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GENERALIZED KILLING STRUCTURE JACOBI OPERATOR FOR REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS

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Abstract. In this paper, first we introduce a new notion of generalized Killing structure Jacobi operator for a real hypersurface M in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S\,(U_2 \cdot U_m)$. Next we prove that there does not exist a Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S\,(U_2 \cdot U_m)$ with generalized Killing structure Jacobi operator.

1. Introduction

In 20th century, classifications with certain geometric problems for real hypersurfaces in complex space form or quaternionic space form were main research subjects in the field of differential geometry (see [19, 20, 22]). Recently, many kinds of geometric problems have been considered for the classification of real hypersurfaces in the complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2 \cdot U_m)$ or complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2 \cdot U_m)$ (see [3, 4, 7, 12, 23, 28–30]). Indeed, the complex space form and the complex (hyperbolic) two-plane Grassmannians mentioned above can be regarded as typical examples of Hermitian symmetric spaces.

In general, a Hermitian symmetric space $\bar{M}$ is defined by a connected complex manifold with a Hermitian structure. Each point $p \in \bar{M}$ is an isolated fixed point of an involutive holomorphic isometry s_p of $\bar{M}$. A Hermitian symmetric space $\bar{M}$ is a Riemannian symmetric space of even dimension (for more detail, see [11]). By using this property, the classification problem of real hypersurfaces

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with shape operator in Hermitian symmetric spaces have been investigated by Berndt and Suh [3–5, 7], Martinez and Pérez [19], Pérez [21], Suh [26, 27].

Certain parallelism on the other symmetric operators like Ricci operator, structure Jacobi and normal Jacobi operators for real hypersurfaces in Hermitian symmetric spaces are extensively studied. Among them, the study of Ricci operator were considered by Lee, Suh and Woo [15], Pérez and Suh [23], Pérez, Suh and Watanabe [24], Suh [28–30], Suh and Woo [31]. Moreover, the structure Jacobi and normal Jacobi operators for real hypersurfaces in Hermitian symmetric spaces have been undertaken by Lee, Suh and Woo [12, 16, 17].

Based on these results, in this paper, we will consider a new notion of generalized Killing structure Jacobi operator for real hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$. In order to do this, we first define the Killing vector field (often called a Killing field) as follows.

Definition 1.1. Let $(\bar{M}, g)$ be a Riemannian manifold with metric g. A vector field X is said to be a *Killing field* if the Lie derivative with respect to X of the metric g vanishes, that is, $\mathcal{L}_X g = 0$.

As a special case of Killing field for a real hypersurface M in a Riemannian manifold $\bar{M}$, we can give the notion of *isometric Reeb flow*, which means that the Reeb vector field $\xi = -JN$, where N denotes the normal vector field of M, is Killing. By using Lie algebraic methods given in [1], [2] and [9], Berndt-Suh [6] gave a complete classification of real hypersurfaces with isometric Reeb flow in Hermitian symmetric spaces. In [26], Suh considered the notion of isometric Reeb flow for real hypersurfaces in the complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ and give a classification theorem as follows.

Theorem A. *Let M be a connected orientable real hypersurface in the complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then, the Reeb flow on M is isometric if and only if M is locally congruent to an open part of*

($\mathcal{T}_A^*$) *a tube around some totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or*

($\mathcal{H}_A^*$) *a horosphere whose center at infinity is singular.*

As a generalization of such a Killing vector field, Yano (see [33–35]) defined the notion of Killing tensor as follows.

Definition 1.2. A skew symmetric tensor field $T_{i_1\cdots i_r}$ of order r is *Killing* if it satisfies

$$\nabla_{i_1} T_{i_2\cdots i_{r+1}} + \nabla_{i_2} T_{i_1\cdots i_{r+1}} = 0.$$

Blair [8] has applied the notion of Killing tensor to a tensor field T of type $(1,1)$ on a Riemannian manifold $\bar{M}$ and a geodesic γ defined on $\bar{M}$. If we denote by γ' the tangent vector of the geodesic γ, then $T\gamma'$ is parallel along the geodesic γ for the Killing tensor field T. Geometrically, this means that

$(\nabla_{\gamma'} T)\gamma' = 0$ along a geodesic γ on $\bar{M}$. If this is the case for any geodesic on $\bar{M}$, we have

$$(\nabla_X T)X = 0 \quad \text{or equivalently} \quad (\nabla_X T)Y + (\nabla_Y T)X = 0$$

for any vector fields X and Y on $\bar{M}$. In this case we say that the tensor T a *Killing tensor field of type* $(1,1)$.

On the other hand, Heil, Moroianu and Semmelmann [10], Semmelmann [25] have remarked that Killing p-tensors are symmetric p-tensor with vanishing symmetrized covariant derivative and the existing literature on symmetric Killing tensors is huge, especially coming from theoretical physics. Moreover, Semmelmann [25] has asserted that a classical object of differential geometry are Killing vector fields. These are by definition infinitesimal isometries, i.e., the flow of such a vector field preserves a given metric (see also Thompson [32]).

Now, we define a *structure Jacobi tensor* $\mathbb{R}_\xi$ of type (0,2) on $\bar{M}$ given by

$$\mathbb{R}_\xi(X, Y) = g(R_\xi X, Y),$$

where R_ξ is the structure Jacobi operator of type (1,1) and X, Y are vector fields on $\bar{M}$. Furthermore, we can also define:

Definition 1.3. The symmetric structure Jacobi tensor $\mathbb{R}_\xi$ of type (0,2) on $\bar{M}$ is called *generalized Killing* if the equation

$$(\nabla_X \mathbb{R}_\xi)(X, X) = g((\nabla_X R_\xi) X, X) = 0 \tag{1.1}$$

holds for all vector fields $X \in T\bar{M}$.

On the other hand, by virtue of polarization, (1.1) can be rearranged as

$$g((\nabla_X R_\xi) Y, Z) + g((\nabla_Y R_\xi) Z, X) + g((\nabla_Z R_\xi) X, Y) = 0 \tag{1.2}$$

for any vector fields X, Y and Z on $\bar{M}$. We say that the structure Jacobi operator R_ξ is *cyclic parallel* if it satisfies (1.2). For the sake of convenience, (1.2) can be written as

$$\mathfrak{S}_{X,Y,Z}\, g((\nabla_X R_\xi) Y, Z) = 0 \tag{1.3}$$

for any X, Y and $Z \in TM$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to the vector fields X, Y and Z. So, the notion of generalized Killing structure Jacobi tensor of $\bar{M}$ is the same as cyclic parallel structure Jacobi operator of $\bar{M}$. Here, we can give the geometric meaning of the generalized Killing structure Jacobi tensor as follows: When we consider a geodesic γ with initial conditions such that $\gamma(0) = z \in \bar{M}$ and $\dot{\gamma}(0) = X$. Then the structure Jacobi curvature $\mathbb{R}_\xi(\dot{\gamma}, \dot{\gamma}) = g(R_\xi \dot{\gamma}, \dot{\gamma})$ is constant along the geodesic γ of the vector field X (see Semmelmann [25]).

On the other hand, a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ is said to be *Hopf* if the shape operator A of M satisfies $A\xi = \alpha\xi$, $\alpha = g(A\xi, \xi)$, for the Reeb vector field $\xi = -JN$, where N denotes a unit normal vector field on M.

From such a view point, in a direction of generalized Killing structure Jacobi operator for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ we gave an important result. In fact, recently, for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with generalized Killing structure Jacobi operator Lee, Suh, and Woo [17] gave a classification theorem as follows:

Theorem B. *Let M be a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the structure Jacobi operator R_ξ of M is generalized Killing if and only if M is locally congruent to an open part of a tube of $r = \frac{\pi}{4\sqrt{2}}$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Motivated by this result, it is natural to consider a generalized Killing structure Jacobi operator for real hypersurfaces M in $SU_{2,m}/S(U_2 \cdot U_m)$. Then we can assert the following:

Main Theorem. *There does not exist a connected Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with generalized Killing structure Jacobi operator.*

On the other hand, the symmetric tensor T on M is said to be *parallel* if the tensor T satisfies $\nabla T = 0$. If the symmetric tensor T is parallel, then T, naturally, satisfies

$$\mathfrak{S}_{X,Y,Z \in TM}\, g((\nabla_X T)Y, Z) = 0$$

for any tangent vector fields X, Y and Z on M. This is a natural generalization of the parallel symmetric tensor T and can be rephrased as follows:

If the symmetric tensor T is parallel, then naturally T becomes a generalized Killing tensor.

