dot{\widehat{\iota\mu}} \in \mathscr{I}(\Gamma/H^\perp)$. Therefore, from the last equation in (19), we have $\mu' \in \mathscr{I}(H)$, which implies $\mu \in \mathscr{I}(H)$. □

Remark 2. Let H be a closed subgroup of G, $m_H \in \mathfrak{M}(H)$ be a fixed Haar measure of H. There exists a Haar measure $m_{H^\perp}$ of $H^\perp$ such that $\iota m_{H^\perp} = \widehat{\iota m_H}$ ([1, Proposition 6.2]). More precisely, $m_{H^\perp}$ is the Plancherel measure corresponding to $\dot\lambda$ ([1, Corollary 6.2]), where $\lambda = dx$ and $\dot\lambda$ is a measure in $\mathfrak{M}(G/H)$ determined by the formula

$$Tf(\dot x) = \int_H f(x+h)dh \quad (x\in G),$$
$$\int_G f(x)d\lambda(x) = \int_{G/H} Tf(\dot x)d\dot\lambda(\dot x) \quad (f\in C_c(G)).$$

This remark will be used in Examples II and III in the next section.

4. Examples

Example I. By applying Theorem 1 to several "μ"s in $M(G)(\subset \mathscr{I}(G))$, we have the following formulas:

(a) $f(x) = \int_\Gamma (x,\gamma)\hat f(\gamma)d\gamma \quad (f\in L^1(G)\cap C_0(G), \hat f\in L^1(\Gamma))$,
$\mu = \delta_0$, $\hat\mu = d\gamma$: the inversion theorem.

(b) $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x-y)e^{-\frac12 y^2}dy = \frac{1}{\sqrt{2\pi}}\int_{\hat{\mathbb{R}}} e^{ixt}\hat f(t)e^{-\frac12 t^2}dt \quad (f\in L^1(\mathbb{R}), x\in\mathbb{R})$,
$\mu = e^{-\frac12 x^2}dx$, $\hat\mu = e^{-\frac12 t^2}dt$: formula for Gaussian transform.

(c) $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x-y)\frac{a}{\pi(a^2+y^2)}dy = \frac{1}{2\pi}\int_{\hat{\mathbb{R}}} e^{itx}\hat f(t)e^{-a|t|}dt \ (f\in L^1(\mathbb{R}), x\in\mathbb{R})$,
$\mu = \frac{a}{\pi(a^2+x^2)}dx$, $\hat\mu = \frac{1}{\sqrt{2\pi}}e^{-a|t|}dt$, $a>0$: formula for Cauchy distribution.

Although all the formulas above are immediate consequences of [1, Theorem 3.3], Theorem 1 may be of some help to visualize the abundance of the generalized Poisson summation formula introduced by L. Argabright and J. Gil de Lamadrid.

Example II. $\mathbb{R}^d$ denotes the d-dimensional real group and $\mathbb{Z}^d$ is its subgroup consisting of elements with integer coordinates. $\hat{\mathbb{R}}^d$ denotes the dual group of $\mathbb{R}^d$ and $\hat{\mathbb{Z}}^d$ is its subgroup consisting of elements with integer coordinates:

$$\hat{\mathbb{Z}}^d = \{(m_1, \ldots, m_d) : m_k \in \hat{\mathbb{Z}}, k = 1, \ldots, d\} \quad \text{and}$$

$$((x_1, \ldots, x_d), (t_1, \ldots, t_d)) = e^{\sum_{k=1}^d ix_k t_k} \ \big((x_1, \ldots, x_d) \in \mathbb{R}^d; (t_1, \ldots, t_d) \in \hat{\mathbb{R}}^d\big).$$

We fix the Haar measure $\lambda = \frac{1}{(2\pi)^{d/2}} dx_1 \cdots dx_d$ on $\mathbb{R}^d$, and fix the counting measure $\omega = m_{\mathbb{Z}}$ on $\mathbb{Z}^d$. Then the Haar measure on $\mathbb{R}^d/\mathbb{Z}^d$ corresponding to ω is $\dot{\lambda} = \frac{1}{(2\pi)^{d/2}} d\dot{x}_1 \cdots d\dot{x}_d$, where $d\dot{x}_1 \cdots d\dot{x}_d$ is the Lebesgue measure on $\mathbb{R}^d/\mathbb{Z}^d$. It is easy to see that $\mathbb{Z}^\perp = (2\pi\hat{\mathbb{Z}})^d$, which is the dual group of $\mathbb{R}^d/\mathbb{Z}^d$, and the Plancherel measure on $(2\pi\hat{\mathbb{Z}})^d$ corresponding to $\dot{\lambda}$ is $(2\pi)^{d/2} m_{(2\pi\hat{\mathbb{Z}})^d}$. Therefore, by Remark 2, it follows that $\widehat{\iota m_{\mathbb{Z}^d}} = \iota(2\pi)^{d/2} m_{(2\pi\hat{\mathbb{Z}})^d}$. By applying Theorem 1, we have the following:

