J. Korean Math. Soc. **59** (2022), No. 2, pp. 353–365
https://doi.org/10.4134/JKMS.j210246
pISSN: 0304-9914 / eISSN: 2234-3008

RADIUS CONSTANTS FOR FUNCTIONS ASSOCIATED WITH A LIMACON DOMAIN

Nak Eun Cho, Anbhu Swaminathan, and Lateef Ahmad Wani

Abstract. Let $\mathcal{A}$ be the collection of analytic functions f defined in $\mathbb{D} := \{\xi \in \mathbb{C} : |\xi| < 1\}$ such that $f(0) = f'(0) - 1 = 0$. Using the concept of subordination ($\prec$), we define

$$\mathcal{S}_\ell^* := \left\{ f \in \mathcal{A} : \frac{\xi f'(\xi)}{f(\xi)} \prec \Phi_\ell(\xi) = 1 + \sqrt{2}\xi + \frac{\xi^2}{2},\ \xi \in \mathbb{D} \right\},$$

where the function $\Phi_\ell(\xi)$ maps $\mathbb{D}$ univalently onto the region Ω_ℓ bounded by the limacon curve

$$\left(9u^2 + 9v^2 - 18u + 5\right)^2 - 16\left(9u^2 + 9v^2 - 6u + 1\right) = 0.$$

For $0 < r < 1$, let $\mathbb{D}_r := \{\xi \in \mathbb{C} : |\xi| < r\}$ and $\mathcal{G}$ be some geometrically defined subfamily of $\mathcal{A}$. In this paper, we find the largest number $\rho \in (0,1)$ and some function $f_0 \in \mathcal{G}$ such that for each $f \in \mathcal{G}$

$$\mathcal{L}_f(\mathbb{D}_r) \subset \Omega_\ell \quad \text{for every} \quad 0 < r \leq \rho,$$

and

$$\mathcal{L}_{f_0}(\partial\mathbb{D}_\rho) \cap \partial\Omega_\ell \neq \emptyset,$$

where the function $\mathcal{L}_f : \mathbb{D} \to \mathbb{C}$ is given by

$$\mathcal{L}_f(\xi) := \frac{\xi f'(\xi)}{f(\xi)}, \quad f \in \mathcal{A}.$$

Moreover, certain graphical illustrations are provided in support of the results discussed in this paper.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc $\{\xi \in \mathbb{C} : |\xi| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{H} := \mathcal{H}(\mathbb{D})$ be the collection of all holomorphic functions defined on $\mathbb{D}$.

Received April 10, 2021; Revised June 20, 2021; Accepted October 14, 2021.
2020 *Mathematics Subject Classification.* 30C45, 30C80.
Key words and phrases. Subordination, radii problems, lemniscates, limacon, cardioid, nephroid.

The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A01050861). The second and third-named authors were also supported by the Project No. CRG/2019/000200/MS of Science and Engineering Research Board, Department of Science and Technology, New Delhi, India.

Further, let $\mathcal{A}$ consist of $f \in \mathcal{H}$ satisfying the normalizations $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two subfamilies of $\mathcal{A}$. The $\mathcal{G}_1$-radius for the family $\mathcal{G}_2$ is the largest number $\rho \in (0,1)$ such that $r^{-1}f(r\xi) \in \mathcal{G}_1$ for each $f \in \mathcal{G}_2$, where $0 < r \leq \rho$. Moreover, if we can find a function $f_0 \in \mathcal{G}_2$ such that $r^{-1}f_0(r\xi) \notin \mathcal{G}_1$ whenever $r > \rho$, then the radius constant ρ is said to be sharp. The problem of finding the number ρ is called a radius problem. For more information related to radius problems of various kinds, we refer to [7, Chapter 13]. Let $f \in \mathcal{A}$ and

$$\mathcal{L}_f(\xi) := \frac{\xi f'(\xi)}{f(\xi)}. \tag{1}$$

By $\mathcal{S}$ we symbolize the family of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$, i.e., $f \in \mathcal{S}$ if it takes no value more than once. A region $\Omega \subset \mathbb{C}$ is called a starlike region with respect to a point $\zeta_0 \in \Omega$ if for every point $\zeta \in \Omega$, $\{\zeta_0 + t(\zeta - \zeta_0) : 0 \leq t \leq 1\}$ lies in Ω. A function $f \in \mathcal{A}$ is said to be a starlike function with respect to a point $\zeta_0 \in f(\mathbb{D})$ if f maps $\mathbb{D}$ onto a region that is starlike with respect to ζ_0. In the special case $\zeta_0 = 0$, we simply say f is a starlike function. If $\mathcal{S}^*$ denotes the collection of all starlike functions, then the inclusion $\mathcal{S}^* \subsetneq \mathcal{S}$ holds. The functions in $\mathcal{S}^*$ are characterized by the inequality $\operatorname{Re}(\mathcal{L}_f(\xi)) > 0$, see Duren [5, p. 41]. Furthermore, let $\mathcal{S}^*(\alpha)$ consist of $f \in \mathcal{A}$ satisfying $\operatorname{Re}(\mathcal{L}_f(\xi)) > \alpha$ for some $\alpha \in [0,1)$. Clearly, $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*$ with $\mathcal{S}^*(0)$ providing the equality.