Consequently, it is a general notion weaker than usual parallelism. If we apply such a relation to the structure Jacobi operator R_ξ for a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, we can give the following result from our Main Theorem.

Corollary. *There does not exist a connected Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with parallel structure Jacobi operator.*

2. The complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$

In this section we summarize basic material about complex hyperbolic two-plane Grassmann manifolds $SU_{2,m}/S(U_2 \cdot U_m)$, for details we refer to [3–5, 7, 26–28].

The Riemannian symmetric space $SU_{2,m}/S(U_2 \cdot U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, becomes a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2 \cdot U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K, respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m} A I_{2,m}$, where $I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$, I_2 and I_m denotes the identity (2×2)-matrix and $(m \times m)$-matrix, respectively. Then $\langle X, Y \rangle = -B(X, \theta Y)$ becomes a positive definite $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2 \cdot U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2 \cdot U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2 \cdot U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2 \cdot U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2 \cdot U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis J_1, J_2, J_3 of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2 \cdot U_m)$, is selfadjoint and satisfies $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X \mid \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_o SU_{2,m}/S(U_2 \cdot U_m)$ of $SU_{2,m}/S(U_2 \cdot U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2 \cdot U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_o SU_{2,m}/S(U_2 \cdot U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular. If there exists more than one maximal abelian subspaces of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

In Section 4, we will prove that under the given condition, the normal vector field N is singular tangent, that is, the Reeb vector field ξ belongs to either the maximal quaternionic subbundle $\mathcal{Q}$ or its orthogonal complement $\mathcal{Q}^\perp$ (see [17, 18]).

3. Real hypersurfaces in complex hyperbolic two-plane Grassamannian $SU_{2,m}/S(U_2 \cdot U_m)$

Let M be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, that is, a hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with real codimension one. It implies that the normal bundle T^*M of M in $SU_{2,m}/S(U_2 \cdot U_m)$

is given $T^*M = \text{span}\{N\}$, where N is a unit normal vector field of M in $SU_{2,m}/S(U_2 \cdot U_m)$. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi-Civita connection of (M, g). As mentioned in Section 2, complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ have the Kähler structure J and quaternionic Kähler structure $\mathfrak{J} = \text{span}\{J_1, J_2, J_3\}$. From these structures, let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector field X of a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$, where ϕX and $\phi_\nu X$ denote the tangential components of JX and $J_\nu X$, respectively.

From the Kähler structure J of $SU_{2,m}/S(U_2 \cdot U_m)$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \ \ \eta(\xi) = 1, \ \ \phi\xi = 0, \quad \text{and} \quad \eta(X) = g(X, \xi)$$

for any vector field X on M and $\xi = -JN$. If M is orientable, then the vector field ξ is globally defined and said to be the induced *Reeb vector field* on M.

Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then, each J_ν induces a local almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, on M. It satisfies

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \ \phi_\nu\xi_\nu = 0, \ \eta_\nu(\xi_\nu) = 1, \ \text{and} \ \eta_\nu(X) = g(X, \xi_\nu)$$

for any vector field X tangent to M and $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Moreover, it is known that the almost contact metric structure J_ν, $\nu = 1, 2, 3$ satisfies $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$ ($\nu = \text{mod}\, 3$). From this property, we get

$$\begin{aligned} &\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}. \end{aligned}$$

The tangential and normal components of the commuting identity $JJ_\nu X = J_\nu JX$ give

$$\phi\phi_\nu X - \phi_\nu\phi X = \eta_\nu(X)\xi - \eta(X)\xi_\nu \quad \text{and} \quad \eta_\nu(\phi X) = \eta(\phi_\nu X).$$

The last equation implies $\phi_\nu\xi = \phi\xi_\nu$.

Moreover, from the parallelisms of Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ (i.e., $\bar{\nabla}_X J = 0$ and $\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$, respectively), together with Gauss and Weingarten formulas, it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.1}$$

$$\begin{aligned} (\nabla_X\phi_\nu)Y &= -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu, \\ \nabla_X\xi_\nu &= q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX. \end{aligned} \tag{3.2}$$

Using the explicit expression for the Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2 \cdot U_m)$ in [3] the Codazzi equation takes the form

$$
\begin{aligned}
&(\nabla_X A)Y - (\nabla_Y A)X \\
&= -\frac{1}{2}\Big[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
&\qquad + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\big\} \\
&\qquad + \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\} \\
&\qquad + \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu\Big]
\end{aligned} \tag{3.3}
$$

for any vector fields X and Y on M. Moreover, we have the equation of Gauss as follows:

$$
\begin{aligned}
&R(X,Y)Z \\
&= -\frac{1}{2}\Big[g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
&\qquad + \sum_{\nu=1}^{3}\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\
&\qquad + \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y, Z)\phi_\nu\phi X - g(\phi_\nu\phi X, Z)\phi_\nu\phi Y\} \\
&\qquad - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\
&\qquad - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y, Z) - \eta(Y)g(\phi_\nu\phi X, Z)\}\,\xi_\nu\Big] \\
&\quad + g(AY, Z)AX - g(AX, Z)AY
\end{aligned} \tag{3.4}
$$

for any tangent vector fields X, Y and Z on M.

On the other hand, the Jacobi operator field with respect to X in a Riemannian manifold $\bar{M}$ is defined by $\bar{R}_X = \bar{R}(\,\cdot\,, X)X$, where $\bar{R}$ denotes the Riemannian curvature tensor of $\bar{M}$. We will call the Jacobi operator on a real hypersruface M in $\bar{M}$ with respect to ξ the *structure Jacobi operator* on M. Thus, from (3.4) the structure Jacobi operator R_ξ of M in $SU_{2,m}/S(U_2 \cdot U_m)$ is given by

$$
\begin{aligned}
R_\xi(X) &= R(X,\xi)\xi \\
&= -\frac{1}{2}\Big[X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu\Big\}
\end{aligned} \tag{3.5}
$$

$$- \sum_{\nu=1}^{3}\Big\{3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\Big\}\Big]$$
$$+ \alpha AX - \eta(AX)A\xi,$$

where the function α is defined by $\alpha = g(A\xi,\xi)$ and said to be the *Reeb function* on M (see [31]).

Finally, we denote by $\mathcal{C}$ and $\mathcal{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle TM on M, respectively. That is, $\mathcal{C}$ is the orthogonal complement in TM of the real span of ξ, and $\mathcal{Q}$ the orthogonal complement in TM of the real span of $\{\xi_1,\xi_2,\xi_3\}$. Hereafter, unless otherwise stated, we want to use these basic equations mentioned above frequently without referring to them explicitly.

4. Key lemma

Let M be a Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2\cdot U_m)$. Hereafter, unless otherwise stated, we consider that X and Y are any tangent vector fields on M. With the assumption of M being Hopf, together with the Codazzi equation, we obtain (see [3, 15, 16]):

$$Y\alpha = (\xi\alpha)\eta(Y) + 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y) \tag{4.1}$$

and

$$\begin{aligned} A\phi AY = {} & \frac{\alpha}{2}(A\phi+\phi A)Y + \sum_{\nu=1}^{3}\big\{\eta(Y)\eta_\nu(\xi)\phi\xi_\nu + \eta_\nu(\xi)\eta_\nu(\phi Y)\xi\big\} \\ & - \frac{1}{2}\phi Y - \frac{1}{2}\sum_{\nu=1}^{3}\big\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y\big\} \end{aligned} \tag{4.2}$$

for any vector field Y on M.

