J. Korean Math. Soc. **59** (2022), No. 2, pp. 407–420
https://doi.org/10.4134/JKMS.j210371
pISSN: 0304-9914 / eISSN: 2234-3008

GOLDBACH-LINNIK TYPE PROBLEMS WITH UNEQUAL POWERS OF PRIMES

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Abstract. It is proved that every sufficiently large even integer can be represented as a sum of two squares of primes, two cubes of primes, two fourth powers of primes and 17 powers of 2.

1. Introduction

In the 1950s, Linnik [7, 8] proved that every sufficiently large even integer can be represented as a sum of two primes and K powers of 2, where K is an absolute constant. In 1975, Gallagher [1] established an asymptotic formula for the number of such representations. Based on the work of Gallagher [1], Liu, Liu and Wang [10] first established the explicit value of K and showed that $K = 54000$ is acceptable. Afterwards, the value of K was improved by many authors (see [3, 5, 6, 11, 13, 17]). The best result so far is due to Pintz and Ruzsa [14], who proved that $K = 8$ is acceptable.

In 2017, motivated by the works of Linnik [7, 8] and Gallagher [1], Liu [9] considered the problem on the representation of the large even integer N in the form

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_k}, \tag{1.1}$$

where p_i are prime numbers and v_j are positive integers. He proved that (1.1) is solvable for $k = 41$. In 2019, by employing the techniques in Zhao [18], Lü [12] improved the value of k to 24. Very recently, motivated by Platt and Trudgian [15], Zhao [19] refined Lü's result and showed that $k = 22$ is acceptable.

In this paper, by improving the estimates for the singular series and the related integral over the minor arcs, we can obtain the following sharper result:

Theorem 1. *Every sufficiently large even integer is a sum of two squares of primes, two cubes of primes, two fourth powers of primes and* 17 *powers of* 2.

Received June 1, 2021; Revised August 30, 2021; Accepted December 6, 2021.
2020 *Mathematics Subject Classification.* Primary 11P05, 11P55.
Key words and phrases. Waring-Goldbach problem, Hardy-Littlewood method.
This work was supported by The National Natural Science Foundation of China (grant no. 11771333).

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2. Notation and outline of the method

In this paper, we assume that N is a sufficiently large even integer. We fix a positive constant η satisfying $\eta \leq 10^{-100}$. Let ε be an arbitrarily small positive number where the value of ε may change from line to line. The letter p, with or without subscript, is reserved for a prime number. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$. As usual, $\varphi(n)$ stands for Euler's function and $d(n)$ denotes the number of divisors of n.

We plan to investigate the sum

$$\mathcal{R}(k,N) = \sum_{\substack{N=p_1^2+p_2^2+p_3^3+p_4^3+p_5^4+p_6^4+2^{v_1}+\cdots+2^{v_k} \\ \frac{P_2}{2}\leq p_1,p_2\leq P_2,\ \frac{P_3}{2}\leq p_3,p_4\leq P_3, \\ \frac{P_4}{2}\leq p_5,p_6\leq P_4,\ 1\leq v_1,\ldots,v_k\leq L}} (\log p_1)\cdots(\log p_6), \tag{2.1}$$

where

$$P_2 = \sqrt{(1-\eta)N},\ P_3 = \left(\frac{\eta N}{2}\right)^{\frac{1}{3}},\ P_4 = \left(\frac{\eta N}{2}\right)^{\frac{1}{4}} \tag{2.2}$$

and

$$L = \frac{\log(N/\log N)}{\log 2}.$$

The exponents of P_j are natural, since the summation (2.1) is solvable when $p_1, p_2 \leq N^{\frac{1}{2}}$, $p_3, p_4 \leq N^{\frac{1}{3}}$ and $p_5, p_6 \leq N^{\frac{1}{4}}$. In order to apply the circle method, we set

$$S_i(\alpha) = \sum_{\frac{P_i}{2}\leq p\leq P_i} e(p^i\alpha)\log p,\quad H(\alpha) = \sum_{1\leq v\leq L} e(2^v\alpha).$$

As in [9], let

$$Q_1 = N^{\frac{3}{20}-2\varepsilon},\ Q_2 = N^{\frac{17}{20}+\varepsilon}. \tag{2.3}$$

Then we can define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as

$$\mathfrak{M} = \bigcup_{q\leq Q_1} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q,a),\quad \mathfrak{M}(q,a) = \left(\frac{a}{q}-\frac{1}{qQ_2}, \frac{a}{q}+\frac{1}{qQ_2}\right], \tag{2.4}$$

$$\mathfrak{m} = \left[\frac{1}{Q_2}, 1+\frac{1}{Q_2}\right] \backslash \mathfrak{M}.$$