For $f, g \in \mathcal{H}$, we say that f is *subordinate* to g, written as $f \prec g$, if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ satisfying $w(0) = 0$ such that $f(\xi) = g(w(\xi))$. In particular, if the function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For further details related to subordination, we refer to [2,13]. Using subordination, Ma and Minda [10] introduced

$$\mathcal{S}^*(\Phi) := \{f \in \mathcal{A} : \mathcal{L}_f(\xi) \prec \Phi(\xi)\}, \tag{2}$$

where the function $\Phi : \mathbb{D} \to \mathbb{C}$ satisfies (i) Φ is analytic univalent with $\operatorname{Re}(\Phi) > 0$, (ii) Φ sends $\mathbb{D}$ onto a region that is starlike with respect to $\Phi(0) = 1$, (iii) $\Phi(\mathbb{D})$ is symmetric about the real-line, and (iv) $\Phi'(0) > 0$. Obviously, for each Φ satisfying (i)-(iv), the containment $\mathcal{S}^*(\Phi) \subseteq \mathcal{S}^*$ holds. In the recent past, several Ma-Minda type families of functions having nice geometrical properties have been introduced and discussed by many geometric function theorists. Here we mention only a few of them and for a comprehensive list of such families, we refer to [19,20]. The family $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+\xi})$ associated with the lemniscate of Bernoulli was introduced in [17] which was later on generalized to $\mathcal{S}^*_L(\alpha) := \mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+\xi})$ for some $\alpha \in [0,1)$ in [9]. The family $\mathcal{S}^*_{RL} := \mathcal{S}^*(\Phi_{RL})$, where

$$\Phi_{RL}(\xi) := \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-\xi}{1+2(\sqrt{2}-1)\xi}}, \qquad \xi \in \mathbb{D} \tag{3}$$

was introduced in [11]. The family $\mathcal{S}^*(e^\xi)$ was introduced and discussed in [12] and generalized to $\mathcal{S}^*_{\alpha,e} := \mathcal{S}^*(\alpha + (1-\alpha)e^\xi)$ in [9]. The family $\mathcal{S}^*_C := \mathcal{S}^*(1 + 4\xi/3 + 2\xi^2/3)$ associated with a cardioid has been introduced in [15]. For $0 \leq \alpha < 1$, the family $\mathcal{B}(\alpha) := \mathcal{S}^*(1 + \xi/(1-\alpha\xi^2))$ associated with the lemniscate of Booth was introduced in [3,8]. The family $\mathcal{S}^*_\ell := \mathcal{S}^*(1 + \sqrt{2}\xi + \xi^2/2)$ associated with a limacon was introduced in [22]. Very recently, the family $\mathcal{S}^*_{Ne} := \mathcal{S}^*(1 + \xi - \xi^3/3)$ associated with a nephroid was introduced and discussed in [19, 20].

1.1. The family $\mathcal{S}^*_\ell$

Among the families mentioned above, our focus in this paper is on the family $\mathcal{S}^*_\ell := \mathcal{S}^*(\Phi_\ell)$, where the function $\Phi_\ell(\xi) := 1 + \sqrt{2}\xi + \xi^2/2$ sends $\mathbb{D}$ univalently onto the interior of a dimpled-curve called *limacon* (Figure 1) given by

$$\left(9u^2 + 9v^2 - 18u + 5\right)^2 - 16\left(9u^2 + 9v^2 - 6u + 1\right) = 0. \tag{4}$$

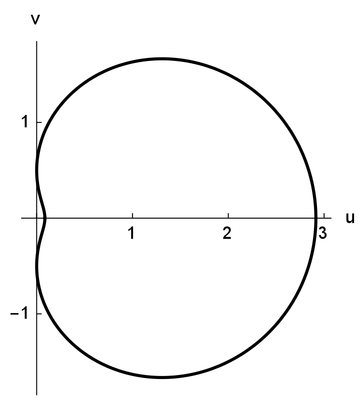


FIGURE 1. The limacon curve $\left(9u^2 + 9v^2 - 18u + 5\right)^2 - 16\left(9u^2 + 9v^2 - 6u + 1\right) = 0$.

Apart from other results, the authors in [22] discussed the structural formula and a few coefficient estimates including the Fekete-Szegö problem for functions belonging to $\mathcal{S}^*_\ell$. Later on, the authors in [21] constructed certain examples of functions belonging to the family $\mathcal{S}^*_\ell$ and discussed several subordination results related to it. In this paper, we consider the family $\mathcal{S}^*_\ell$ and study some radius results. The study of radius problems for families of functions with special geometries is continuing to be an active area of research in the theory of univalent functions. We redefine the radius problem for the family $\mathcal{S}^*_\ell$ in geometrical terms as follows.

Definition. Let $\mathbb{D}_r := \{\xi : |\xi| < r\}$ and $\Omega_\ell := \Phi_\ell(\mathbb{D})$. By $\mathscr{R}_\ell(\mathcal{G})$ for the family $\mathcal{G} \subset \mathcal{A}$ we mean the largest number $\rho \in (0,1)$ such that each $f \in \mathcal{G}$ satisfies

$$\mathcal{L}_f(\mathbb{D}_r) \subset \Omega_\ell \quad \text{for every} \quad 0 < r \leq \rho,$$

and

$$\mathcal{L}_{f_0}(\partial\mathbb{D}_\rho) \cap \partial\Omega_\ell \neq \emptyset \quad \text{for some} \quad f_0 \in \mathcal{G},$$

where $\mathcal{L}_f(\xi)$ is defined in (1).