In order to consider the generalized Killing structure Jacobi operator, let us calculate the formula $(\nabla_X R_\xi)Y$ for any tangent vector fields X and Y on M. From (3.5) and our assumption of M being Hopf, it follows that

$$\begin{aligned} & 2(\nabla_X R_\xi)Y \\ & = g(\phi AX,Y)\xi + \eta(Y)\phi AX - 2\alpha\eta(Y)(\nabla_X A)\xi - 2\alpha\eta(Y)A\phi AX \\ & \quad + 2\eta((\nabla_X A)\xi)AY + 2\alpha(\nabla_X A)Y - 2\alpha\eta((\nabla_X A)Y)\xi - 2\alpha g(AY,\phi AX)\xi \\ & \quad + \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX,Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\ & \qquad + 3g(\phi_\nu AX,\phi Y)\phi_\nu\xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi + 3\eta_\nu(\phi Y)\phi_\nu\phi AX \\ & \qquad - 3\alpha\eta_\nu(\phi Y)\eta(X)\xi_\nu + 4\eta_\nu(\xi)\eta_\nu(\phi Y)AX \\ & \qquad - 4g(AX,Y)\eta_\nu(\xi)\phi_\nu\xi + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big]. \end{aligned} \tag{4.3}$$

Replacing the vector fields X and Y by Z and X in (4.3), respectively, let us take the inner product of the obtained equation with Y. Then by using the equation of Codazzi, we have

$$
\begin{aligned}
&2g((\nabla_Z R_\xi)X, Y)\\
&= -g(A\phi X, Z)g(\xi, Y) - \eta(X)g(A\phi Y, Z)\\
&\quad + \Big\{2(\xi\alpha)\eta(Z) + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Z)\Big\}g(AX, Y)\\
&\quad + \sum_{\nu=1}^{3}\Big[-g(A\phi_\nu X, Z)\eta_\nu(Y) + 2\eta(X)\eta_\nu(Y)g(A\phi\xi_\nu, Z)\\
&\qquad - 2\alpha\eta(Y)\Big\{(\xi\alpha)\eta(X)\eta(Z) - 2\eta(X)\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi\xi_\nu, Z)\Big\}\\
&\qquad + 2\alpha g(A\phi AX, Z)\eta(Y) + 2\alpha\eta(X)g(A\phi AY, Z)\\
&\qquad - 2\alpha\eta(X)\eta(Z)\Big\{(\xi\alpha)\eta(Y) - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi\xi_\nu, Y)\Big\}\\
&\qquad - \eta_\nu(X)g(A\phi_\nu Y, Z) + 3g(A\phi_\nu\phi X, Z)\eta_\nu(\phi Y)\\
&\qquad - 3\eta(X)\eta_\nu(\phi Y)g(A\xi_\nu, Z) + 3\eta_\nu(\phi X)g(A\phi\phi_\nu Y, Z)\\
&\qquad + 3\alpha\eta_\nu(\phi X)\eta_\nu(\phi Y)\eta(Z) + 4\eta_\nu(\xi)\eta_\nu(\phi X)g(AY, Z)\\
&\qquad + 4\eta_\nu(\xi)\eta_\nu(\phi Y)g(AX, Z) - 2g(\phi_\nu\phi X, Y)g(A\phi\xi_\nu, Z)\Big]\\
&\quad + \alpha\Big[2g((\nabla_X A)Y, Z) - \eta(X)g(\phi Y, Z) - \eta(Z)g(\phi X, Y) - 2\eta(Y)g(\phi X, Z)\\
&\qquad + \sum_{\nu=1}^{3}\Big\{-\eta_\nu(X)g(\phi_\nu Y, Z) - \eta_\nu(Z)g(\phi_\nu X, Y) - 2\eta_\nu(Y)g(\phi_\nu X, Z)\Big\}\\
&\qquad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)g(\phi\phi_\nu Y, Z) - \eta_\nu(\phi Z)g(\phi_\nu\phi X, Y)\Big\}\\
&\qquad + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(Y)\eta_\nu(\phi Z) - \eta_\nu(\phi X)\eta_\nu(Y)\eta(Z)\Big\}\Big]
\end{aligned}
\tag{4.4}
$$

for any tangent vector fields X, Y, and Z on M.

Now let us use the symmetric property of $\nabla_Y R_\xi$, that is, $g((\nabla_Y R_\xi)Z, X) = g(Z, (\nabla_Y R_\xi)X)$ in (4.4) and the equation of Codazzi. Then after deleting the vector field Z from the obtained equation, we can rearrange the generalized Killing structure Jacobi operator as follows:

$$
\begin{aligned}
0 &= g(\phi AX, Y)\xi + \eta(Y)\phi AX\\
&\quad + 2\eta((\nabla_X A)\xi)AY + 2\alpha(\nabla_X A)Y - 2\alpha\eta((\nabla_X A)Y)\xi
\end{aligned}
$$

$$
\begin{aligned}
& - 2\alpha g(AY, \phi AX)\xi - 2\alpha\eta(Y)(\nabla_X A)\xi - 2\alpha\eta(Y)A\phi AX \\
& + g(\phi AY, X)\xi + \eta(X)\phi AY + 2\eta((\nabla_Y A)\xi)AX \\
& + 2\alpha(\nabla_Y A)X - 2\alpha\eta((\nabla_Y A)X)\xi - 2\alpha g(AX, \phi AY)\xi \\
& - 2\alpha\eta(X)(\nabla_Y A)\xi - 2\alpha\eta(X)A\phi AY - \eta(Y)A\phi X - \eta(X)A\phi Y \\
& + 2(\xi\alpha)g(AX,Y)\xi - 4(\xi\alpha)g(AX,Y)\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu + 2\alpha(\nabla_X A)Y \\
& - \alpha\eta(X)\phi Y - \alpha g(\phi X, Y)\xi - 2\alpha\eta(Y)\phi X - 4\alpha(\xi\alpha)\eta(X)\eta(Y)\xi \\
& + 2\alpha\eta(Y)A\phi AX + 2\alpha\eta(X)A\phi AY \\
& + \sum_{\nu=1}^{3}\big[\, g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
& \qquad + 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi + 3\eta_\nu(\phi Y)\phi_\nu\phi AX \\
& \qquad - 3\alpha\eta_\nu(\phi Y)\eta(X)\xi_\nu + 4\eta_\nu(\xi)\eta_\nu(\phi Y)AX \\
& \qquad - 4\eta_\nu(\xi)g(AX,Y)\phi_\nu\xi + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\big] \\
& + \sum_{\nu=1}^{3}\big[\, g(\phi_\nu AY, X)\xi_\nu - 2\eta(X)\eta_\nu(\phi AY)\xi_\nu + \eta_\nu(X)\phi_\nu AY \\
& \qquad + 3g(\phi_\nu AY, \phi X)\phi_\nu\xi + 3\eta(X)\eta_\nu(AY)\phi_\nu\xi + 3\eta_\nu(\phi X)\phi_\nu\phi AY \\
& \qquad - 3\alpha\eta_\nu(\phi X)\eta(Y)\xi_\nu + 4\eta_\nu(\xi)\eta_\nu(\phi X)AY \\
& \qquad - 4\eta_\nu(\xi)g(AY,X)\phi_\nu\xi + 2\eta_\nu(\phi AY)\phi_\nu\phi X\big] \\
& + \sum_{\nu=1}^{3}\big[- \eta_\nu(Y)A\phi_\nu X + 2\eta(X)\eta_\nu(Y)A\phi\xi_\nu - \eta_\nu(X)A\phi_\nu Y \\
& \qquad + 3\eta_\nu(\phi Y)A\phi_\nu\phi X - 3\eta(X)\eta_\nu(\phi Y)A\xi_\nu + 3\eta_\nu(\phi X)A\phi\phi_\nu Y \\
& \qquad - 3\alpha\eta_\nu(\phi X)\eta_\nu(Y)\xi + 4\eta_\nu(\xi)\eta_\nu(\phi X)AY \\
& \qquad + 4\eta_\nu(\xi)\eta_\nu(\phi Y)AX - 2g(\phi_\nu\phi X, Y)A\phi\xi_\nu\big] \\
& + \alpha\sum_{\nu=1}^{3}\big[- \eta_\nu(X)\phi_\nu Y - g(\phi_\nu X, Y)\xi_\nu - 2\eta_\nu(Y)\phi_\nu X \\
& \qquad + \eta_\nu(\phi X)\phi\phi_\nu Y + g(\phi_\nu\phi X, Y)\phi\xi_\nu + 4\eta(X)\eta(Y)\eta_\nu(\xi)\phi\xi_\nu \\
& \qquad - \eta(X)\eta_\nu(Y)\phi\xi_\nu - \eta_\nu(\phi X)\eta_\nu(Y)\xi - 4\eta(X)\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\big],
\end{aligned} \tag{4.5}
$$

where we have used (3.3) and (4.1). Then, by virtue of (4.5) and basic equations given in Section 3, we can prove the following:

Lemma 4.1. *Let M be a Hopf real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with generalized Killing structure Jacobi operator. Then the Reeb vector field ξ belongs to either the maximal quaternionic subbundle $\mathcal{Q}$ or its orthogonal complement $\mathcal{Q}^\perp$.*

Proof. In order to prove this lemma, we put

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \quad \text{such that} \quad \eta(X_0)\eta(\xi_1) \neq 0 \tag{4.6}$$

for some unit vectors $X_0 \in \mathcal{Q}$ and $\xi_1 \in \mathcal{Q}^\perp$.

Together with (4.6) and a Hopf hypersurface condition, if $\alpha = g(A\xi, \xi)$ vanishes on M, then (4.1) implies $\eta(\xi_1)\phi\xi_1 = 0$. This gives ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$. So we may assume that α is non-vanishing.

Lee and Loo [14] show that if M is Hopf, then the Reeb function α is constant along the direction of structure vector field ξ, that is, $\xi\alpha = 0$. Also in [16], we see that $\xi\alpha = 0$ gives the distribution $\mathcal{Q}$- and the $\mathcal{Q}^\perp$-component of the Reeb vector field ξ is invariant by the shape operator A, that is,

$$AX_0 = \alpha X_0, \quad \text{and} \quad A\xi_1 = \alpha\xi_1.$$

In addition, from (4.6) and $\phi\xi = 0$, we have

$$\begin{cases} \phi X_0 = -\eta(\xi_1)\phi_1 X_0, \\ \phi\xi_1 = \phi_1\xi = \eta(X_0)\phi_1 X_0, \\ \phi_1\phi X_0 = \eta(\xi_1)X_0. \end{cases}$$

The equation (4.2) yields $\alpha A\phi X_0 = (\alpha^2 - 2\eta^2(X_0))\phi X_0$ by substituting $X_0 \in \mathcal{Q}$ instead of X. Since we assumed that the Reeb function α is non-vanishing, it becomes

$$A\phi X_0 = \sigma\phi X_0, \quad \text{where} \quad \sigma = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}.$$

Putting $X = X_0$ and $Y = \xi_1$ in (4.5), we have

$$\begin{aligned} 0 = {} & \alpha\eta(\xi_1)\phi X_0 + \alpha\phi_1 X_0 \\ & + 2\alpha\eta((\nabla_{X_0}A)\xi)\xi_1 + 2\alpha(\nabla_{X_0}A)\xi_1 - 2\alpha\eta((\nabla_{X_0}A)\xi_1)\xi \\ & - 2\alpha\eta(\xi_1)(\nabla_{X_0}A)\xi - 2\alpha^2\sigma\eta(\xi_1)X_0 + \alpha\eta(X_0)\phi\xi_1 + 3\alpha\eta(X_0)\phi_1\xi \\ & + 2\alpha\eta((\nabla_{\xi_1}A)\xi)X_0 + 2\alpha(\nabla_{\xi_1}A)X_0 - 2\alpha\eta((\nabla_{\xi_1}A)X_0)\xi \\ & - 2\alpha\eta(X_0)(\nabla_{\xi_1}A)\xi - 2\alpha^2\sigma\eta(X_0)\phi\xi_1 \\ & - \sigma\eta(\xi_1)\phi X_0 - \sigma\eta(X_0)\phi\xi_1 - \sigma\phi_1 X_0 + 2\sigma\eta(X_0)\phi\xi_1 \\ & + 2\alpha(\nabla_{X_0}A)\xi_1 - \alpha\eta(X)\phi\xi_1 - 2\alpha\eta(\xi_1)\phi X_0 \\ & - 2\alpha\phi_1 X_0 - \alpha\eta(X_0)\phi\xi_1 + 4\alpha\eta(X_0)\eta^2(\xi_1))\phi\xi_1 \\ & + 2\alpha^2\sigma\eta(\xi_1)X_0 + 2\alpha^2\sigma\eta(X_0)\xi_1. \end{aligned} \tag{4.7}$$

On the other hand, taking the covariant derivative with respect to the Levi-Civita connection ∇ of M to the assumption of $A\xi_1 = \alpha\xi_1$ and using (3.2), we get

$$\begin{aligned} (\nabla_X A)\,\xi_1 &= (X\alpha)\xi_1 + \alpha\nabla_X\xi_1 - A\,(\nabla_X\xi_1) \\ &= (X\alpha)\xi_1 + \alpha\,\{q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX\} \\ &\quad - q_3(X)A\xi_2 + q_2(X)A\xi_3 - A\phi_1 AX \end{aligned}$$

$$= 4\eta(\xi_1)\, g(\phi\xi, X)\xi_1 + \alpha\{q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX\} - q_3(X)A\xi_2 + q_2(X)A\xi_3 - A\phi_1 AX.$$

Moreover, by using the similar method given in [13] we obtain $q_\nu(\xi) = q_\nu(\xi_1) = q_\nu(X_0) = 0$ for $\nu = 2, 3$. Thus we have

$$(\nabla_X A)\,\xi_1 = 4\eta(\xi_1)\, g(\phi\xi, X)\xi_1 + \alpha\phi_1 AX - A\phi_1 AX \tag{4.8}$$

and

$$\begin{cases} (\nabla_{X_0} A)\,\xi_1 = \alpha\{q_3(X_0)\xi_2 - q_2(X_0)\xi_3 + \alpha\phi_1 X_0\} \\ \qquad\qquad - q_3(X_0)A\xi_2 + q_2(X_0)A\xi_3 - \alpha\sigma\phi_1 X_0, \\ \qquad = (\alpha^2 - \alpha\sigma)\phi_1 X_0, \\ (\nabla_{X_0} A)\,\xi = (X_0\alpha)\xi - (\alpha^2 - \alpha\sigma)\phi X_0, \\ (\nabla_{\xi_1} A)\,\xi = (\xi_1\alpha)\xi - (\alpha^2 - \alpha\sigma)\phi\xi_1, \\ (\nabla_{\xi_1} A)X_0 = (\nabla_{X_0} A)\xi_1. \end{cases} \tag{4.9}$$