By orthogonality, we get

$$\begin{aligned} \mathcal{R}(k,N) &= \int_0^1 S_2(\alpha)^2 S_3(\alpha)^2 S_4(\alpha)^2 H(\alpha)^k e(-N\alpha)d\alpha \\ &= \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) S_2(\alpha)^2 S_3(\alpha)^2 S_4(\alpha)^2 H(\alpha)^k e(-N\alpha)d\alpha \\ &= I(k,\mathfrak{M},N) + I(k,\mathfrak{m},N), \end{aligned} \tag{2.5}$$

where

$$I(k,\mathfrak{X},N)=\int_{\mathfrak{X}} S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2H(\alpha)^k e(-N\alpha)d\alpha. \tag{2.6}$$

In the following sections, we shall prove

$$I(k,\mathfrak{M},N)\geq 0.0295049P_3^2P_4^2L^k, \tag{2.7}$$

$$|I(k,\mathfrak{m},N)|\leq 0.58814u^kP_3^2P_4^2L^k+O(P_3^2P_4^2L^{k-1}), \tag{2.8}$$

where $u=0.833783$.

3. The lower bound for $I(k,\mathfrak{M},N)$

The purpose of this section is to obtain the lower bound for $I(k,\mathfrak{M},N)$. We first state some auxiliary results. Let

$$C_j(q,a)=\sum_{\substack{m=1\\(m,q)=1}}^{q} e\left(\frac{am^j}{q}\right),$$

$$B(n,q)=\sum_{\substack{a=1\\(a,q)=1}}^{q} C_2^2(q,a)C_3^2(q,a)C_4^2(q,a)e\left(-\frac{an}{q}\right),$$

$$A(n,q)=\frac{B(n,q)}{\varphi^6(q)},\qquad \mathfrak{S}(n)=\sum_{q=1}^{\infty}A(n,q),$$

$$\mathfrak{J}(n)=\sum_{\substack{m_1+\cdots+m_6=n\\ \left(\frac{P_2}{2}\right)^2\leq m_1,m_2\leq P_2^2,\ \left(\frac{P_3}{2}\right)^3\leq m_3,m_4\leq P_3^3,\\ \left(\frac{P_4}{2}\right)^4\leq m_5,m_6\leq P_4^4}} (m_1m_2)^{-\frac{1}{2}}(m_3m_4)^{-\frac{2}{3}}(m_5m_6)^{-\frac{3}{4}}.$$

Lemma 3.1. *Let $\mathfrak{M}$ be defined as (2.4) with Q_1,Q_2 determined by (2.3). Then for $(1-\eta)N\leq n\leq N$, we have*

$$\int_{\mathfrak{M}} S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2e(-n\alpha)d\alpha$$
$$=\frac{1}{2^2\cdot 3^2\cdot 4^2}\mathfrak{S}(n)\mathfrak{J}(n)+O\left(N^{\frac{7}{6}}L^{-1}\right). \tag{3.1}$$

Here $\mathfrak{S}(n)\gg 1$ for $n\equiv 0 \pmod 2$ and $N^{\frac{7}{6}}\ll\mathfrak{J}(n)\ll N^{\frac{7}{6}}$.

Proof. Note that Q_1,Q_2 are selected as the same values as in [9]. Therefore, the desired conclusion follows from [9, Lemma 2.1]. □

Lemma 3.2. *When $(a,p)=1$, we have*

(i) $|C_j(p,a)|\leq (j-1)p^{\frac{1}{2}}+1$,
(ii) $C_3(p,a)=-1$ *if* $p\equiv 2 \pmod 3$.

Proof. For (i), see [16, Lemma 4.3]. For (ii), note that $p \equiv 2 \pmod 3$ and $(a,p)=1$. Then it follows from [16, Lemma 4.3] that

$$\sum_{x=1}^{p} e\left(\frac{ax^3}{p}\right) = 0.$$

Hence

$$C_3(p,a) = \sum_{x=1}^{p-1} e\left(\frac{ax^3}{p}\right) = -1.$$

□

Lemma 3.3. *We have*

$$\prod_{p\geq 11} (1 + A(n,p)) \geq 0.902346.$$

Proof. For $11 \leq p \leq 199$, we can directly calculate $\min\limits_{1\leq n\leq p}(1 + A(n,p))$ on PC and obtain that

$$1+A(n,11) \geq 0.999503,\ 1+A(n,13) \geq 0.925347, \ldots, 1+A(n,199) \geq 0.999997.$$