For recent works on radius problems, we refer to [1,3,4,6,11,12,15,16,18,19].

2. Main results

In order to prove our results, the following lemma is required.

Lemma 2.1. *Let $(3-2\sqrt{2})/2 < a < (3+2\sqrt{2})/2$, and let $\mathfrak{r}_a$ and $\mathfrak{R}_a$ be defined as*

$$\mathfrak{r}_a = \begin{cases} a + \sqrt{2} - 3/2, & (3-2\sqrt{2})/2 < a \leq 3/2, \\ 3/2 + \sqrt{2} - a, & 3/2 \leq a < (3+2\sqrt{2})/2, \end{cases}$$

and

$$\mathfrak{R}_a = \begin{cases} 3/2 + \sqrt{2} - a, & (3-2\sqrt{2})/2 < a \leq (1+\sqrt{2})/2, \\ \frac{1}{2}\sqrt{\frac{a(2a-1)^2}{a-1}}, & (1+\sqrt{2})/2 \leq a < (3+2\sqrt{2})/2. \end{cases}$$

Then

$$\{w \in \mathbb{C} : |w-a| < \mathfrak{r}_a\} \subseteq \Omega_\ell \subseteq \{w \in \mathbb{C} : |w-a| < \mathfrak{R}_a\}.$$

Proof. From the definition of the function $\Phi_\ell(\xi)$, it is easy to verify that the limacon curve (4) has following parametric equations:

$$u(t) = 1 + \sqrt{2}\cos t + \frac{\cos 2t}{2}, \quad v(t) = \sqrt{2}\sin t + \frac{\sin 2t}{2}, \qquad t \in (-\pi, \pi].$$

The square of the distance from the point $(a, 0)$ to the points on the limacon curve (4) is

$$\begin{aligned} \zeta(t) :=& (u(t)-a)^2 + (v(t))^2 \\ =& -2(a-1)y^2 + \sqrt{2}(3-2a)y + (a-1)a + \frac{9}{4} =: g(y), \end{aligned}$$

where $y = \cos t$, $t \in (-\pi, \pi]$. Since the limacon curve is symmetric about the real line, it is sufficient to take $t \in [0, \pi]$. Simple calculation shows that $g'(y) = 0$ implies

$$y = \frac{3-2a}{2\sqrt{2}(a-1)} =: y_0(a).$$

It is easy to verify that the number $y_0(a)$ lies between -1 and 1 if and only if $a \geq \left(1+\sqrt{2}\right)/2 \approx 1.20711$, see Figure 2. Moreover, for this a, we have

$g''(y_0(a)) < 0$. This shows that the function $g(y)$ is increasing in $[-1, y_0(a)]$ and decreasing in $[y_0(a), 1]$ for $\left(1+\sqrt{2}\right)/2 \le a < \left(3+2\sqrt{2}\right)/2$. Hence

$$\min_{0\le t\le \pi} \zeta(t) = \min\{g(-1),\, g(1)\} \quad \text{for} \quad \frac{1+\sqrt{2}}{2} \le a < \frac{3+2\sqrt{2}}{2}$$

and

$$\max_{0\le t\le \pi} \zeta(t) = g(y_0(a)) = \frac{(1-2a)^2 a}{4(a-1)} \quad \text{for} \quad \frac{1+\sqrt{2}}{2} \le a < \frac{3+2\sqrt{2}}{2}.$$

Also

$$g(1) - g(-1) = 2\sqrt{2}(3-2a),$$

so that

$$g(-1) \le g(1) \quad \text{if} \quad a \le \frac{3}{2}$$

and

$$g(1) \le g(-1) \quad \text{if} \quad a \ge \frac{3}{2}.$$

Therefore, we conclude that

$$\mathfrak{r}_a = \min_{0\le t\le \pi} \sqrt{\zeta(t)} = \begin{cases} \sqrt{g(-1)} = a - \left(\frac{3}{2} - \sqrt{2}\right), & \frac{1+\sqrt{2}}{2} < a \le \frac{3}{2}, \\ \sqrt{g(1)} = \frac{3}{2} + \sqrt{2} - a, & \frac{3}{2} \le a < \frac{3+2\sqrt{2}}{2} \end{cases}$$

and

$$\begin{aligned} \mathfrak{R}_a = \max_{0\le t\le \pi} \sqrt{\zeta(t)} &= \sqrt{g(y_0(a))} \\ &= \frac{1}{2}\sqrt{\frac{a(2a-1)^2}{a-1}} \quad \text{for} \quad \frac{1+\sqrt{2}}{2} \le a < \frac{3+2\sqrt{2}}{2}. \end{aligned}$$