Using (4.8) and (4.9), then (4.7) becomes

$$\begin{aligned} 0 = & -\alpha\eta^2(\xi_1)\phi_1 X_0 + \alpha\phi_1 X_0 + 2\alpha(\alpha^2 - \alpha\sigma)\phi_1 X_0 - 2\alpha\eta((\nabla_{X_0} A)\xi_1)\xi \\ & - 2\alpha\eta(\xi_1)((X_0\alpha)\xi + (\alpha^2 - \alpha\sigma)\eta(\xi_1)\phi_1 X_0) - 2\alpha^2\sigma\eta(\xi_1)X_0 \\ & + \alpha\eta^2(X_0)\phi_1 X_0 + 3\alpha\eta^2(X_0)\phi_1 X_0 + 2\alpha\eta((\nabla_{\xi_1} A)\xi)X_0 \\ & + 2\alpha(\alpha^2 - \alpha\sigma)\phi_1 X_0 - 2\alpha\eta((\nabla_{\xi_1} A)X_0)\xi - 2\alpha\eta(X_0)(\xi_1\alpha)\xi \\ & + 2\alpha\eta(X_0)(\alpha^2 - \alpha\sigma)\eta(X_0)\phi_1 X_0 - 2\alpha^2\sigma\eta^2(X_0)\phi_1 X_0 \\ & + \sigma\eta^2(\xi_1)\phi_1 X_0 - \sigma\eta^2(X_0)\phi_1 X_0 - \sigma\phi_1 X_0 + 2\sigma\eta^2(X_0)\phi_1 X_0 \\ & + 2\alpha(\alpha^2 - \alpha\sigma)\phi_1 X_0 - 2\alpha\eta^2(X_0)\phi_1 X_0 + 2\alpha\eta(\xi_1)^2\phi_1 X_0 - 2\alpha\phi_1 X_0 \\ & + 4\alpha\eta^2(\xi_1)\eta^2(X_0)\phi_1 X_0 + 2\alpha^2\sigma\eta(\xi_1)X_0 + 2\alpha^2\sigma\eta(X_0)\xi_1. \end{aligned} \tag{4.10}$$

Taking the inner product of (4.10) with ξ_1, then we have

$$\begin{aligned} 0 &= 2\alpha\eta((\nabla_{X_0} A)\xi) - 2\alpha\eta((\nabla_{X_0} A)\xi_1)\eta(\xi_1) \\ &\quad - 2\alpha\eta((\nabla_{\xi_1} A)X_0)\eta(\xi_1) + 2\alpha^2\sigma\eta(X_0) \\ &= 2\alpha^2\sigma\eta(X_0). \end{aligned} \tag{4.11}$$

Since $\sigma = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}$ and $\alpha\eta(X_0) \neq 0$, (4.11) gives us

$$\alpha^2 = 2\eta^2(X_0). \tag{4.12}$$

Taking the inner product of (4.10) with $\phi_1 X_0$, then we have

$$0 = -4\eta^4(X_0) + \left\{4\alpha^2 - 6\alpha\sigma + 5\right\}\eta^2(X_0) + 6\alpha^2 - 6\alpha\sigma - 2\alpha^2 + 2\alpha\sigma.$$

Since $\sigma = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}$ and $\eta^2(X_0) \neq 0$, we have

$$0 = 12\eta^2(X_0) - 2\alpha^2 + 9. \tag{4.13}$$

Using (4.12) and (4.13), we have

$$\eta^2(X_0) = -\frac{9}{8}.$$

This gives us a contradiction. So, we assert that ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$. □

5. The Reeb vector field $\xi \in \mathcal{Q}^\perp$

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with generalized Killing structure Jacobi operator. Then by Lemma 4.1 we shall make an investigation into two cases depending on ξ belongs to either distribution $\mathcal{Q}^\perp$ or distribution $\mathcal{Q}$, respectively. So, in this section let us consider the case $\xi \in \mathcal{Q}^\perp$ (i.e., $JN \in \mathfrak{J}N$ where N is a unit normal vector field on M in $SU_{2,m}/S(U_2 \cdot U_m)$). Since $\mathcal{Q}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$, we may put $\xi = \xi_1$. By using this equation we obtain:

Lemma 5.1. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$ and $\xi \in \mathcal{Q}^\perp$. Then*

(i) $\phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX$ *and*

(ii) $A\phi X = 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X$.

Proof. Differentiating $\xi = \xi_1$ along any direction $X \in TM$ and using (3.2), it gives

$$\phi AX = \nabla_X \xi = \nabla_X \xi_1 = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX. \tag{5.1}$$

Taking the inner product with ξ_2 and ξ_3 in (5.1), respectively, gives

$$q_3(X) = 2\eta_3(AX) \quad \text{and} \quad q_2(X) = 2\eta_2(AX).$$

Then (5.1) can be revised:

$$\phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX.$$

From this, by applying the inner product with any tangent vector Y, we have

$$g(\phi AX, Y) = 2\eta_3(AX)g(\xi_2, Y) - 2\eta_2(AX)g(\xi_3, Y) + g(\phi_1 AX, Y).$$

Then, by using the symmetric (resp. skew-symmetric) property of the shape operator A (resp. the structure tensor field ϕ), we have

$$-g(X, A\phi Y) = 2g(X, A\xi_3)g(\xi_2, Y) - 2g(X, A\xi_2)g(\xi_3, Y) - g(Y, A\phi_1 X)$$

for any tangent vector fields X and Y on M. Then it can be rewritten as below:

$$A\phi X = 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X.$$ □

From now on, by using this lemma, let us consider our classification problem with respect to the notion of generalized Killing structure Jacobi operator of a real hypersurface with $\xi \in \mathcal{Q}^\perp$ in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$.

In order to do this, putting $X=\xi$ into (4.3), and replacing Y as X, we have

$$
\begin{aligned}
2(\nabla_\xi R_\xi)X &= 2(\xi\alpha)AX + 2\alpha(\nabla_\xi A)X - 4\alpha(\xi\alpha)\eta(X)\xi \\
&\quad - 4\alpha\sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi\big\} \\
&\quad + 4\alpha\sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)\eta_\nu(\phi X)\xi - \eta_\nu(\xi)\eta(X)\phi_\nu\xi\big\}
\end{aligned}
\tag{5.2}
$$

for any tangent vector field X on M.

On the other hand, putting $Y=\xi$ into (4.3), we have

$$
\begin{aligned}
2(\nabla_X R_\xi)\xi &= \phi AX - 2\alpha A\phi AX \\
&\quad - \sum_{\nu=1}^{3}\big\{g(\phi_\nu AX,\xi)\xi_\nu - \eta_\nu(\xi)\phi_\nu AX\big\} \\
&\quad + \sum_{\nu=1}^{3}\big\{3\eta_\nu(AX)\phi_\nu\xi - 8\eta_\nu(\xi)g(AX,\xi)\phi_\nu\xi\big\}.
\end{aligned}
\tag{5.3}
$$

By using these equations, we assert:

Lemma 5.2. *Let M be a real hypersurface in $SU_{2,m}/S(U_2\cdot U_m)$, $m\geq 3$ with generalized Killing structure Jacobi operator. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then the shape operator A commutes with the structure operator ϕ, that is, $A\phi=\phi A$.*

Proof. By our assumption $\xi\in\mathcal{Q}^\perp$, we may put $\xi=\xi_1$. Substituting $Z=\xi$ into (1.2), then the generalized Killing structure Jacobi operator R_ξ of M becomes

$$
g\left((\nabla_X R_\xi)Y,\xi\right) + g\left((\nabla_Y R_\xi)\xi,X\right) + g\left((\nabla_\xi R_\xi)X,Y\right) = 0. \tag{5.4}
$$

From (5.2) and (5.3), the equation (5.4) follows

$$
\begin{aligned}
0 &= g(\phi AX,Y) + g(\phi_1 AX) + \eta_2(Y)\eta_3(AX) - \eta_3(Y)\eta_2(AX) \\
&\quad + \sum_{\nu=1}^{3}\big[-g(\phi_\nu AY,\xi)g(\xi_\nu,X) + 3\eta_\nu(AY)g(\phi_\nu\xi,X)\big] \\
&\quad + \sum_{\nu=1}^{3}\big[-4\alpha\eta_\nu(\phi X)g(\xi_\nu,Y) + 4\alpha\eta_\nu(X)g(\phi_\nu\xi,Y)\big] + 2\alpha g(\nabla_\xi A)X,Y)
\end{aligned}
\tag{5.5}
$$

for any vector fields X and Y on M in $SU_{2,m}/S(U_2\cdot U_m)$. Then it can be rewritten as follows:

$$
\begin{aligned}
0 &= \phi AX + \phi_1 AX + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 \\
&\quad - A\phi X - A\phi_1 X - 2\eta_3(X)A\xi_2 + 2\eta_2(X)A\xi_3 + 2\alpha(\nabla_\xi A)X.
\end{aligned}
\tag{5.6}
$$

By (i) (resp. (ii)) in Lemma 5.1, we have

$$0 = 2\phi AX - 2A\phi X + 2\alpha(\nabla_\xi A)X \tag{5.7}$$

for any tangent vector field X on M.