Thus

$$\prod_{11\leq p\leq 199} (1 + A(n,p)) \geq 0.916851. \tag{3.2}$$

For $199 < p \leq 10^5$, if $p \equiv 2 \pmod 3$ and $(a,p) = 1$, then we can deduce from Lemma 3.2(i) and (ii) that

$$\begin{aligned} 1 + A(n,p) \geq & 1 - \frac{\sum\limits_{a=1}^{p-1} |C_2^2(p,a)C_4^2(p,a)|}{(p-1)^6} \\ \geq & 1 - \frac{(\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}. \end{aligned} \tag{3.3}$$

If $p \equiv 1 \pmod 3$, then it follows from Lemma 3.2(i) that

$$1 + A(n,p) \geq 1 - \frac{(\sqrt{p}+1)^2(2\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}. \tag{3.4}$$

Combining (3.3)-(3.4), we can deduce from numerical calculation that

$$\begin{aligned} \prod_{199<p\leq 10^5} (1 + A(n,p)) \geq & \prod_{\substack{199<p\leq 10^5 \\ p\equiv 1 \pmod 3}} \left(1 - \frac{(\sqrt{p}+1)^2(2\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}\right) \\ & \times \prod_{\substack{199<p\leq 10^5 \\ p\equiv 2 \pmod 3}} \left(1 - \frac{(\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}\right) \\ \geq & 0.98425 \times 0.999989 \geq 0.984239. \end{aligned} \tag{3.5}$$

For $p > 10^5$, it follows from [9, Section 3, p. 443] that

$$\prod_{p>10^5}(1+A(n,p)) \geq \prod_{p>10^5}\left(1-\frac{1}{(p-1)^2}\right)^{37} \geq 0.99994. \tag{3.6}$$

Now, we can conclude from (3.2) and (3.5)-(3.6) that

$$\prod_{p\geq 11}(1+A(n,p)) \geq 0.916851 \times 0.984239 \times 0.99994 \geq 0.902346. \tag{3.7}$$

□

Lemma 3.4. *Let* $\Xi(N,k) = \{(1-\eta)N \leq n \leq N : n = N - 2^{v_1} - \cdots - 2^{v_k},$ $1 \leq v_1, \ldots, v_k \leq L\}$. *Then for* $k \geq 17$ *and* $N \equiv 0 \pmod 2$, *we have*

$$\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} \mathfrak{S}(n) \geq 1.80321L^k. \tag{3.8}$$

Proof. Since $A(n,q)$ is multiplicative and $A(n,p^j) = 0$ for $j \geq 2$ (see [9, (3.3)]), we have

$$\mathfrak{S}(n) = \prod_{p\geq 2}(1+A(n,p)). \tag{3.9}$$

Set $C = 0.902346$. Then by applying Lemma 3.3, we can get

$$\begin{aligned}\mathfrak{S}(n) &= \prod_{2\leq p\leq 7}(1+A(n,p)) \prod_{11\leq p}(1+A(n,p))\\ &\geq C \prod_{2\leq p\leq 7}(1+A(n,p)).\end{aligned} \tag{3.10}$$

Note that $1 + A(n,2) = 2$ for $n \equiv 0 \pmod 2$. Then for $q = \prod_{3\leq p\leq 7} p = 105$, we obtain

$$\begin{aligned}\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} \mathfrak{S}(n) &\geq 2C \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} \prod_{3\leq p\leq 7}(1+A(n,p))\\ &= 2C \sum_{1\leq j\leq q} \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2\\ n\equiv j \pmod q}} \prod_{3\leq p\leq 7}(1+A(n,p))\\ &= 2C \sum_{1\leq j\leq q} \prod_{3\leq p\leq 7}(1+A(j,p)) \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2\\ n\equiv j \pmod q}} 1.\end{aligned} \tag{3.11}$$

Let S denote the innermost sum in (3.11). Noting that $N \equiv 0 \pmod 2$, we have

$$S = \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2\\ n\equiv j \pmod q}} 1 = \sum_{\substack{1\leq v_1,\ldots,v_k\leq L\\ N-2^{v_1}-\cdots-2^{v_k}\equiv 0 \pmod 2\\ N-2^{v_1}-\cdots-2^{v_k}\equiv j \pmod q}} 1$$

$$= \sum_{\substack{1\le v_1,\dots,v_k\le L\\ 2^{v_1}+\cdots+2^{v_k}\equiv N-j \pmod q}} 1. \tag{3.12}$$

Let $\rho(q)$ denote the smallest positive integer ρ such that $2^\rho \equiv 1 \pmod q$. Thus

$$S = \left(\frac{L}{\rho(q)} + O(1)\right)^k \sum_{\substack{1\le v_1,\dots,v_k\le \rho(q)\\ 2^{v_1}+\cdots+2^{v_k}\equiv N-j \pmod q}} 1$$

$$= \left(\frac{L}{\rho(q)} + O(1)\right)^k \frac{1}{q}\sum_{r=1}^{q} e\left(\frac{r(j-N)}{q}\right)\left(\sum_{1\le v\le\rho(q)} e\left(\frac{r2^v}{q}\right)\right)^k. \tag{3.13}$$

