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HISTORIC BEHAVIOR FOR FLOWS WITH THE GLUING ORBIT PROPERTY

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Abstract. We consider the set of points with historic behavior (which is also called the irregular set) for continuous flows and suspension flows. In this paper under the hypothesis that $(X_t)_t$ is a continuous flow on a d-dimensional Riemaniann closed manifold M ($d \geq 2$) with gluing orbit property, we prove that the set of points with historic behavior in a compact and invariant subset Δ of M is either empty or is a Baire residual subset on Δ. We also prove that the set of points with historic behavior of a suspension flows over a homeomorphism satisfyng the gluing orbit property is either empty or Baire residual and carries full topological entropy.

1. Introduction

Our main goal here is to study the set of points with historic behavior for continuous flows satisfying gluing orbit property and suspension flows over homeomorphisms satisfying the gluing orbit property. In the sense of historic behavior, Ruelle in [16] says that a point x has *historic behavior* if the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ does not converge in the weak* topology. More precisely, let M be a compact metric space, $f : M \to M$ a continuous map and $\varphi : M \to \mathbb{R}^d$ ($d \geq 1$) a continuous observable, a point $x \in M$ has *historic behavior* with respect to φ if the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))$$

does not converge. The set of points with historic behavior with respect to φ is denoted by I_φ. Let us recall the definition of points with historic behavior for continuous time. Let $(X_t)_t$ be a continuous flow on compact metric space M and $\varphi : M \to \mathbb{R}^d$ ($d \geq 1$) a continuous observable. We say that a point $x \in M$

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has *historic behavior* with respect to φ if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi \circ X_s(x)\, ds$$

does not exist. We will keep denoting I_φ for set of points with historic behavior of $(X_t)_t$ with respect to φ. It is also known as *irregular set* of $(X_t)_t$ with respect to φ. We can similarly define the set of points with historic behavior for quotients of Birkhoff averages with applications to the case of suspension flows, see [3]. The notion of historic behavior allows some applications, including rotation sets studied in [9, 12, 13].

Note that by Birkhoff's Ergodic Theorem I_φ has zero measure with respect to any invariant probability measure. If the observable φ is cohomologous to a constant (i.e., there exist a bounded function ϕ and a constant c such that $\varphi = \phi - \phi \circ f + c$), then the set I_φ is empty. On the other hand, Takens in [17] claimed that the set of points with historic behaviour is not negligible, from the topological viewpoint. This fact was first observed by Pesin and Pitskel in [15], they show that these points carry full topological pressure for full shift. Moreover, Barreira and Schmeling in [4] show that from the point of view of dimension theory it as large as the whole space.

We say that $B \subset M$ is Baire residual if it contains a countable intersection of open and dense subsets of M. Barreira, Li and Valls prove in [2] that for continuous map with specification property and continuous observable the set of points with historic behavior is either empty or is Baire residual. Thompson proves in [18] under the same assumptions that irregular set is either empty or carries full topological pressure. Moreover, the author and Varandas in [12] showed that for continuous maps with gluing orbit property and continuous observables the set of points with historic behavior is either empty or it is a Baire residual and it carries full topological pressure. More recently, Araujo and Pinheiro in [1] proved that the set of point with wild historic behavior (a notion more general of points with historic behavior) for wide classes of dynamical models is a topologically generic subset. In [10] it is studied historic behavior for geometric Lorenz flows. A recently published article by Carvalho and Varandas ([8]) establishs a sufficient condition for a continuous map and flow on a compact metric space to have a Baire residual set of points with historic behavior. Many more results about the set of points with historic behavior are known. But, concerning to the continuous time, the set of points with historic behavior is less studied.