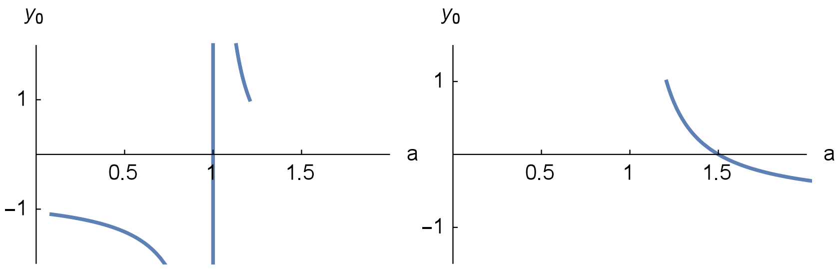
Furthermore, for $\left(3-2\sqrt{2}\right)/2 < a \le \left(1+\sqrt{2}\right)/2$, a simple verification (graphically as well) reveals that $g(y)$ is increasing whenever $-1 \le y \le 1$. Thus, in this case

$$\mathfrak{r}_a = \min_{0\le t\le \pi} \sqrt{\zeta(t)} = \sqrt{g(-1)} = a - \left(\frac{3}{2} - \sqrt{2}\right)$$

and

$$\mathfrak{R}_a = \max_{0\le t\le \pi} \sqrt{\zeta(t)} = \sqrt{g(1)} = \frac{3}{2} + \sqrt{2} - a.$$

Combining, we obtain the desired results. □



(A) $a \in \left(\frac{3-2\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right)$ (B) $a \in \left(\frac{1+\sqrt{2}}{2}, \frac{3+2\sqrt{2}}{2}\right)$

FIGURE 2. Plots of $y_0(a) = \frac{3-2a}{2\sqrt{2}(a-1)}$.

Recall that the Janowski family $\mathcal{S}^*[A,B]$ is the collection of $f \in \mathcal{A}$ satisfying $\mathcal{L}_f(\xi) \in \mathcal{P}[A,B]$, where

$$\mathcal{P}[A,B] := \left\{ p(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n : p(\xi) \prec \frac{1+A\xi}{1+B\xi},\ \xi \in \mathbb{D} \right\}.$$

Geometrically, a function $p \in \mathcal{H}$ satisfying $p(0) = 1$ belongs to $\mathcal{P}[A,B]$ if and only if the region $p(\mathbb{D})$ lies inside the open disk which has the line-segment $\left[\frac{1-A}{1-B}, \frac{1+A}{1+B}\right]$ as its diameter.

Theorem 2.2. *For the Janowski function family $\mathcal{S}^*[A,B]$, we have*

$$\mathscr{R}_\ell\left(\mathcal{S}^*[A,B]\right) = \rho = \begin{cases} \min\left\{1, \frac{2\sqrt{2}-1}{2A+(2\sqrt{2}-3)B}\right\}, & \text{if } \ 0 \le B < A \le 1, \\ \min\left\{1, \frac{2\sqrt{2}+1}{2A-(2\sqrt{2}+3)B}\right\}, & \text{if } \ -1 \le B < A \le 1. \end{cases}$$

To prove this theorem, we make use of the following lemma.

Lemma 2.3 ([14, Lemma 2.1, p. 267])**.** *If $p \in \mathcal{P}[A,B]$, then for $|\xi| = r < 1$,*

$$\left| p(\xi) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2}.$$

Proof. Let $f \in \mathcal{S}^*[A,B]$. Then $\mathcal{L}_f \in \mathcal{P}[A,B]$, and Lemma 2.3 yields

$$\left| \mathcal{L}_f(\xi) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2}. \tag{5}$$

The inequality (5) represents a disk with center at $(1-ABr^2)/(1-B^2r^2)$ and radius $(A-B)r/(1-B^2r^2)$.

Case 1. For $0 \le B < A \le 1$, we have

$$B^2 \le AB \implies 1 - B^2 \ge 1 - AB \implies (1-AB)/(1-B^2) \le 1,$$

and hence $(1-ABr^2)/(1-B^2r^2) \leq 1$. Thus, by Lemma 2.1, the disk (5) lies completely inside Ω_ℓ if

$$\frac{(A-B)r}{1-B^2r^2} \leq \frac{1-ABr^2}{1-B^2r^2} + \sqrt{2} - \frac{3}{2}.$$

An easy simplification yields $r \leq \frac{2\sqrt{2}-1}{2A+(2\sqrt{2}-3)B}$. Further, consider the function

$$f_0(\xi) := \begin{cases} \xi\,(1+B\xi)^{\frac{A-B}{B}}, & \text{if } B \neq 0, \\ \xi e^{A\xi}, & \text{if } B = 0. \end{cases}$$

The function $f_0(\xi)$ satisfies $\mathcal{L}_{f_0}(\xi) = (1+A\xi)/(1+B\xi)$, implying that $f_0 \in \mathcal{S}^*[A,B]$. Also, for $\xi_0 = -\frac{2\sqrt{2}-1}{2A+(2\sqrt{2}-3)B} \in \partial\mathbb{D}_\rho$, we have

$$\mathcal{L}_{f_0}(\xi_0) = \frac{3}{2} - \sqrt{2} \in \partial\Omega_\ell,$$

i.e.,

$$\mathcal{L}_{f_0}(\partial\mathbb{D}_\rho) \cap \partial\Omega_\ell = \left\{\frac{3}{2} - \sqrt{2}\right\} \neq \emptyset.$$

This proves that $\rho = \min\left\{1, \frac{2\sqrt{2}-1}{2A+(2\sqrt{2}-3)B}\right\}$ if $0 \leq B < A \leq 1$.