Since $q = 105$, we can get $\rho(q) = 12$. Write $f(r) = \left|\sum_{1\le v\le\rho(q)} e\left(\frac{r2^v}{q}\right)\right|$. With the help of a computer, it is easy to check that

$$\max_{1\le r<q-1} f(r) = f(7) = 6 \quad \text{and} \quad f(q) = \rho(q) = 12. \tag{3.14}$$

Therefore, we can get

$$S \ge \left(\frac{L}{\rho(q)} + O(1)\right)^k \frac{1}{q}\left(\rho^k(q) - (q-1)\left(\max_{1\le r<q-1} f(r)\right)^k\right)$$

$$\ge \frac{L^k}{q}\left(1-(q-1)\left(\frac{\max_{1\le r<q-1} f(r)}{\rho(q)}\right)^k\right) + O(L^{k-1})$$

$$\ge \frac{L^k}{105}\left(1-104\times\left(\frac{1}{2}\right)^{17}\right) + O(L^{k-1}) \ge 0.009516L^k, \tag{3.15}$$

where the bound $k \ge 17$ is used. Combining (3.11) and (3.15), we obtain

$$\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} \mathfrak{S}(n) \ge 2C\times 0.009516L^k \sum_{1\le j\le q}\prod_{3\le p\le 7}(1+A(j,p)). \tag{3.16}$$

On considering the facts $q = 3\times5\times7$ and $A(j,p) = A(j_1,p)$ for $j \equiv j_1 \pmod p$, we have

$$\sum_{1\le j\le q}\prod_{3\le p\le 7}(1+A(j,p))$$

$$= \sum_{1\le j\le q}(1+A(j,3))(1+A(j,5))(1+A(j,7))$$

$$= \sum_{1\le j_1\le 3}\sum_{1\le j_2\le 5}\sum_{1\le j_3\le 7}(1+A(j_1,3))(1+A(j_2,5))(1+A(j_3,7))$$

$$= \prod_{3\le p\le 7}\left(\sum_{1\le j\le p}(1+A(j,p))\right). \tag{3.17}$$

Moreover, from the definition of $A(j,p)$, we have

$$\begin{aligned}
&\sum_{1\le j\le p}(1+A(j,p))\\
&= p+\sum_{1\le j\le p}\frac{1}{(p-1)^6}\sum_{1\le a\le p-1}C_2^2(p,a)C_3^2(p,a)C_4^2(p,a)e\left(-\frac{aj}{p}\right)\\
&= p+\frac{1}{(p-1)^6}\sum_{1\le a\le p-1}C_2^2(p,a)C_3^2(p,a)C_4^2(p,a)\sum_{1\le j\le p}e\left(-\frac{aj}{p}\right)\\
&= p, \qquad (3.18)
\end{aligned}$$

where the bound $\sum_{1\le j\le p} e(-\frac{aj}{p}) = 0$ is used in the last step. Now we can conclude from (3.16)-(3.18) that

$$\begin{aligned}
\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}}\mathfrak{S}(n) &\ge 2C\times 0.009516L^k\prod_{3\le p\le 7}\left(\sum_{1\le j\le p}(1+A(j,p))\right)\\
&= 2C\times 0.009516L^k\prod_{3\le p\le 7}p\ge 1.80321L^k. \qquad (3.19)
\end{aligned}$$

□

We remark that the primary role of taking $q=\prod_{3<p\le 7}p$ is to deduce (3.11) and (3.17). It is easy to verify that taking $q=\prod_{3\le p\le 7}p$ is the optimal choice. Changing the number of primes contained in q will reduce the lower bound in (3.8).

Lemma 3.5. *For $(1-\eta)N\le n\le N$, we have*

$$\mathfrak{J}(n) > (3\pi-180\eta)P_3^2P_4^2. \tag{3.20}$$

Proof. This is [12, Lemma 3.1]. □

Proposition 3.1. *We have*

$$I(k,\mathfrak{M},N)\ge 0.0295049P_3^2P_4^2L^k. \tag{3.21}$$

Proof. Note that $N\equiv 0 \pmod 2$ and $H(\alpha)^k e(-N\alpha) = \sum_{\substack{n\in\Xi(N,k)\\ n\equiv N \pmod 2}} e(-n\alpha)$. Then we can deduce from Lemma 3.1 and Lemmas 3.4-3.5 that

$$I(k,\mathfrak{M},N) = \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}}\int_{\mathfrak{M}} S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2e(-n\alpha)d\alpha$$