We will recall the definiton of specification and gluing orbit property. First, in discrete time setting, let $f : M \to M$ be a continuous map on a compact metric space M. We say that f satisfies the *specification property* if for any $\epsilon > 0$ there exists an integer $m = m(\epsilon) \geq 1$ so that for any points $x_1, x_2, \ldots, x_k \in M$ and for any positive integers $n_1, \ldots, n_k$ and $0 \leq p_1, \ldots, p_{k-1}$ with $p_i \geq m(\epsilon)$ there exists a point $y \in X$ such that $d(f^j(y), f^j(x_1)) \leq \epsilon$ for every $0 \leq j \leq n_1$

and

$$d(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(y), f^j(x_i)) \leq \epsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

The definition of gluing orbit property, introduced in [7], is as follows. Let $f : M \to M$ be a continuous map on a compact metric space M. We say that f satisfies the *gluing orbit property* if for any $\epsilon > 0$ there exists an integer $m = m(\epsilon) \geq 1$ so that for any points $x_1, x_2, \dots, x_k \in M$ and any positive integers $n_1, \dots, n_k$ there are $0 \leq p_1, \dots, p_{k-1} \leq m(\epsilon)$ and a point $y \in M$ so that $d(f^j(y), f^j(x_1)) \leq \epsilon$ for every $0 \leq j \leq n_1$ and

$$d(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(y), f^j(x_i)) \leq \epsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

Now we recall the gluing orbit property for flows, introduced in [7]. Let $(X_t)_t$ be a continuous flow on a compact manifold M and $\Delta \subset M$ be a $(X_t)_t$-invariant subset. We say that $(X_t)_{t \geq 0}$ satisfies the *gluing orbit property* if for any $\epsilon > 0$ there exists $K = K(\epsilon) > 0$ such that for any points $x_1, x_2, \dots, x_k \in M$ and times $t_1, \dots, t_k \geq 0$ there are $0 \leq p_1, \dots, p_{k-1} \leq K(\epsilon)$ and a point $y \in M$ so that

$$d(X_t(y)), X_t(x_1)) < \epsilon \ \text{ for all } t \in [0, t_1]$$

and

$$d(X_{t+\sum_{j=1}^{i-1} t_j + p_j}(y), X_t(x_i)) < \epsilon \ \text{ for all } t \in [0, t_i]$$

for every $2 \leq i \leq k$.

Note that the gluing orbit property is clearly a topological invariant and is weaker than specification. Other evidence is that under the gluing orbit property the dynamical is not necessarily topologically mixing but it is transitive. A continuous flow with denseness of periodic orbits and the shadowing property satisfies a gluing orbit property, see [5, 6] for more details and examples.

In our first result here, we consider a continuous flow with gluing orbit property on some compact and invariant subset. Under these circumstances we prove that the set of points with historic behavior is large from the topological viewpoint, if it is not empty. The second result is about suspension flow over homeomorphisms with gluing orbit property. We prove that if the set of points with historic behavior is non-empty, then it is large from the topological viewpoint.

This paper is organized as follows. In Section 2 we state our main results. The Section 3 is the preliminary, where we recall the definitions that appear in the results and we make some comments. The proofs of the main results occupy Section 4.

2. Statement of the main results

Our first main result extend Theorem D of [12] for continuous time dynamics. We describes the set of points with historic behavior of a continuous flow with gluing orbit property. More precisely:

Theorem 2.1. *Let M be a d-dimensional Riemaniann closed manifold with $d \geq 2$ and $(X_t)_t$ be a continuous flow on M with the gluing orbit property on compact and invariant subset Δ. If $\varphi : M \to \mathbb{R}^d$ $(d \geq 1)$ is a continuous observable, then $I_\varphi \cap \Delta$ is either empty or it is a Baire residual subset of Δ.*

The conclusion of Theorem 2.1 can be written as follows: either there is $v \in \mathbb{R}^d$ so that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(X_t(x))\, dt = v$$

for all $x \in \Delta$, or the set points $x \in \Delta$ so that

$$(\frac{1}{T} \int_0^T \varphi(X_t(x))\, dt)_{T\geq 1}$$

accumulates in a non-trivial connected subset of $\mathbb{R}^d$ is Baire residual on Δ (the expresion non-trivial means that the set is not a singleton).