Case 2. Let $-1 \leq B < A \leq 1$. Then $B \leq 0$ implies that $(1-ABr^2)/(1-B^2r^2) \geq 1$. Therefore, in this case, the disk (5) lies in the interior of Ω_ℓ if

$$\frac{(A-B)r}{1-B^2r^2} \leq \frac{3}{2} + \sqrt{2} - \frac{1-ABr^2}{1-B^2r^2}.$$

Solving, we get the desired value of the radius constant ρ. Choosing $\xi_0 = \rho$ and $f_0 \in \mathcal{S}^*[A,B]$ defined above, it is easy to verify that $\mathcal{L}_{f_0}(\xi_0) = 3/2 + \sqrt{2} \in \partial\Omega_\ell$. □

Specializing A and B in Theorem 2.2, the followings are obtained.

Corollary 2.4. $\mathscr{R}_\ell\,(\mathcal{S}^*) = (2\sqrt{2}+1)/(2\sqrt{2}+5) \approx 0.489042$.

Corollary 2.5. $\mathscr{R}_\ell\,(\mathcal{C}) = (2\sqrt{2}+1)/(2\sqrt{2}+3) \approx 0.656854$.

Before going to the next result, we mention that $\mathcal{S}^*_L(\alpha) \subset \mathcal{S}^*_\ell$ for $\alpha \geq 3/2 - \sqrt{2} \approx 0.0857864$, and $\mathcal{S}^*_{\alpha,e} \subset \mathcal{S}^*_\ell$ for $\alpha \geq \left(3 - 2\left(e - \sqrt{2}\right)\right)/(2-2e) \approx -0.114028$ (see [21, Theorem 3]). Therefore, $\mathscr{R}_\ell(\mathcal{G}) = 1$ for (i) $\mathcal{G} = \mathcal{S}^*_L(\alpha)$ whenever $\alpha \geq 3/2 - \sqrt{2}$ and (ii) $\mathcal{G} = \mathcal{S}^*_{\alpha,e}$ whenever $\alpha \geq \left(3 - 2\left(e - \sqrt{2}\right)\right)/(2-2e)$.

Theorem 2.6. *For the function families $\mathcal{B}(\alpha)$, $\mathcal{S}^*_L(\alpha)$, and $\mathcal{S}^*_{\alpha,e}$ in Section 1, we have*

(i) $\mathscr{R}_\ell\,(\mathcal{B}(\alpha)) = \frac{-1+\sqrt{-4\sqrt{2}\alpha+9\alpha+1}}{2\sqrt{2}\alpha-\alpha} =: \rho_1(\alpha),\ 0 < \alpha < 1.$

(ii) $\mathscr{R}_\ell\,(\mathcal{S}^*_L(\alpha)) = \frac{\left(\sqrt{2}-\frac{1}{2}\right)\left(2(1-\alpha)-\left(\sqrt{2}-\frac{1}{2}\right)\right)}{(1-\alpha)^2} =: \rho_2(\alpha),\ 0 \leq \alpha \leq (3-2\sqrt{2})/2.$
In particular, $\mathscr{R}_\ell\,(\mathcal{S}^*_L) = 3\sqrt{2} - 13/4$.

(iii) $\mathscr{R}_\ell\left(\mathcal{S}^*_{\alpha,e}\right) = \log\left(\frac{-\alpha+\sqrt{2}+\frac{1}{2}}{1-\alpha}\right) =: \rho_3(\alpha),\ -\frac{1}{e-1} \leq \alpha \leq \frac{3-2\left(e-\sqrt{2}\right)}{2(1-e)}$.

The estimates on the radius constants in (i) *and* (ii) *are best possible.*

Proof. (i) Let $f \in \mathcal{B}(\alpha)$. Then $\mathcal{L}_f(\xi) \prec G_\alpha(\xi) := 1 + \xi/(1-\alpha\xi^2)$ and hence, for $|\xi| = r$, we have

$$|\mathcal{L}_f(\xi) - 1| \leq \max_{|\xi|=r}\left|\frac{\xi}{1-\alpha\xi^2}\right| \leq \frac{r}{1-\alpha r^2}. \tag{6}$$

In view of Lemma 2.1, the disk (6) lies completely inside the limacon region Ω_ℓ if $r/(1-\alpha r^2) \leq \sqrt{2} - 1/2$, or if $\left(\sqrt{2}-1/2\right)\alpha r^2 + r - \left(\sqrt{2}-1/2\right) \leq 0$. This gives $r \leq \mathscr{R}_\ell\left(\mathcal{B}(\alpha)\right) = \rho_1(\alpha)$. For sharpness, consider the function

$$f_{\mathcal{B}}(\xi) = \begin{cases} \xi\left(\frac{1+\sqrt{\alpha}\xi}{1-\sqrt{\alpha}\xi}\right)^{1/(2\sqrt{\alpha})}, & \alpha \in (0,1), \\ \xi e^{\xi}, & \alpha = 0. \end{cases}$$