$$= \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} \left(\frac{1}{2^2\cdot 3^2\cdot 4^2}\mathfrak{S}(n)\mathfrak{J}(n)+O\left(N^{\frac{7}{6}}L^{-1}\right)\right)$$

$$\geq \frac{3\pi-180\eta}{2^2\cdot 3^2\cdot 4^2}P_3^2P_4^2 \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} \mathfrak{S}(n)+O\left(N^{\frac{7}{6}}L^{-1}\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} 1\right)$$

$$\geq 0.02950495P_3^2P_4^2L^k+O\left(N^{\frac{7}{6}}L^{k-1}\right)$$

$$\geq 0.0295049P_3^2P_4^2L^k, \tag{3.22}$$

where the trivial bound $\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 (\mathrm{mod}\ 2)}} 1 \ll L^k$ is used. □

4. The upper bound for $|I(k,\mathfrak{m},N)|$

In this section, we will give the upper bound for $|I(k,\mathfrak{m},N)|$. For this purpose, we need to introduce a further division of the minor arcs $\mathfrak{m}$. Let

$$\mathcal{E}(u)=\{\alpha\in(0,1]:|H(\alpha)|\geq uL\}. \tag{4.1}$$

Then we have

$$\begin{aligned} I(k,\mathfrak{m},N) &= \int_{\mathfrak{m}} S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2H(\alpha)^k e(-N\alpha)d\alpha \\ &= \left(\int_{\mathfrak{m}\backslash\mathcal{E}(u)}+\int_{\mathfrak{m}\bigcap\mathcal{E}(u)}\right) S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2H(\alpha)^k e(-N\alpha)d\alpha \\ &= I(k,\mathfrak{m}\backslash\mathcal{E}(u),N)+I(k,\mathfrak{m}\cap\mathcal{E}(u),N). \end{aligned} \tag{4.2}$$

The first term in (4.2) will be evaluated by the following Lemma 4.1(i) while the second term will be evaluated by Lemma 4.1(ii) and Lemmas 4.5-4.6.

Lemma 4.1. *We have*

$$\text{(i)}\quad \int_0^1 |S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2|d\alpha \leq 0.58814P_3^2P_4^2,$$

$$\text{(ii)}\quad \int_0^1 |S_2(\alpha)^2S_4(\alpha)^4|d\alpha \ll NL^c,$$

where c is an absolute constant.

Proof. This is [12, Lemma 2.2]. □

Proposition 4.1. *We have*

$$|I(k,\mathfrak{m}\backslash\mathcal{E}(u),N)|\leq 0.58814u^kP_3^2P_4^2L^k. \tag{4.3}$$

Proof. Note that $|H(\alpha)|<uL$ for $\alpha\in\mathfrak{m}\backslash\mathcal{E}(u)$. Then by Lemma 4.1(i), we have

$$|I(k,\mathfrak{m}\backslash\mathcal{E}(u),N)|\leq (uL)^k\int_{\mathfrak{m}\backslash\mathcal{E}(u)}|S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2|d\alpha$$

$$\le (uL)^k \int_0^1 |S_2(\alpha)^2 S_3(\alpha)^2 S_4(\alpha)^2| d\alpha$$

$$\le 0.58814 u^k P_3^2 P_4^2 L^k. \tag{4.4}$$

□

Lemma 4.2. *For $\alpha \in \mathfrak{m}$, we have*

$$S_2(\alpha) \ll N^{\frac{7}{16}+\varepsilon}. \tag{4.5}$$

Proof. This is [9, Lemma 2.4]. □

Lemma 4.3. *Define the multiplicative function $w_k(q)$ by*

$$w_k(p^{ku+v}) = \begin{cases} kp^{-u-\frac{1}{2}}, & \text{when } u \ge 0 \text{ and } v = 1; \\ p^{-u-1}, & \text{when } u \ge 0 \text{ and } 2 \le v \le k \end{cases}$$

and let

$$\mathcal{L}(\gamma) = \sum_{q \le P_3^{\frac{3}{4}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{|\alpha - \frac{a}{q}| \le N} \frac{w_3^2(q) \left| \sum_{\frac{P_4}{2} \le p \le P_4} e(p^4(\alpha+\gamma)) \log p \right|^2}{1 + P_3^3 |\alpha - \frac{a}{q}|} d\alpha.$$

Then we have uniformly for $\gamma \in \mathbb{R}$ that

$$\mathcal{L}(\gamma) \ll N^{-\frac{1}{2}+\varepsilon}.$$

Proof. Write $\alpha = \frac{a}{q} + \lambda$. Then we have

$$\mathcal{L}(\gamma)$$

$$\le \sum_{q \le P_3^{\frac{3}{4}}} \int_{|\lambda| \le N} \frac{w_3^2(q) \sum_{1 \le a \le q} \left| \sum_{\frac{P_4}{2} \le p \le P_4} e(p^4(\frac{a}{q}) + p^4(\lambda+\gamma)) \log p \right|^2}{1 + P_3^3 |\lambda|} d\lambda. \tag{4.6}$$