Let M be a d-dimensional Riemaniann closed manifold $(d \geq 2)$. For $L > 0$, denote $\mathfrak{X}^0(M)$ the set of continuous vector fields $X : M \to TM$ and $\mathfrak{X}^{0,1}_L(M)$ the set of Lipschitz continuous vector fields $X : M \to TM$ with Lipschitz constant $\leq L$. We endow $\mathfrak{X}^0(M)$ and $\mathfrak{X}^{0,1}_L(M)$ with the C^0-topology, i.e., given $X, Y \in \mathfrak{X}^{0,1}_L(M)$, X is ϵ-close Y if $\max_{x\in M} \|X(x) - Y(x))\| < \epsilon$. This is a Baire space (cf. [5]). Let $(X_t)_t$ be a continuous flow on M. We say that $p \in M$ is a *non-wandering* point for $(X_t)_t$ if for any neighbourhood U of p and any $\eta > 0$, there exists $T > \eta$ such that $X_T(U) \cap U \neq \emptyset$. Let us denote by $\Omega(X)$ the set of non-wandering points of $(X_t)_t$. Given $x, y \in M$ we say that $x \approx y$ if for any $\delta > 0$ and $T > 1$ there exists a (δ, T)-pseudo-orbit $[x_i, t_i]_{i=1,\dots,k}$ such that $x_1 = x$ and $X_{t_k}(x_k) = y$. The relation $\approx$ is an equivalence relation. Each of the equivalence classes of $\approx$ is called a chain recurrence class. We observe that chain recurrence classes are disjoint, compact and invariant subsets of M.

Now we turn our attention to suspension flows over continuous maps defined on a compact metric space satisfying the gluing orbit property. Theorem F of [7] shows that if the roof function satisfies a bounded distortion property, then the suspension flow satisfies the gluing orbit property, so applying Theorem 2.1 we have that the set of points with historic behavior is either empty or is a Baire residual. In the case of suspension flow we denote by $\widehat{I}_\Phi$ the set of points with historic behavior of a potential Φ. The next result was inspired by Thompson [18] which proves a similar result under the assumption of specification property.

Theorem 2.2. *Let M be a compact metric space and $f : M \to M$ be a continuous map on M with gluing orbit property. Assume that $(X_t)_t$ is a suspension flow over f, that $r : M \to (0, \infty[$ is a continuous roof function bounded away from zero and $\Phi : M_r \to \mathbb{R}^d$ is a continuous observable. If the set of points with historic behavior of Φ is non-empty, then it is a Baire residual subset of M_r and carries full topological entropy on M_r. In other words, $h_{\widehat{I}_\Phi}((X_t)_t) = h_{top}((X_t)_t)$.*

3. Preliminaries

3.1. Suspension flows

Let M be a compact metric space, $f : M \to M$ be a continuous map and $r : M \longrightarrow [0, \infty)$ be a continuous roof function bounded away from zero. We define the *suspension flow* $(X_t)_t$ over f by

$$X_t(x, s) = (x, s + t),$$

whenever the expression is well defined, acting on

$$M_r = \{(x, t) \in M \times \mathbb{R}_+ : 0 \le t \le r(x)\}/ \approx,$$

where $\approx$ is the equivalence relation given by $(x, r(x)) \approx (f(x), 0)$ for all $x \in M$. Since r is continuous, $(X_t)_t$ defines a flow on M_r which is continuous with respect to the Bowen-Walters distance (see [7]). In local coordinates, $(X_t)_t$ coincides with the flow along the vertical direction. More precisely,

$$X_t(x, s) = \Big(f^k(x), s + t - \sum_{j=0}^{k-1} r(f^j(x))\Big),$$

where $k = k(x, s, t)$ is determined by $\sum_{j=0}^{k-1} r(f^j(x)) \le s + t < \sum_{j=0}^{k} r(f^j(x))$.

Given a continuous map $\Phi : M_r \to \mathbb{R}$, we associate the function $\varphi : M \to \mathbb{R}$ defined by $\varphi(x) = \int_0^{r(x)} \Phi(x, s)\, ds$. Note that φ is continuous. For μ-invariant measure in M we define the measure μ_r by

$$\int_{M_r} \Phi \, d\,\mu_r = \int_M \varphi \, d\,\mu \;/ \int r \, d\,\mu.$$

3.2. Pressure and entropy

First, we recall the definition of topological pressure topological entropy for discrete time setting. Let (M, d) be a compact metric space, $f : M \to M$ a continuous map and $\psi : M \to \mathbb{R}$ a continuous potential. Given $n \in \mathbb{N}$ and $\epsilon > 0$ we say that $E \subset M$ is (n, ϵ)-separated if given $x, y \in E$, there exist $j \in \{0, \ldots, n-1\}$ such that $d(f^j(x), f^j(y)) \ge \epsilon$. In the other words, if $x \in E$, then $B(x, n, \epsilon)$ does not contain any other element of E, where $B(x, n, \epsilon) := \{y \in M : d(f^j(x), f^j(y)) < \epsilon \text{ for all } 0 \le j \le n-1\}$ is said dynamic ball from center x, length n and radius $\epsilon > 0$. We define the *topological pressure* of f with respect to ψ by