On the other hand, putting $X = \xi$ into the equation of Codazzi and substitute Y as X, we have

$$2(\nabla_X A)\xi - 2(\nabla_\xi A)X = \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \tag{5.8}$$

Since M is Hopf and $\xi \in \mathcal{Q}^\perp$, we get

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX \tag{5.9}$$

and

$$2A\phi AX = \alpha(A\phi + \phi A)X - \phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2. \tag{5.10}$$

By (5.9) and (5.10), the equation (5.8) becomes

$$2(X\alpha)\xi + \alpha\phi AX - \alpha A\phi X - 2(\nabla_\xi A)X = 0.$$

So, we have $2(\nabla_\xi A)X = 2(X\alpha)\xi + \alpha(\phi A - A\phi)X$. Moreover, taking the inner product of (5.6) with ξ and using (5.5), we have $X\alpha = (\xi\alpha)\eta(X)$. Hence, we have

$$(\nabla_\xi A)X = (\xi\alpha)\eta(X)\xi + \frac{\alpha}{2}\phi AX - \frac{\alpha}{2}A\phi X.$$

From [14], we know that $\xi\alpha = 0$ under the condition of Hopf hypersurface. Thus the above equation leads to

$$2\alpha(\nabla_\xi A)X = \alpha^2(\phi AX - A\phi X).$$

From this, the equation (5.7) becomes

$$\begin{aligned} 0 &= 2\phi AX - 2A\phi X + \alpha^2\phi AX - \alpha^2 A\phi X \\ &= (\alpha^2 + 2)(\phi AX - A\phi X). \end{aligned}$$

Since $(\alpha^2 + 2)$ is non-vanishing on M, it means that $\phi AX - A\phi X = 0$, which completes the proof of Lemma 5.2. □

Through algebraic calculations, we see that *the notion of isometric Reeb flow is equivalent to the fact that the shape operator A of M satisfies $A\phi = \phi A$.* In fact, taking the Lie derivative for the metric tensor field g of type (0,2) along the Reeb direction ξ and using $\mathcal{L}_\xi(g(X,Y)) = \nabla_\xi(g(X,Y))$, together with $\mathcal{L}_X Y = [X,Y] = \nabla_X Y - \nabla_Y X$ and (3.1), we obtain

$$\begin{aligned} 0 &= (\mathcal{L}_\xi g)(X,Y) \\ &= \mathcal{L}_\xi(g(X,Y)) - g(\mathcal{L}_\xi X, Y) - g(X, \mathcal{L}_\xi Y) \\ &= \nabla_\xi(g(X,Y)) - g([\xi,X],Y) - g(X,[\xi,Y]) \\ &= g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y) - g(\nabla_\xi X, Y) + g(\nabla_X \xi, Y) \\ &\quad - g(X, \nabla_\xi Y) + g(X, \nabla_Y \xi) \\ &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \end{aligned}$$

$$= g(\phi SX, Y) + g(X, \phi SY) = g((\phi S - S\phi)X, Y)$$

for any tangent vector fields X and Y on M. Thus, Lemma 5.2, consequently, assures that *a real hypersurface M with generalized Killing structure Jacobi operator in complex hyperbolic two-plane Grassmannians satisfying $\xi \in \mathcal{Q}^\perp$ has isometric Reeb flow.* Therefore, by Theorem A in the introduction, we assert that a real hypersurface M with the assumptions given in Lemma 5.2 is locally congruent to one of the following real hypersurfaces:

($\mathcal{T}_A^*$) a tube over a totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$

or

($\mathcal{H}_A^*$) a horosphere in $SU_{2,m}/S(U_2 \cdot U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.

Therefore, by virtue of Lemma 5.2 we conclude that if $\xi \in \mathcal{Q}^\perp$, then M is of ($\mathcal{T}_A^*$) or ($\mathcal{H}_A^*$), where M is a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, satisfying generalized Killing structure Jacobi operator. Such real hypersurfaces of type ($\mathcal{T}_A^*$) and ($\mathcal{H}_A^*$) in $SU_{2,m}/S(U_2 \cdot U_m)$ are denoted by M_A. In [5], Berndt and Suh gave some information related to the shape operator A of ($\mathcal{T}_A^*$) and ($\mathcal{H}_A^*$) as follows.

Proposition A. *Let M_A be a connected real hypersurface of type ($\mathcal{T}_A^*$) or ($\mathcal{H}_A^*$) in complex hyperbolic two-plane Grassamnnian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then one of the following statements holds*:

(a) *M_A is Hopf.*
(b) *The maximal complex subbundle $\mathcal{C}$ of TM_A and the maximal quaternionic subbundle $\mathcal{Q}$ of TM_A are both invariant under the shape operator S of M_A.*
(c) *The normal vector field N of M_A in $SU_{2,m}/S(U_2 \cdot U_m)$ is singular satisfying $JN \in \mathfrak{J}N$.*
(d) *All eigenvalues of M_A are constant as follows.*
   - ($\mathcal{T}_A^*$) *has exactly four distinct constant principal curvatures*

   $$\alpha = 2\coth(2r),\ \beta = \coth(r),\ \lambda_1 = \tanh(r),\ \lambda_2 = 0,$$

   and the corresponding principal curvature spaces are

   $$T_\alpha = \mathrm{span}\{\xi\},\ T_\beta = \mathrm{span}\{\xi_2, \xi_3\},\ T_{\lambda_1} = E_{-1},\ T_{\lambda_2} = E_{+1}.$$

   The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).
   - ($\mathcal{H}_A^*$) *has exactly three distinct constant principal curvatures*

   $$\alpha = 2,\ \beta = 1,\ \lambda = 0$$

   with corresponding principal curvature spaces

   $$T_\alpha = \mathrm{span}\{\xi\},\ T_\beta = \mathrm{span}\{\xi_2, \xi_3\} \oplus E_{-1},\ T_\lambda = E_{+1}.$$

Here, E_{+1} and E_{-1} are the eigenbundles of $\phi\phi_1|_{\mathcal{Q}}$ with respect to the eigenvalues $+1$ and -1, respectively.

(e) *The Reeb flow of M_A is isometric.*

From (a) and (b) of Proposition A, we see that the model space M_A is Hopf with $\xi \in \mathcal{Q}^{\perp}$. So, in the remaining part of this section, by using Proposition A let us check if the structure Jacobi operator R_ξ on a real hypersurface M_A of type $(\mathcal{T}_A^*)$ (or $(\mathcal{H}_A^*)$, resp.) satisfies the condition of generalized Killing. In order to do this, we assume that the structure Jacobi operator R_ξ of M_A is generalized Killing. Then, (4.5) becomes