It is easy to see that

$$\sum_{1 \le a \le q} \left| \sum_{\frac{P_4}{2} \le p \le P_4} e\left(p^4 \left(\frac{a}{q}\right) + p^4(\lambda+\gamma)\right) \log p \right|^2$$

$$= \sum_{\frac{P_4}{2} \le p_1, p_2 \le P_4} (\log p_1)(\log p_2) e((p_1^4 - p_2^4)(\lambda+\gamma)) \sum_{1 \le a \le q} e\left(\frac{(p_1^4 - p_2^4)a}{q}\right)$$

$$\le (\log N)^2 q \sum_{\substack{\frac{P_4}{2} \le p_1, p_2 \le P_4 \\ p_1^4 \equiv p_2^4 \pmod{q} \\ (p_1 p_2, q)=1}} 1 + (\log N)^2 q \sum_{\substack{\frac{P_4}{2} \le p_1, p_2 \le P_4 \\ p_1^4 \equiv p_2^4 \pmod{q} \\ p_1 | q,\, p_2 | q}} 1. \tag{4.7}$$

Note that $q \le P_3^{\frac{3}{4}}$. Thus

$$(\log N)^2 q \sum_{\substack{\frac{P_4}{2} \le p_1, p_2 \le P_4 \\ p_1^4 \equiv p_2^4 \pmod q \\ p_1 | q,\, p_2 | q}} 1 \ll (\log N)^2 q d(q)^2 \ll P_3^{\frac{3}{4}+\varepsilon}. \tag{4.8}$$

Moreover,

$$q \sum_{\substack{\frac{P_4}{2} \le p_1, p_2 \le P_4 \\ p_1^4 \equiv p_2^4 \pmod q \\ (p_1 p_2, q)=1}} 1 \ll \frac{P_4^2}{q} \sum_{\substack{1 \le n_1, n_2 < q \\ n_1^4 \equiv n_2^4 \pmod q \\ (n_1 n_2, q)=1}} 1 \ll P_4^2 \sum_{\substack{1 \le n < q \\ n^4 \equiv 1 \pmod q}} 1. \tag{4.9}$$

Write $q = q_1^{r_1} q_2^{r_2} \cdots q_s^{r_s}$ (prime factorization). Then by [2, Theorem 122] and [4, p. 45], we can get

$$\begin{aligned} \sum_{\substack{1 \le n < q \\ n^4 \equiv 1 \pmod q}} 1 &= \prod_{1 \le i \le s} \sum_{\substack{1 \le n < q_i^{r_i} \\ n^4 \equiv 1 \pmod{q_i^{r_i}}}} 1 \\ &\ll \prod_{1 \le i \le s} (4, \phi(q_i^{r_i})) \ll 4^s \ll d^3(q). \end{aligned} \tag{4.10}$$

Now we can deduce from (4.6)-(4.10) that

$$\begin{aligned} \mathcal{L}(\gamma) &\ll \sum_{q \le P_3^{\frac{3}{4}}} w_3^2(q) \int_{|\lambda| \le N} \frac{P_4^2 d^3(q) \log^2 N}{1 + |\lambda| P_3^3} d\lambda \\ &\ll P_4^{2+\varepsilon} \sum_{q \le P_3^{\frac{3}{4}}} w_3^2(q) d^3(q) \left(\int_{|\lambda| \le \frac{1}{P_3^3}} 1 d\lambda + \int_{\frac{1}{P_3^3} \le |\lambda| \le N} \frac{1}{|\lambda| P_3^3} d\lambda \right) \\ &\ll P_4^{2+\varepsilon} P_3^{-3} (\log N) \sum_{q \le P_3^{\frac{3}{4}}} w_3^2(q) d^3(q) \ll N^{-\frac{1}{2}+\varepsilon}, \end{aligned} \tag{4.11}$$

where we used [18, Lemma 2.1] in the last step. □

Lemma 4.4. *Let*

$$\mathcal{M}(q,a) = \left\{ \alpha : |q\alpha - a| \le P_3^{-\frac{9}{4}} \right\}$$

and let $\mathcal{M}$ *be the union of the intervals* $\mathcal{M}(q,a)$ *for* $1 \le a \le q \le P_3^{\frac{3}{4}}$, $(a,q)=1$. *Suppose that* $G(\alpha)$ *and* $h(\alpha)$ *are integrable functions of period one. Then we have*