$$P(f, \psi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_E \sum_{x \in E} e^{S_n \psi(x)},$$

where $S_n\psi(x) := \sum_{j=0}^{n-1} \psi(f^j(x))$ and the supremum is taken over every (n, ϵ)-separated sets E contained in M. In the case that $\psi \equiv 0$, we obtain the *topological entropy* of f, denoted by $h_{top}(f)$.

We present the definitions of entropy for flows. Let $(X_t)_t$ be a flow on a compact metric space M. Given $x \in M$, $T > 0$ and $\epsilon > 0$, we call dynamic ball of center x, length T and radius $\epsilon > 0$ to the set $B(x,T,\epsilon) = \{y \in M : d(X_t(x), X_t(y)) < \epsilon \text{ for all } 0 \leq t \leq T\}$. We say that $E \subset M$ is a set (T,ϵ)-separated if the dynamic ball $B(x,T,\epsilon)$ of each $x \in E$ does not contain any other element of E. If $s(T,\epsilon)$ denotes the maximal cardinality of a (T,ϵ)-separated $E \subset M$, then define the *topological entropy of the flow* $(X_t)_t$ by

$$h_{top}((X_t)_t) = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log s(T,\epsilon).$$

Now, we define relative pressure and relative entropy in the discrete time setting: Let M be a compact metric space, $f : M \to M$ a continuous map and $Z \subset M$ be an f-invariant Borel set. Given $s \in \mathbb{R}$ and $\psi : M \to \mathbb{R}$ a continuous potential. Define

$$Q(Z,\psi,s,\epsilon,N,\Gamma) = \sum_{B_{n_i}(x_i,\epsilon)\in\Gamma} e^{-s\,n_i + S_{n_i}\psi(B_{n_i}(x_i,\epsilon))}$$

and

$$M(Z,\psi,s,\epsilon,N) = \inf_{\Gamma} \left\{Q(Z,\psi,s,\epsilon,N,\Gamma)\right\},$$

where $S_{n_i}\psi(B_{n_i}(x_i,\epsilon)) := \sup_{x\in B_{n_i}(x_i,\epsilon)} \sum_{k=0}^{n_i-1} \psi(f^k(x))$ where the infimum is taken over all countable collections $\Gamma = \{B_{n_i}(x_i,\epsilon)\}_i$ that cover Z and so that $n_i \geq N$. Since the function $M(Z,\psi,s,\epsilon,N)$ is non-decreasing in N the limit $m(Z,\psi,s,\epsilon) = \lim_{N\to\infty} M(Z,\psi,s,\epsilon,N)$ does exist. Let

$$P_Z(f,\psi,\epsilon) = \inf\{s \in \mathbb{R} \colon m(Z,\psi,s,\epsilon) = 0\} = \sup\{s \in \mathbb{R} \colon m(Z,\psi,s,\epsilon) = \infty\}.$$

The existence of $P_Z(f,\psi,\epsilon)$ follows by the Carathéodory structure [14]. The *topological pressure* on Z with respect to ψ (and f) is defined by

$$P_Z(f,\psi) = \lim_{\epsilon\to 0} P_Z(f,\psi,\epsilon).$$

We set $h_Z(f,\epsilon) = P_Z(f,0,\epsilon)$ for every $\epsilon > 0$ and define the *topological entropy* on Z with respect f by $h_Z(f) = P_Z(f,0)$.