$$\begin{aligned} S_{\iota m_{\mathbb{Z}^d}}(\mathbb{R}^d) := \Bigg\{ & f \in L^1(\mathbb{R}^d) : \sup_{y \in \mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} |f(y-n)| < \infty, \\ & \lim_{x \to 0} \sup_{y \in \mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} |f(x+y-n) - f(y-n)| = 0, \\ & \sum_{(m_1, \ldots, m_d) \in \hat{\mathbb{Z}}^d} |\hat{f}(2\pi m_1, \ldots, 2\pi m_d)| < \infty \Bigg\}, \end{aligned}$$

with norm

$$\begin{aligned} \|f\|_{\iota m_{\mathbb{Z}^d}} = \|f\|_1 &+ \sup_{y \in \mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} \big|f(y-n)\big| \\ &+ (2\pi)^{\frac{d}{2}} \sum_{(m_1, \ldots, m_d) \in \hat{\mathbb{Z}}^d} |\hat{f}(2\pi m_1, \ldots, 2\pi m_d)| \ (f \in S_{\iota m_{\mathbb{Z}^d}}(\mathbb{R}^d)) \end{aligned}$$

is a Segal algebra, and for all $f \in S_{\iota m_{\mathbb{Z}^d}}(\mathbb{R}^d)$ and for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, the Poisson summation formula holds:

$$\sum_{n \in \mathbb{Z}^d} f(x-n) = (2\pi)^{\frac{d}{2}} \sum_{(m_1, \ldots, m_d) \in \hat{\mathbb{Z}}^d} e^{\sum_{k=1}^d 2\pi i x_k m_k} \hat{f}(2\pi m_1, \ldots, 2\pi m_d).$$

Example III. Let $G = \mathbb{R}^n$ be the n-dimensional real group with the dual group $\hat{\mathbb{R}}^n$, and let $0 < m < n$. We fix a Haar measure on $\mathbb{R}^n$: $\lambda = \frac{1}{(2\pi)^{n/2}} dx_1 \cdots dx_n$. Let

$$H = \big\{(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \in \mathbb{R}^n : x_{m+1} = \cdots = x_n = 0\big\} (\cong \mathbb{R}^m)$$

be a closed subgroup of $\mathbb{R}^n$ with the annihilator

$$H^\perp = \left\{(t_1, \ldots, t_m, t_{m+1}, \ldots, t_n) \in \hat{\mathbb{R}}^n : t_1 = \cdots = t_m = 0\right\}(\cong \hat{\mathbb{R}}^{n-m}).$$

We fix a Haar measure $\omega = \frac{1}{(2\pi)^{m/2}} dx_1 \cdots dx_m$ on H. Then $\iota\omega \in \mathfrak{M}_T(\mathbb{R}^n) \cap \mathfrak{M}_B(\mathbb{R}^n)$ by Theorem 3. Obviously,

$$\mathbb{R}^n/H = \mathbb{R}^{n-m} \text{ and } \dot{\lambda} = \frac{1}{(2\pi)^{(n-m)/2}} dx_{m+1} \cdots dx_n.$$

Also $H^\perp \cong \widehat{\mathbb{R}^n/H} \cong \hat{\mathbb{R}}^{n-m}$. Then the Plancherel measure corresponding to $\dot{\lambda}$ is $\tilde{\omega} = \frac{1}{(2\pi)^{(n-m)/2}} dt_{m+1} \cdots dt_n$ and, by Remark 2, $\widehat{\iota\omega} = \iota\tilde{\omega}$ follows.

By Theorem 1, we have the following:

$$S_{\iota\omega}(\mathbb{R}^n) = \Bigg\{ f \in L^1(\mathbb{R}^n) : \||f| * \iota\omega\|_\infty < \infty,\ \lim_{y\to 0} \||f - f_y| * \iota\omega\|_\infty = 0,$$
$$\int_{H^\perp} |\hat{f}(0, \ldots, 0, t_{m+1}, \ldots, t_n)| \frac{1}{(2\pi)^{(n-m)/2}} dt_{m+1} \cdots dt_n < \infty \Bigg\},$$

with norm

$$\begin{aligned} &\|f\|_{\iota\omega} \\ &= \|f\|_1 + \||f| * (\iota\omega)\|_\infty \\ &\quad + \int_{H^\perp} |\hat{f}(0, \ldots, 0, t_{m+1}, \ldots, t_n)| \frac{1}{(2\pi)^{(n-m)/2}} dt_{m+1} \cdots dt_n \ (f \in S_{\iota\omega}(\mathbb{R}^n)) \end{aligned}$$

is a Segal algebra, and for all $f \in S_{\iota\omega}(\mathbb{R}^n)$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, the generalized Poisson summation formula for $\iota\omega$ holds:

(21)

$$\begin{aligned} &\int_H f(x_1 - y_1, \ldots, x_m - y_m, x_{m+1}, \ldots, x_n) \frac{1}{(2\pi)^{m/2}} dy_1 \cdots dy_m \\ &= \int_{H^\perp} e^{i(x_{m+1}t_{m+1} + \cdots + x_n t_n)} \hat{f}(0, \ldots, 0, t_{m+1}, \ldots, t_n) \frac{1}{(2\pi)^{(n-m)/2}} dt_{m+1} \cdots dt_n. \end{aligned}$$

Remark 3. Of course, the equation (21) is not new, since it an immediate consequence of [1, Theorem 3.3].

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