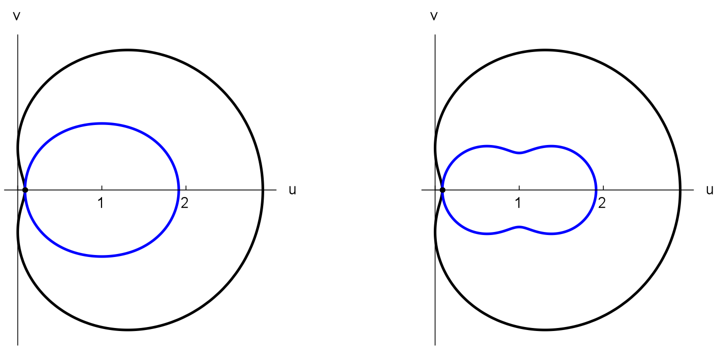
It is easy to verify that $\mathcal{L}_{f_{\mathcal{B}}}(\xi) = G_\alpha(\xi)$, and hence $f_{\mathcal{B}} \in \mathcal{B}(\alpha)$. Also, a simple calculation shows that

$$\mathcal{L}_{f_{\mathcal{B}}}(\xi_1) = \frac{3}{2} - \sqrt{2} \in \partial\Omega_\ell \quad \text{for } \xi_1 = -\rho_1(\alpha).$$

Therefore,

$$\mathcal{L}_{f_{\mathcal{B}}}\left(\partial\mathbb{D}_{\rho_1(\alpha)}\right) \cap \partial\Omega_\ell = \left\{\frac{3}{2} - \sqrt{2}\right\} \neq \emptyset.$$

This proves that $\mathscr{R}_\ell\left(\mathcal{B}(\alpha)\right) = \rho_1(\alpha)$. Figure 3 shows that the radius constant $\rho_1(\alpha)$ is best possible.



(A) $\alpha = 0.1$ (B) $\alpha = 0.9$

FIGURE 3. Sharpness of $\rho_1(\alpha)$.

(ii) Let $f \in \mathcal{S}^*_L(\alpha)$. Then $\mathcal{L}_f(\xi) \prec \alpha + (1-\alpha)\sqrt{1+\xi}$ and hence

$$|\mathcal{L}_f(\xi) - 1| \leq \left|\alpha + (1-\alpha)\sqrt{1+\xi} - 1\right|$$

$$= (1-\alpha)\left|1-\sqrt{1+\xi}\right|$$
$$\leq (1-\alpha)\left(1-\sqrt{1-r}\right), \quad |\xi| = r.$$

Applying Lemma 2.1, we have $f \in \mathcal{S}^*_\ell$ if

$$(1-\alpha)\left(1-\sqrt{1-r}\right) \leq \sqrt{2} - 1/2,$$

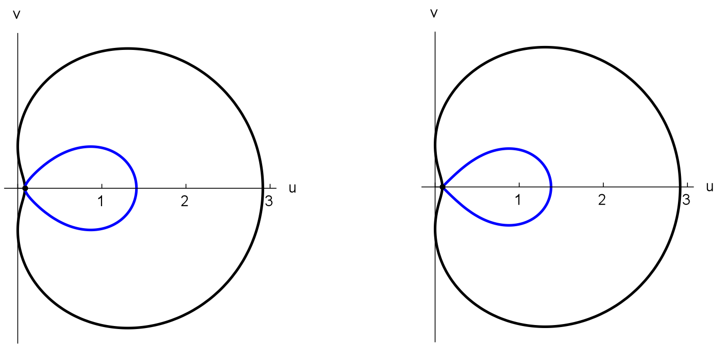
which further gives $r \leq \rho_2(\alpha)$. Since the function

$$f_L(\xi) = \xi + \frac{1}{2}(1-\alpha)\xi^2 + \frac{1}{16}(1-\alpha)(1-2\alpha)\xi^3 + \cdots$$

satisfies $\mathcal{L}_{f_L}(\xi) = \alpha + (1-\alpha)\sqrt{1+\xi}$, it follows that $f_L(\xi)$ is a member of $\mathcal{S}^*_L(\alpha)$. In addition,

$$\mathcal{L}_{f_L}(\xi_2) = \frac{3}{2} - \sqrt{2} \in \partial\Omega_\ell \quad \text{for} \quad \xi_2 = -\rho_2(\alpha) \in \partial\mathbb{D}_{\rho_2}.$$

This shows that $\mathscr{R}_\ell\left(\mathcal{S}^*_L(\alpha)\right) = \rho_2(\alpha)$, as illustrated in Figure 4.



(A) $\alpha = 0$ (B) $\alpha = \frac{1}{2}\left(3 - 2\sqrt{2}\right)$

FIGURE 4. Sharpness of $\rho_2(\alpha)$.

(iii) Since $f \in \mathcal{S}^*_{\alpha,e}$ gives $\mathcal{L}_f(\xi) \prec \alpha + (1-\alpha)e^\xi$. Therefore, for $|\xi| = r$, we have

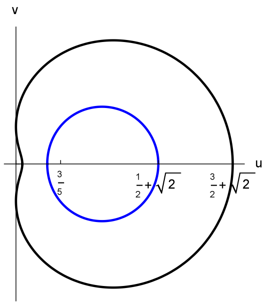
$$|\mathcal{L}_f(\xi) - 1| \leq (1-\alpha)\left|e^\xi - 1\right| \leq (1-\alpha)\left(e^r - 1\right) \leq (2\sqrt{2} - 1)/2$$

provided $r \leq \rho_3(\alpha)$. Thus, $f \in \mathcal{S}^*_\ell$ if $|\xi| = r \leq \rho_3(\alpha)$. Since, for $\Phi_{\alpha,e}(\xi) = \alpha + (1-\alpha)e^\xi$,