We define relative entropy in the continuous time setting: Let $Z \subset M$ be an arbitrary Borel set. Let $(X_t)_t$ be a continuous flow on M. Consider the finite collections $\Gamma = \{B_{t_i}(x_i,\epsilon)\}_i$, where $t_i > 0$, $x_i \in M$ and $B_{t_i}(x_i,\epsilon) = \{x \in M : d(X_s(x), X_s(y)) < \epsilon \text{ for all } s \in [0,t_i)\}$. Given $T > 0$, for $s \in R$ and $\psi \in C^0(M,\mathbb{R})$ define:

$$Q(Z,\psi,s,\Gamma) = \sum_{B_{t_i}(x_i,\epsilon)\in\Gamma} e^{-s\,t_i} \text{ and } M(Z,s,\epsilon,T) = \inf_{\Gamma}\ Q(Z,s,\Gamma),$$

where the infimum is taken over all countable collections of the form $\{B_{t_i}(x_i,\epsilon)\}_i$ with $x_i \in M$ such that Γ covers Z and $t_i \geq T$ for all i. Define

$$m(Z,s,\epsilon) = \lim_{T\to\infty} M(Z,s,\epsilon,T).$$

The existence of the limit follows by Caratheodory structure [14]. We can show that

$$h(Z,\epsilon) = \inf\{s : m(Z,s,\epsilon) = 0\} = \sup\{s : m(Z,s,\epsilon) = \infty\}.$$

Define the *topological entropy* on Z with respect to $(X_t)_t$ by

$$h_Z((X_t)_t) = \lim_{\epsilon\to 0} h(Z,\epsilon).$$

4. Proof of the theorems

In this section we prove the main results.

4.1. Proof of Theorem 2.1

The proof follows a strategy of Li and Wu in [11]. Let $(X_t)_t$ be a continuous flow on M with the gluing orbit property in a compact and invariant subset Δ of M and let $\varphi : M \to \mathbb{R}^d$ be a continuous function. For $d \geq 2$, we define

$$\mathcal{L}_\varphi = \{\vec{v} \in \mathbb{R}^d : A_\varphi(\vec{v}) \neq \emptyset\},$$

where

$$A_\varphi(\vec{v}) := \{x \in \Delta : \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi \circ X_s(x)\, ds = \vec{v}\}.$$

Recall that the set of points with historic behavior of $(X_t)_t$ with respect to φ,

$$I_\varphi = \left(x \in M : \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi\circ X_s(x) ds \, \text{does not exist} \right).$$

If $I_\varphi \cap \Delta = \emptyset$ we are done. Thus, in what follows we suppose that $I_\varphi \cap \Delta$ is non-empty.

Let D be a countable and dense subset on Δ. Given $\epsilon > 0$ fixed let $K(\epsilon)$ be given by the gluing orbit property on Δ. For $w \in \mathcal{L}_\varphi$, $\delta > 0$ and $n \in \mathbb{N}$ let

$$P(w,\delta,t) = \left\{ x \in \ \Delta : \left\| \frac{1}{t}\int_0^t \varphi(X_r(x))\, dr - w \right\| < \delta \right\}.$$

Clearly, for $w \in \mathcal{L}_\varphi$ and any $\delta > 0$ the set $P(w,\delta,t)$ is not empty for every sufficiently large t.

Note that as $I_\varphi \cap \Delta \neq \emptyset$, then there are $u, v \in \mathcal{L}_\varphi$ distinct. Let $\{\delta_k\}_{k\geq 1} \searrow 0$ be a sequence of positive real numbers and let $\{t_k\}_{k\geq 1} \nearrow \infty$ be a sequence of integers with $t_k \gg K_k$, where $K_k := K(\epsilon/2^k)$, (meaning $\lim_{k\to\infty} \frac{K_k}{t_k} = 0$) so that $P(u, \delta_{2j-1}, t_{2j-1}) \neq \emptyset$ and $P(v, \delta_{2j}, t_{2j}) \neq \emptyset$ for all $j \geq 1$.

Given $q \in D$ and $k \geq 1$, let $W_0 = \{q\}$. For $j \geq 1$ let W_{2j-1} be a maximal $(t_{2j-1}, 8\epsilon)$-separated subset of $P(u, \delta_{2j-1}, t_{2j-1})$ and let W_{2j} be a maximal $(t_{2j}, 8\epsilon)$-separated subset of $P(v, \delta_{2j}, t_{2j})$. Choose a sequence of integers $\{N_k\}_{k\geq 1}$ so that

$$\text{(1)} \quad \lim_{k\to\infty} \frac{t_k + K_k}{N_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{N_1(t_1+K_1) + \cdots + N_k(t_k+K_k)}{N_{k+1}} = 0.$$