$$\begin{aligned}
0 = {} & g(\phi AX, Y)\xi + \eta(Y)\phi AX \\
& + 2\alpha(\nabla_X A)Y - 2\alpha\eta((\nabla_X A)Y)\xi - 2\alpha g(AY, \phi AX)\xi \\
& - 2\alpha^2\eta(Y)\phi AX + g(\phi AY, X)\xi + \eta(X)\phi AY \\
& + 2\alpha(\nabla_Y A)X - 2\alpha\eta((\nabla_Y A)X)\xi - 2\alpha g(AX, \phi AY)\xi \\
& - 2\alpha^2\eta(X)\phi AY - \eta(Y)A\phi X - \eta(X)A\phi Y \\
& + 2\alpha(\nabla_X A)Y - \alpha\eta(X)\phi Y - \alpha g(\phi X, Y)\xi - 2\alpha\eta(Y)\phi X \\
& + \sum_{\nu=1}^{3}\big[\, g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu \\
& \qquad + \eta_\nu(Y)\phi_\nu AX + 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi \\
& \qquad + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi + 3\eta_\nu(\phi Y)\phi_\nu\phi AX \\
& \qquad - 3\alpha\eta_\nu(\phi Y)\eta(X)\xi_\nu + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\big] \\
& + \sum_{\nu=1}^{3}\big[\, g(\phi_\nu AY, X)\xi_\nu - 2\eta(X)\eta_\nu(\phi AY)\xi_\nu \\
& \qquad + \eta_\nu(X)\phi_\nu AY + 3g(\phi_\nu AY, \phi X)\phi_\nu\xi \\
& \qquad + 3\eta(X)\eta_\nu(AY)\phi_\nu\xi + 3\eta_\nu(\phi X)\phi_\nu\phi AY \\
& \qquad - 3\alpha\eta_\nu(\phi X)\eta(Y)\xi_\nu + 2\eta_\nu(\phi AY)\phi_\nu\phi X\big] \\
& + \sum_{\nu=1}^{3}\big[- \eta_\nu(Y)A\phi_\nu X + 2\eta(X)\eta_\nu(Y)A\phi\xi_\nu \\
& \qquad - \eta_\nu(X)A\phi_\nu Y + 3\eta_\nu(\phi Y)A\phi_\nu\phi X \\
& \qquad - 3\eta(X)\eta_\nu(\phi Y)A\xi_\nu + 3\eta_\nu(\phi X)A\phi\phi_\nu Y \\
& \qquad - 3\alpha\eta_\nu(\phi X)\eta_\nu(Y)\xi - 2g(\phi_\nu\phi X, Y)A\phi\xi_\nu\big] \\
& + \alpha\sum_{\nu=1}^{3}\big[- \eta_\nu(X)\phi_\nu Y - g(\phi_\nu X, Y)\xi_\nu - 2\eta_\nu(Y)\phi_\nu X \\
& \qquad + \eta_\nu(\phi X)\phi\phi_\nu Y + g(\phi_\nu\phi X, Y)\phi\xi_\nu \\
& \qquad - \eta(X)\eta_\nu(Y)\phi\xi_\nu - \eta_\nu(\phi X)\eta_\nu(Y)\xi\big],
\end{aligned} \tag{5.11}$$

where we have used that the structure vector field ξ of M_A belongs to the distribution $\mathcal{Q}^{\perp}$ and $(\nabla_X A)\xi = \alpha\phi AX - A\phi AX$ for any tangent vector field X on M. Since the tangent bundle TM_A of M_A is given by $TM_A = T_\alpha \oplus T_\beta \oplus E_{-1} \oplus E_{+1}$, let us consider the case $Y = \xi(= \xi_1) \in T_\alpha$. Then (5.11) can be rewritten as

$$\begin{aligned} 0 = {}& 2\phi AX + 2\alpha(\nabla_\xi A)X - 2A\phi X + 2\alpha^2\phi AX - 2\alpha A\phi AX \\ & - 2\alpha\phi X + 4\alpha\eta_2(X)\xi_3 - 4\alpha\eta_3(X)\xi_2. \end{aligned} \tag{5.12}$$

In addition, by the equation of Codazzi (5.12) gives

$$\begin{aligned} 0 = {}& (2 + 2\alpha^2)\phi AX - (2 + 2\alpha^2)A\phi X \\ & - \alpha\phi X + \alpha\phi_1 X + 2\alpha\eta_2(X)\xi_3 - 2\alpha\eta_3(X)\xi_2 \end{aligned} \tag{5.13}$$

for any tangent vector field X on TM_A.

Putting $X = \xi_2 \in T_\beta$ in (5.13) gives

$$4\alpha\xi_3 = 0. \tag{5.14}$$

Bearing in mind Proposition A, in the case of $(\mathcal{T}_A^*)$ (resp., $(\mathcal{H}_A^*)$), we have $\alpha = 2\coth(2r)$ (resp., $\alpha = 2$). In both cases, we know that the Reeb function α is non-vanishing. From this fact, (5.14) gives $\xi_3 = 0$, which gives a contradiction.

Summing up these observations, we assert that the structure Jacobi operator R_ξ of real hypersurfaces M_A of two kinds of model spaces $(\mathcal{T}_A^*)$ and $(\mathcal{H}_A^*)$ in $SU_{2,m}/S(U_2 \cdot U_m)$ does not satisfy the property of generalized Killing.

6. The Reeb vector field $\xi \in \mathcal{Q}$

Let M be a Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ with generalized Killing structure Jacobi operator. Then, by virtue of Lemma 4.1 and the facts in Section 5, we know that the Reeb vector field ξ belongs to the maximal quaternionic subbundle $\mathcal{Q}$ of the tangent bundle TM on M. So, in this section let us consider $\xi \in \mathcal{Q}$ (i.e., $JN \perp \mathfrak{J}N$). In [27], Suh gave a complete classification of Hopf real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ satisfying $\xi \in \mathcal{Q}$ as follows.

Theorem C. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with the Reeb vector field belonging to the maximal quaternionic subbundle $\mathcal{Q}$. Then one of the following statements holds*

- $(\mathcal{T}_B^*)$ *M is an open part of a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$,*
- $(\mathcal{H}_B^*)$ *M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$, or*
- $(\mathcal{E})$ *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$.*

Then by the assumption of the generalized Killing structure Jacobi operator in our Main Theorem, a Hopf hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ is locally congruent to an open part of one of the model spaces mentioned in Theorem C. Hereafter, unless otherwise stated, such model spaces of type of $(\mathcal{T}_B^*)$, $(\mathcal{H}_B^*)$ and $(\mathcal{E})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ are denoted by M_B.

Moreover, Berndt and Suh [3] gave some geometric properties for the model space M_B as follows.

Proposition B. *Let M_B be a real hypersurface of type $(\mathcal{T}_B^*)$ (resp. $(\mathcal{H}_B^*)$ or $(\mathcal{E})$) in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then one of the following statements holds:*

(a) *M_B is Hopf.*

(b) *The maximal complex subbundle $\mathcal{C}$ of TM_B and the maximal quaternionic subbundle $\mathcal{Q}$ of TM_B are both invariant under the shape operator A of M_B.*

(c) *The normal vector filed N of M_B is singular. In particular, it satisfies $JN \perp \mathfrak{J}N$.*

(d) *M_B has distinct principal curvatures as follows.*

- *$(\mathcal{T}_B^*)$ has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures*

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r),\ \beta = \sqrt{2}\coth(\sqrt{2}r),\ \gamma = 0,$$
$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh(\frac{1}{\sqrt{2}}r),\ \lambda_2 = \frac{1}{\sqrt{2}}\coth(\frac{1}{\sqrt{2}}r),$$

and the corresponding principal curvature spaces are

$$T_\alpha = \text{span}\{\xi\}, \quad T_\beta = \text{span}\{\xi_1, \xi_2, \xi_3\},$$
$$T_\gamma = \text{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are invariant under $\mathfrak{J}$ and are mapped onto each other by J. In particular, the quaternionic dimension of $SU_{2,m}/S(U_2U_m)$ must be even.

- *$(\mathcal{H}_B^*)$ has exactly three distinct constant principal curvatures*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = \text{span}\{\xi, \xi_1, \xi_2, \xi_3\}, \quad T_\gamma = \text{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\},$$
$$T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

- *$(\mathcal{E})$ has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = \mathrm{span}\{\xi, \xi_1, \xi_2, \xi_3\}, \quad T_\gamma = \mathrm{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\},$$
$$T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ *is another (possibly nonconstant) principal curvature function, then* $JT_\mu \subset T_\lambda$ *and* $\mathfrak{J}T_\mu \subset T_\lambda$. *Thus, the corresponding multiplicities are*

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

From Proposition B, we see that the model space M_B is a Hopf real hypersurface with $\xi \in \mathcal{Q}$ in $SU_{2,m}/S(U_2 \cdot U_m)$. Finally, let us check whether the structure Jacobi operator R_ξ of M_B satisfies (1.3).