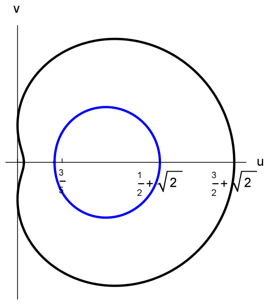
$$\Phi_{\alpha,e}(\rho_3(\alpha)) = \frac{1}{2} + \sqrt{2} < \frac{3}{2} + \sqrt{2} = \Phi_\ell(1),$$

we conclude that the radius constant $\rho_3(\alpha)$ can be improved further. See Figure 5 for graphical illustrations. Observe that, for each $-\frac{1}{e-1} \leq \alpha \leq \frac{3-2\left(e-\sqrt{2}\right)}{2(1-e)}$, the domain $\Phi_{\alpha,e}(|\xi| < \rho_3(\alpha))$ is properly contained in the limacon domain and

$$\Phi_{\alpha,e}(\partial\mathbb{D}_{\rho_3(\alpha)}) \cap \partial\Omega_\ell = \emptyset.$$



(A) $\alpha = -1/(e-1)$



(B) $\alpha = \left(3-2(e-\sqrt{2})\right)/(2(1-e))$

$\square$

FIGURE 5. Non-sharpness of $\rho_3(\alpha)$.

Theorem 2.7. *For the families* $\mathcal{S}^*_{RL}$ *and* $\mathcal{S}^*_C$ *we have*:

(i) $\mathscr{R}_\ell\left(\mathcal{S}^*_{RL}\right) = \rho_{RL} := \frac{1}{158}\left(65+63\sqrt{2}\right) \approx 0.975288$,

(ii) $\mathscr{R}_\ell\left(\mathcal{S}^*_C\right) = \rho_c := \frac{1}{2}\left(\sqrt{1+6\sqrt{2}}-2\right) \approx 0.539909$.

The radius constant in (i) *is best possible while as the constant in* (ii) *can be improved* (*see* Figure 6).

Proof. (i) Let $f \in \mathcal{S}^*_{RL}$. Then $\mathcal{L}_f(\xi) \prec \Phi_{RL}(\xi)$, where $\Phi_{RL}(\xi)$ is defined in (3). Thus, for $|\xi| = r < 1$, we have

$$\begin{aligned}|\mathcal{L}_f(\xi)-1| &\leq \left|\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1-\xi}{1+2(\sqrt{2}-1)\xi}}-1\right| \\ &\leq 1-\left(\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right).\end{aligned}$$

In view of Lemma 2.1, the above disk lies within the region Ω_ℓ if

$$1-\left(\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right) \leq \sqrt{2}-\frac{1}{2},$$

or equivalently, if $r \leq \rho_{RL}$. The result is sharp for the function $f_{RL} \in \mathcal{S}^*_{RL}$ given by

$$\begin{aligned}f_{RL}(\xi) = \xi &\left(\frac{\sqrt{1-\xi}+\sqrt{1+2(\sqrt{2}-1)\xi}}{2}\right)^{2\sqrt{2}-2} \\ &\times \exp\left(\sqrt{2\left(\sqrt{2}-1\right)}\ \tan^{-1}\left(\Psi(\xi)\right)\right),\end{aligned}$$

where

$$\Psi(\xi) = \frac{\sqrt{2\left(\sqrt{2}-1\right)}\left(\sqrt{2\left(\sqrt{2}-1\right)\xi+1}-\sqrt{1-\xi}\right)}{2\left(\sqrt{2}-1\right)\sqrt{1-\xi}+\sqrt{2\left(\sqrt{2}-1\right)\xi+1}}.$$

Further, $\mathcal{L}_{f_{RL}}\left(\partial\mathbb{D}_{\rho_{RL}}\right)\cap\partial\Omega_\ell = \{3/2-\sqrt{2}\}$.

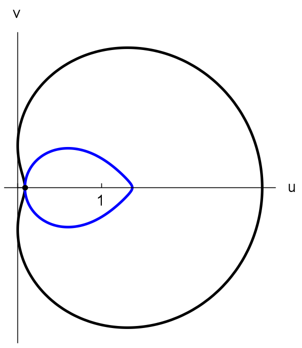
(ii) $f \in \mathcal{S}^*_C$ implies $\mathcal{L}_f(\xi) \prec 1+4\xi/3+2\xi^2/3$, which further gives

$$|\mathcal{L}_f(\xi)-1| \leq 2\left(r^2+2r\right)/3, \qquad |\xi| = r.$$

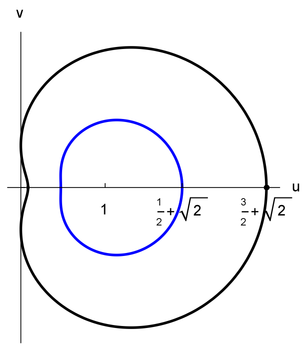
Applying Lemma 2.1, we observe that $f \in \mathcal{S}^*_\ell$ if $2\left(r^2+2r\right)/3 \leq \sqrt{2}-1/2$, or, if $r \leq \rho_c$. For $\Phi_C(\xi) = 1+4\xi/3+2\xi^2/3$, we have

$$\Phi_C(\rho_c) = \frac{1}{2}+\sqrt{2} < \frac{3}{2}+\sqrt{2} = \Phi_\ell(1).$$

Therefore, the domain $\Phi_C(|\xi| < \rho_c)$ is properly contained in the domain bounded by the limacon curve (4). This proves that the radius constant ρ_c is not sharp and hence can be improved further. ☐



(A) Sharpness of ρ_{RL}



(B) Non-sharpness of ρ_c

Figure 6

Acknowledgements. The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1 A3A01050861). The second and third-named authors were also supported by the Project No. CRG/2019/000200/MS of Science and Engineering Research Board, Department of Science and Technology, New Delhi, India. The authors would like to express their gratitude to the anonymous referees for their thoughtful comments and efforts towards improving the paper.