We need the following auxiliary construction. For $k \geq 1$, the gluing orbit property ensures that for every $\underline{x_k} := (x_1^k, \ldots, x_{N_k}^k) \in W_k^{N_k}$ (where $W_k^{N_k}$ means to say that it is the cartesian product of N_k copies of W_k) there exists a point $y = y(\underline{x_k}) \in M$ and transition time functions

$$p_j^k : W_k^{N_k} \times \mathbb{R}_+ \to \mathbb{R}_+,\ j = 1, 2, \ldots, N_k - 1$$

bounded by K_k so that

$$d(X_{e_j+t}(y), X_t(x_j^k)) < \frac{\epsilon}{2^k} \tag{2}$$

for every $t \in [0, t_k]$ and $j = 1, 2, \ldots, N_k - 1$, where

$$e_j = \begin{cases} 0, & \text{if } j = 1, \\ (j-1)\, t_k + \sum_{r=1}^{j-1} p_r^k, & \text{if } j = 2, \ldots, N_k. \end{cases}$$

For $k \geq 1$ and $j \in \{1, 2, \ldots, N_k - 2\}$ we have that $p_j^k = p_j^k(x_1^k, x_2^k, \ldots, x_{N_k}^k, \epsilon)$ is a function that describes the time lag that the orbit of $y = y(\underline{x_k})$ takes to jump from an $\frac{\epsilon}{2^k}$-neighborhood of $X_{t_k}(x_j^k)$ to an $\frac{\epsilon}{2^k}$-neighborhood of x_{j+1}^k, and it is bounded above by K_k.

We order the family $\{W_k\}_{k\geq 1}$ lexicographically: $W_k \prec W_s$ if and only if $k \leq s$. We proceed to make a recursive construction of points in a neighborhood of q that shadow N_k points in the family W_k successively with bounded time lags in between. More precisely, we construct a family $\{L_k(q)\}_{k\geq 0}$ of sets (guaranteed by the gluing orbit property) contained in a neighborhood of q and a family of positive numbers $\{l_k\}_{k\geq 0}$ (also depending on q) corresponding to the time during the shadowing process.

- Let $L_0(q) = \{q\}$ and $l_0 = N_0 = t_0 = 0$;
- Let $L_1(q) = \{z = z(q, y) \in M : \underline{x_1} \in W_1^{N_1}\}$ and $l_1 = p_0^1 + s_1$, where the point $y = y(\underline{x_1})$ is as in (2), $0 \leq p_0^1 \leq K_1 = K(\frac{\epsilon}{2})$ is the time lag that the orbit of z takes to jump from an $\frac{\epsilon}{2}$-neighborhood of q to an $\frac{\epsilon}{2}$-neighborhood of x_1^1 given by the gluing orbit property, $s_1 = N_1 t_1 + \sum_{r=1}^{N_1-1} p_r^1$ and $z(q, y(\underline{x_1}))$ satisfies
$$d(z, q) < \frac{\epsilon}{2} \ \text{ and } d(X_{p_0^1+t}(z), X_t(y(\underline{x_1}))) < \frac{\epsilon}{2}$$
for every $t \in [0, s_1]$. Such a point exists due to the gluing orbit property.
- Let $L_k(q) = \{z = z(z_0, y(\underline{x_k})) \in M\colon \underline{x_k} \in W_k^{N_k}$ and $z_0 \in L_{k-1}\}$ and $l_k = l_{k-1} + p_0^k + s_k$, where the point $y = y(\underline{x_k})$ is defined as in (2), p_0^k is the time lag that the orbit of z takes to jump from an $\frac{\epsilon}{2^{k-1}}$- neighborhood of $X_{t_{k-1}}(x_{N_{k-1}}^{k-1})$ to an $\frac{\epsilon}{2^k}$-neighborhood of x_1^k, it is bounded above by K_k, $s_k = N_k t_k + \sum_{r=1}^{N_k-1} p_r^k$, and z satisfies
$$d(X_t(z), X_t(z_0)) < \frac{\epsilon}{2^k},\ \forall\ t \in [0, l_{k-1}]$$
and
$$d(X_{l_{k-1}+p_0^k+t}(z), X_t(y(\underline{x_k}))) < \frac{\epsilon}{2^k}$$
for all $t \in [0, s_k]$.