$$\int_{\mathfrak{m}} S_3(\alpha) G(\alpha) h(\alpha) d\alpha \ll P_3 \mathcal{J}_0^{\frac{1}{4}} \left(\int_{\mathfrak{m}} |G(\alpha)|^2 d\alpha \right)^{\frac{1}{4}} \mathcal{J}^{\frac{1}{2}} + P_3^{\frac{7}{8}+\varepsilon} \mathcal{J}, \tag{4.12}$$

where

$$\mathcal{J}=\int_{\mathfrak{m}}|G(\alpha)h(\alpha)|d\alpha,\ \ \mathcal{J}_0=\sup_{\beta\in[0,1)}\int_{\mathcal{M}}\frac{w_3^2(q)|h(\alpha+\beta)|^2}{(1+P_3^3|\alpha-\frac{a}{q}|)^2}d\alpha. \tag{4.13}$$

Proof. It follows from [18, Lemma 3.1] with $k=3$. □

Lemma 4.5. *We have*

$$\int_{\mathfrak{m}}|S_2(\alpha)^2S_3(\alpha)^3S_4(\alpha)^2|d\alpha\ll N^{\frac{35}{24}+\varepsilon}. \tag{4.14}$$

Proof. Applying Lemma 4.4 with $G(\alpha)=S_3(-\alpha)S_4(-\alpha)\left|S_2(\alpha)^2S_3(\alpha)\right|$ and $h(\alpha)=S_4(\alpha)$, we have

$$\begin{aligned}\int_{\mathfrak{m}}|S_2(\alpha)^2S_3(\alpha)^3S_4(\alpha)^2|d\alpha&=\int_{\mathfrak{m}}S_3(\alpha)G(\alpha)h(\alpha)d\alpha\\ &\ll P_3\mathcal{J}_0^{\frac14}\left(\int_{\mathfrak{m}}|G(\alpha)|^2d\alpha\right)^{\frac14}\mathcal{J}^{\frac12}+P_3^{\frac78+\varepsilon}\mathcal{J},\end{aligned} \tag{4.15}$$

where

$$\mathcal{J}_0=\sup_{\beta\in[0,1)}\int_{\mathcal{M}}\frac{w_3^2(q)|S_4(\alpha+\beta)|^2}{(1+P_3^3|\alpha-\frac{a}{q}|)^2}d\alpha \tag{4.16}$$

with $\mathcal{M}$ given in Lemma 4.4 and

$$\mathcal{J}=\int_{\mathfrak{m}}|G(\alpha)h(\alpha)|d\alpha=\int_{\mathfrak{m}}|S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2|d\alpha. \tag{4.17}$$

For $\mathcal{J}_0$, by Lemma 4.3, we have

(4.18)

$$\begin{aligned}\mathcal{J}_0&\ll\sup_{\beta\in[0,1)}\sum_{q\le P_3^{\frac34}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{|\alpha-\frac{a}{q}|\le\frac{1}{qP_3^{\frac94}}}\frac{w_3^2(q)\left|\sum_{\frac{P_4}{2}\le p\le P_4}e(p^4(\alpha+\beta))\log p\right|^2}{(1+P_3^3|\alpha-\frac{a}{q}|)^2}d\alpha\\ &\ll\sup_{\beta\in[0,1)}\mathcal{L}(\beta)\ll N^{-\frac12+\varepsilon}.\end{aligned}$$

Applying Cauchy's inequality, Hua's inequality and Lemma 4.2, we obtain

$$\begin{aligned}\int_{\mathfrak{m}}|G(\alpha)|^2d\alpha&\ll\sup_{\alpha\in\mathfrak{m}}|S_2(\alpha)|^3\left(\int_0^1|S_3(\alpha)|^8d\alpha\right)^{\frac12}\left(\int_0^1|S_2(\alpha)|^2|S_4(\alpha)|^4d\alpha\right)^{\frac12}\\ &\ll N^{\frac{21}{16}+\frac56+\frac12+\varepsilon}\ll N^{\frac{127}{48}+\varepsilon},\end{aligned} \tag{4.19}$$

where Lemma 4.1(ii) is used. For $\mathcal{J}$, it follows from Lemma 4.1(i) that

$$\mathcal{J}\ll\int_0^1|S_2(\alpha)^2S_3(\alpha)^2S_4(\alpha)^2|d\alpha\ll N^{\frac76+\varepsilon}. \tag{4.20}$$

Combining (4.15) and (4.18)-(4.20), we have

$$\begin{aligned}\int_{\mathfrak{m}} |S_2(\alpha)^2 S_3(\alpha)^3 S_4(\alpha)^2| d\alpha &\ll N^{\frac{1}{3}-\frac{1}{8}+\frac{127}{192}+\frac{7}{12}+\varepsilon} + N^{\frac{7}{24}+\frac{7}{6}+\varepsilon} \\ &\ll N^{\frac{35}{24}+\varepsilon}. \end{aligned} \tag{4.21}$$