References

[1] R. M. Ali, N. K. Jain, and V. Ravichandran, *Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane*, Appl. Math. Comput. **218** (2012), no. 11, 6557–6565. https://doi.org/10.1016/j.amc.2011.12.033

[2] T. Bulboacă, *Differential Subordinations and Superordinations*, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[3] N. E. Cho, S. Kumar, V. Kumar, and V. Ravichandran, *Differential subordination and radius estimates for starlike functions associated with the Booth lemniscate*, Turkish J. Math. **42** (2018), no. 3, 1380–1399.

[4] N. E. Cho, V. Kumar, S. S. Kumar, and V. Ravichandran, *Radius problems for starlike functions associated with the sine function*, Bull. Iranian Math. Soc. **45** (2019), no. 1, 213–232. https://doi.org/10.1007/s41980-018-0127-5

[5] P. L. Duren, *Univalent functions*, Grundlehren der mathematischen Wissenschaften, 259, Springer-Verlag, New York, 1983.

[6] S. Gandhi and V. Ravichandran, *Starlike functions associated with a lune*, Asian-Eur. J. Math. **10** (2017), no. 4, 1750064, 12 pp. https://doi.org/10.1142/S1793557117500644

[7] A. W. Goodman, *Univalent functions. Vol. II*, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[8] R. Kargar, A. Ebadian, and J. Sokół, *On Booth lemniscate and starlike functions*, Anal. Math. Phys. **9** (2019), no. 1, 143–154. https://doi.org/10.1007/s13324-017-0187-3

[9] K. Khatter, V. Ravichandran, and S. Sivaprasad Kumar, *Starlike functions associated with exponential function and the lemniscate of Bernoulli*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 1, 233–253. https://doi.org/10.1007/s13398-017-0466-8

[10] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

[11] R. Mendiratta, S. Nagpal, and V. Ravichandran, *A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli*, Internat. J. Math. **25** (2014), no. 9, 1450090, 17 pp. https://doi.org/10.1142/S0129167X14500906

[12] R. Mendiratta, S. Nagpal, and V. Ravichandran, *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 1, 365–386. https://doi.org/10.1007/s40840-014-0026-8

[13] S. S. Miller and P. T. Mocanu, *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York, 2000.

[14] V. Ravichandran, F. Rønning, and T. N. Shanmugam, *Radius of convexity and radius of starlikeness for some classes of analytic functions*, Complex Variables Theory Appl. **33** (1997), no. 1-4, 265–280. https://doi.org/10.1080/17476939708815027

[15] K. Sharma, N. K. Jain, and V. Ravichandran, *Starlike functions associated with a cardioid*, Afr. Mat. **27** (2016), no. 5-6, 923–939. https://doi.org/10.1007/s13370-015-0387-7

[16] J. Sokół, *Radius problems in the class $\mathscr{SL}^*$*, Appl. Math. Comput. **214** (2009), no. 2, 569–573. https://doi.org/10.1016/j.amc.2009.04.031

[17] J. Sokół and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. No. **19** (1996), 101–105.

[18] A. Swaminathan and L. A. Wani, *Sufficient conditions and radii problems for a starlike class involving a differential inequality*, Bull. Korean Math. Soc. **57** (2020), no. 6, 1409–1426. https://doi.org/10.4134/BKMS.b191074

[19] L. A. Wani and A. Swaminathan, *Radius problems for functions associated with a nephroid domain*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114** (2020), no. 4, Paper No. 178, 20 pp. https://doi.org/10.1007/s13398-020-00913-4

[20] L. A. Wani and A. Swaminathan, *Starlike and convex functions associated with a nephroid domain*, Bull. Malays. Math. Sci. Soc. **44** (2021), no. 1, 79–104. https://doi.org/10.1007/s40840-020-00935-6

[21] L. A. Wani and A. Swaminathan, *Certain geometric properties of starlike functions related to a limacon domain*, Mathematics in Engineering, Science and Aerospace (MESA) **12** (2021), no. 2, 323–342.

[22] Y. Yunus, S. A. Halim, and A. B. Akbarally, *Subclass of starlike functions associated with a limacon*, in AIP Conference Proceedings 2018 Jun 28 (Vol. 1974, No. 1, p. 030023), AIP Publishing.

Nak Eun Cho
Department of Applied Mathematics
Pukyong National University
Busan 48513, Korea
Email address: necho@pknu.ac.kr

Anbhu Swaminathan
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee 247667, Uttarakhand, India
Email address: mathswami@gmail.com

Lateef Ahmad Wani
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee 247667, Uttarakhand, India
Email address: lateef17304@gmail.com