By construction, for every $k \geq 1$, we have:

$$l_k = \sum_{s=1}^{k} N_s t_s + \sum_{s=1}^{k} \sum_{r=0}^{N_s-1} p_r^s. \tag{3}$$

Remark 4.1. Note that l_k and s_k are functions (since these depend on p_j^k), the p_j^k are bounded by K_k and by definition of N_k (cf. (1)) and definition of l_k (cf. (3)) one has that $\frac{l_k}{N_{k+1}} \leq \frac{\sum_{s=1}^{k} N_s(t_s+K_s)}{N_{k+1}}$ tends to zero as $k \to \infty$.

For every $k \geq 0$, $q \in D$ and $\epsilon > 0$ define

$$\mathcal{R}_k(q,\epsilon) = \bigcup_{z \in L_k(q)} B_{l_k}\Big(z, \frac{\epsilon}{2^k}\Big), \tag{4}$$

$$\mathcal{R}(q,\epsilon) = \bigcap_{k=0}^{\infty} \mathcal{R}_k(q,\epsilon)$$

and

$$\mathcal{R}(\epsilon) = \bigcup_{q \in D} \mathcal{R}_k(q,\epsilon),$$

where $B_{l_k}(z,\delta)$ is the set of points $y \in \Delta$ so that $d(X_\alpha(z), X_\alpha(y)) < \delta$ for all $\alpha \in [0, l_{k-1}]$ and $d(X_\beta(z), X_\beta(y)) \leq \delta$ for all $\beta \in [l_{k-1}, l_k]$. By construction $\mathcal{R}_{k+1}(q,\epsilon) \subset \mathcal{R}_k(q,\epsilon)$ for every $k \geq 0$, $q \in D$ and $\epsilon > 0$. Finally, we define the set

$$\mathcal{R}' = \bigcup_{j=1}^{\infty} \mathcal{R}(\frac{1}{j}) = \bigcup_{j=1}^{\infty} \bigcup_{q \in D} \bigcap_{k=0}^{\infty} \bigcup_{z \in L_k(q)} B_{l_k}\Big(z, \frac{1}{j2^k}\Big).$$

Now we define the following auxiliar set

$$\mathcal{R} = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{q \in D} \bigcup_{z \in L_k(q)} B_{l_k}\Big(z, \frac{1}{j2^k}\Big).$$

Note that $\mathcal{R} \subset \mathcal{R}'$. The following lemma ensures that $\mathcal{R}$ is a Baire generic subset of Δ.

Lemma 4.2. *$\mathcal{R}$ is a G_δ-set and it is dense in Δ.*

Proof. First we prove denseness. It is enough to show that $\mathcal{R} \cap B(x,r) \neq \emptyset$ for every $x \in \Delta$ and $r > 0$. In fact, given $x \in \Delta$ and $r > 0$, there exist $j \in \mathbb{N}$ with $2/j < r$ and $q \in D$ such that $d(x,q) < 1/j$. Choose a point $y \in \mathcal{R}(q, \frac{1}{j})$, it holds that $d(q,y) < \frac{1}{j}$ because $\mathcal{R}(q,\frac{1}{j}) \subset B(q,\frac{1}{j})$. Therefore, $d(x,y) \leq d(x,q) + d(q,y) < 2/j < r$. This ensures that $\mathcal{R} \cap B(x,r) \neq \emptyset$.

Now, we prove that $\mathcal{R}$ is a G_δ-set. It suffices to show that $R(q,\epsilon)$ is a G_δ-set for any $\epsilon > 0$ and any $q \in D$. Thus, fix $\epsilon > 0$ and $q \in D$. For any $k \geq 1$, consider the open set

$$G_k(q,\epsilon) := \bigcup_{z\in L_k(q)} \widetilde{B}\Big(z, \frac{\epsilon}{2^k}\Big),$$

where $\widetilde{B}(z,\frac{\epsilon}{2^k})$ is the set of points $y\in\Delta$ so that $d(X_\alpha(z),X_\alpha(y))<\delta$ for all $\alpha\in[0,l_k]$. It is clear that $G_k(q,\epsilon)\subset\mathcal{R}_k(q,\epsilon)$ for any $k\geq 1$. We claim that $\mathcal{R}_{k+1}(q,\epsilon)\subset G_k(q,\epsilon)$ for any $k\geq 1$, in which case we conclude that

$$\bigcup_{q\in D}\bigcup_{j=1}^{\infty} R_k(q,\frac{1}{j}) = \bigcup_{q\in D}\bigcup_{j=1}^{\infty} G_k(q,\frac{1}{j}).$$

Consequently we obtain that $\mathcal{R}$ is a countable intersection of open sets, hence it is a G_δ-set.