In order to check this problem, we suppose that the structure Jacobi operator R_ξ of M_B is generalized Killing. Since $\xi \in \mathcal{Q}$, the equation (4.5) is written as

$$\begin{aligned}
0 = {} & g(\phi AX, Y)\xi + \eta(Y)\phi AX \\
& + 2\eta((\nabla_X A)\xi)AY + 2\alpha(\nabla_X A)Y - 2\alpha\eta((\nabla_X A)Y)\xi \\
& - 2\alpha g(AY, \phi AX)\xi - 2\alpha\eta(Y)(\nabla_X A)\xi - 2\alpha\eta(Y)A\phi AX \\
& + g(\phi AY, X)\xi + \eta(X)\phi AY + 2\eta((\nabla_Y A)\xi)AX + 2\alpha(\nabla_Y A)X \\
& - 2\alpha\eta((\nabla_Y A)X)\xi - 2\alpha g(AX, \phi AY)\xi - 2\alpha\eta(X)(\nabla_Y A)\xi \\
& - 2\alpha\eta(X)A\phi AY - \eta(Y)A\phi X - \eta(X)A\phi Y \\
& + 2\alpha(\nabla_X A)Y - \alpha\eta(X)\phi Y - \alpha g(\phi X, Y)\xi - 2\alpha\eta(Y)\phi X \\
& + 2\alpha\eta(Y)A\phi AX + 2\alpha\eta(X)A\phi AY \\
& + \sum_{\nu=1}^{3} \big[g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
& \qquad + 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi + 3\eta_\nu(\phi Y)\phi_\nu\phi AX \\
& \qquad - 3\alpha\eta_\nu(\phi Y)\eta(X)\xi_\nu + 2\eta_\nu(\phi AX)\phi_\nu\phi Y \big] \\
& + \sum_{\nu=1}^{3} \big[g(\phi_\nu AY, X)\xi_\nu - 2\eta(X)\eta_\nu(\phi AY)\xi_\nu + \eta_\nu(X)\phi_\nu AY \\
& \qquad + 3g(\phi_\nu AY, \phi X)\phi_\nu\xi + 3\eta(X)\eta_\nu(AY)\phi_\nu\xi + 3\eta_\nu(\phi X)\phi_\nu\phi AY \\
& \qquad - 3\alpha\eta_\nu(\phi X)\eta(Y)\xi_\nu + 2\eta_\nu(\phi AY)\phi_\nu\phi X \big] \\
& + \sum_{\nu=1}^{3} \big[-\eta_\nu(Y)A\phi_\nu X + 2\eta(X)\eta_\nu(Y)A\phi\xi_\nu - \eta_\nu(X)A\phi_\nu Y \\
& \qquad + 3\eta_\nu(\phi Y)A\phi_\nu\phi X - 3\eta(X)\eta_\nu(\phi Y)A\xi_\nu + 3\eta_\nu(\phi X)A\phi\phi_\nu Y \\
& \qquad - 3\alpha\eta_\nu(\phi X)\eta_\nu(Y)\xi - 2g(\phi_\nu\phi X, Y)A\phi\xi_\nu \big]
\end{aligned} \tag{6.1}$$

$$+\,\alpha\sum_{\nu=1}^{3}\big[-\eta_\nu(X)\phi_\nu Y-g(\phi_\nu X,Y)\xi_\nu-2\eta_\nu(Y)\phi_\nu X+\eta_\nu(\phi X)\phi\phi_\nu Y$$
$$+\,g(\phi_\nu\phi X,Y)\phi\xi_\nu-\eta(X)\eta_\nu(Y)\phi\xi_\nu-\eta_\nu(\phi X)\eta_\nu(Y)\xi\big]$$

for any tangent vector fields X and Y on TM_B.

Putting $Y=\xi\in T_\alpha$ into (6.1), we have

$$\begin{aligned}0=\ &\phi AX+2\alpha(\nabla_\xi A)X-A\phi X+2\alpha(\nabla_X A)\xi-2\alpha\phi X\\&+\sum_{\nu=1}^{3}\big[g(\phi_\nu AX,\xi)\xi_\nu-2\eta_\nu(\phi AX)\xi_\nu+3\eta_\nu(AX)\phi_\nu\xi\\&\quad+g(\phi_\nu A\xi,X)\xi_\nu+\eta_\nu(X)\phi_\nu A\xi+3g(\phi_\nu A\xi,\phi X)\phi_\nu\xi\\&\quad-3\alpha\eta_\nu(\phi X)\xi_\nu-\eta_\nu(X)A\phi_\nu\xi+3\eta_\nu(\phi X)A\phi\phi_\nu\xi\\&\quad-2g(\phi_\nu\phi X,\xi)A\phi\xi_\nu-\alpha\eta_\nu(X)\phi_\nu\xi-\alpha g(\phi_\nu X,\xi)\xi_\nu\\&\quad+\alpha\eta_\nu(\phi X)\phi\phi_\nu\xi+\alpha g(\phi_\nu\phi X,\xi)\phi_\nu\xi\big].\end{aligned}\tag{6.2}$$

Using $A\phi\xi_\nu=0$ and $A\xi_\nu=\beta\xi_\nu$, together with $\phi\phi_\nu\xi=\phi^2\xi_\nu=-\xi_\nu+\eta(\xi_\nu)\xi=-\xi_\nu$, then (6.2) becomes

$$\begin{aligned}0=\ &\phi AX+2\alpha(\nabla_\xi A)X-A\phi X+2\alpha(\nabla_X A)\xi-2\alpha\phi X\\&+(2\alpha+3\beta)\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\xi-3(2\alpha+\beta)\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu.\end{aligned}\tag{6.3}$$

On the other hand, from the Codazzi equation and our assumption of M_B being Hopf, we get

$$\begin{aligned}2\alpha(\nabla_\xi A)X&=2\alpha(\nabla_X A)\xi-\alpha\phi X+\alpha\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu\xi-3\eta_\nu(\phi X)\xi_\nu\big\}\\&=2\alpha^2\phi AX-2\alpha A\phi AX-\alpha\phi X\\&\quad+\alpha\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu\xi-3\eta_\nu(\phi X)\xi_\nu\big\}.\end{aligned}\tag{6.4}$$

Merging (6.3) and (6.4), we have

$$\begin{aligned}0=\ &\phi AX-A\phi X-3\alpha\phi X+4\alpha^2\phi AX-4\alpha A\phi AX\\&+3(\alpha+\beta)\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\xi-3(3\alpha+\beta)\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu\end{aligned}\tag{6.5}$$

for any tangent vector field X on TM_B.

If we put $X=\xi_1\in T_\beta$ in (6.5) and take the inner product with $\phi_1\xi$, then we have

$$4\beta(1+\alpha^2)=0.\tag{6.6}$$

On the other hand, putting $X = \phi_1\xi \in T_\gamma$ in (6.5) yields

$$\beta\xi_1 + 3\alpha\xi_1 - 3(3\alpha + \beta)\xi_1 = 0,$$

where we have used $A\phi_1\xi = 0$ and $\phi^2\xi_1 = -\xi_1$. Since ξ_1 is unit, this implies $\beta = -3\alpha$. Substituting this fact into (6.6) gives

$$-12\alpha(1 + \alpha^2) = 0.$$

Since the Reeb function α of M_B is non-vanishing, it makes a contradiction. In fact, the Reeb function α of M_B is given by

$$\alpha = \begin{cases} \sqrt{2}\tanh(\sqrt{2}r) & \text{on } (\mathcal{T}_B^*), \\ \sqrt{2} & \text{on } (\mathcal{H}_B^*), \\ \sqrt{2} & \text{on } (\mathcal{E}), \end{cases}$$

respectively.

From these facts, we conclude that real hypersurfaces M_B of types $(\mathcal{T}_B^*)$, $(\mathcal{H}_B*)$ or $(\mathcal{E})$ cannot satisfy the condition of generalized Killing structure Jacobi. Therefore we obtain a non-existence theorem for the case $\xi \in \mathcal{Q}$.

Summing up Lemma 4.1 and all the facts in Sections 5 and 6, we can assert a non-existence result in our Main Theorem.

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