□

Lemma 4.6. *Let $\mathcal{E}(u)$ be defined as* (4.1). *Write meas$(\mathcal{E}(u))$ for the measure of the set $\mathcal{E}(u)$. Then we have*

$$meas(\mathcal{E}(0.833783)) \le N^{-\frac{2}{3}-10^{-10}}. \tag{4.22}$$

Proof. For any $\lambda > 0$ and $\varepsilon > 0$, we can deduce from [13, Section 7] that

$$meas(\mathcal{E}(u)) \le e^{\frac{(\psi(\lambda)-\lambda u+\varepsilon)\log N}{\log 2}}, \tag{4.23}$$

where $\psi(\lambda)$ is defined in [13, Theorem 2]. Following the procedure of [13, Sections 4-6] with $k = 40$, $L = 2^{30}$, $\lambda = 1.1$, $\varepsilon = 10^{-100}$, we obtain

$$\psi(1.1) \le 0.4550627. \tag{4.24}$$

Now combining (4.23)-(4.24), we have

$$meas(\mathcal{E}(0.833783)) \le N^{\frac{\psi(1.1)-1.1\times 0.833783+10^{-100}}{\log 2}} \le N^{-0.666667}.$$

□

Proposition 4.2. *Let $u = 0.833783$. Then we have*

$$|I(k, \mathfrak{m} \cap \mathcal{E}(u), N)| \ll N^{\frac{7}{6}-\varepsilon} \ll P_3^2 P_4^2 L^{k-1}. \tag{4.25}$$

Proof. By Hölder's inequality, Hua's inequality and Lemmas 4.5-4.6, we have

$$\begin{aligned} |I(k, \mathfrak{m} \cap \mathcal{E}(u), N)| &\ll L^k \left(\int_0^1 |S_2(\alpha)^2 S_4(\alpha)^4| d\alpha\right)^{\frac{1}{6}} \left(\int_0^1 |S_2^4(\alpha)| d\alpha\right)^{\frac{1}{12}} \\ &\quad\times \left(\int_{\mathfrak{m}} |S_2(\alpha)^2 S_3(\alpha)^3 S_4(\alpha)^2| d\alpha\right)^{\frac{2}{3}} \left(\int_{\mathcal{E}(0.833783)} 1 d\alpha\right)^{\frac{1}{12}} \\ &\ll N^{\frac{1}{6}+\frac{1}{12}+\frac{35}{36}-\frac{1}{18}-10^{-12}+\varepsilon} \ll N^{\frac{7}{6}-\varepsilon}, \end{aligned} \tag{4.26}$$

where Lemma 4.1(ii) and the trivial bound $H(\alpha) \ll L$ are used.

Now combining (4.2) and Propositions 4.1-4.2 with $u = 0.833783$, we have

$$\begin{aligned} |I(k, \mathfrak{m}, N)| &\le |I(k, \mathfrak{m}\backslash\mathcal{E}(u), N)| + |I(k, \mathfrak{m} \cap \mathcal{E}(u), N)| \\ &\le 0.58814 u^k P_3^2 P_4^2 L^k + O(P_3^2 P_4^2 L^{k-1}). \end{aligned} \tag{4.27}$$

□

5. Proof of Theorem 1

On recalling notations defined in Section 2, we have

$$\begin{aligned}\mathcal{R}(k,N) &= \int_0^1 S_2(\alpha)^2 S_3(\alpha)^2 S_4(\alpha)^2 H(\alpha)^k e(-N\alpha)d\alpha \\ &= \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) S_2(\alpha)^2 S_3(\alpha)^2 S_4(\alpha)^2 H(\alpha)^k e(-N\alpha)d\alpha \\ &\geq I(k,\mathfrak{M},N) - |I(k,\mathfrak{m},N)|. \end{aligned} \tag{5.1}$$

When $k \geq 17$ and $u = 0.833783$, we can deduce from (4.27) and Proposition 3.1 that

$$\begin{aligned}\mathcal{R}(k,N) &\geq (0.0295049 - 0.58814 \times 0.833783^{17}) P_3^2 P_4^2 L^k + O(P_3^2 P_4^2 L^{k-1}) \\ &> 0.002 P_3^2 P_4^2 L^k.\end{aligned}$$

Now the proof of Theorem 1 is completed.

Acknowledgments. We thank the referees for their time and comments. The author would like to express the most sincere gratitude to Professor Yingchun Cai for his valuable advice and constant encouragement.

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