Now we proceed to prove the previous claim. Given $y\in\mathcal{R}_{k+1}(q,\epsilon)$, there exists $z\in L_{k+1}(q)$ such that $y\in B_{l_{k+1}}(z,\frac{\epsilon}{2^{k+1}})$. By definition of $L_{k+1}(q)$, there exists $z_0\in L_k(q)$ such that $d(X_t(z),X_t(z_0))<\frac{\epsilon}{2^{k+1}}$ for all $t\in[0,l_k]$. Thus,

$$\begin{aligned} d(X_t(z_0),X_t(y)) &\leq d(X_t(z),X_t(z_0)) + d(X_t(z),X_t(y)) \\ &< \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2^k} \end{aligned}$$

for all $t\in[0,l_k]$. This proves that $y\in G_k(q,\epsilon)$, proves the claim and completes the proof of the lemma. □

We must show that $\mathcal{R}\subset I_\varphi$. It is sufficient to prove that $\mathcal{R}(\epsilon,q)\subset I_\varphi$ for any $\epsilon>0$ and any $q\in D$. For any $\eta>0$ write

$$\mathrm{var}(\varphi,\eta) := \sup\{\|\varphi(x)-\varphi(y)\| : d(x;y)<\eta\}$$

which, by compactness, satisfies $\mathrm{var}(\varphi,\eta)\to 0$ as $\eta\to 0$. We need one more lemma:

Lemma 4.3. *Let $u,v\in\mathcal{L}_\varphi$ distinct. For every $k\geq 1$ the following following sentences hold*:

(i) *If k is odd and $y=y(\underline{x_k})$, then*

$$\Big\|\int_0^{s_k}\varphi(X_r(y))\,dr - s_k u\Big\| \leq N_k t_k\Big(\mathrm{var}(\varphi,\frac{\epsilon}{2^k})+\delta_k\Big) + 2(N_k-1)K_k\|\varphi\|_\infty;$$

(ii) *If k is even, then*

$$\Big\|\int_0^{s_k}\varphi(X_r(y))\,dr - s_k v\Big\| \leq N_k t_k\Big(\mathrm{var}(\varphi,\frac{\epsilon}{2^k})+\delta_k\Big) + 2(N_k-1)K_k\|\varphi\|_\infty.$$

Proof. Let $k\geq 1$ be fixed and assume that it is odd (the case when it is even is completely analogous). By construction of W_k and relation (2) there exists $(x_1^k,\ldots,x_{N_k}^k)\in W_k^{N_k}$ so that

$$d(X_{\mathrm{e}_j+t}(y),X_t(x_j^k)) < \frac{\epsilon}{2^k},$$

and

$$\begin{aligned}\Big\| \int_0^{t_k} \varphi(X_{e_j+r}(y))\,dr - t_k u \Big\| &\le \Big\| \int_0^{t_k} \varphi(X_{e_j+r}(y))\,dr - \int_0^{t_k} \varphi(X_r(x_j^k))\,dr \Big\| \\ &\quad + \Big\| \int_0^{t_k} \varphi(X_r(x_j^k))\,dr - t_k u \Big\| \\ &\le t_k \Big(\operatorname{var}(\varphi, \frac{\epsilon}{2^k}) + \delta_k \Big) \end{aligned} \tag{5}$$

for every $j = 1, 2, \dots, N_k - 1$.

On the other hand, as $\|u\| \le \|\varphi\|_\infty$, we also have that

$$\Big\| \int_0^{p_j^k} \varphi(X_{e_j+t_k+r}(y))\,dr - p_j^k u \Big\| \le K_k(\|\varphi_\infty\| + \|u\|) \le 2K_k \|\varphi\|_\infty \tag{6}$$

for every $j = 1, 2, \dots, N_k - 1$.

Moreover, decomposing the time interval $[0, s_k - 1]$ according to shadowing times and transition times

$$[0, s_k - 1] = \bigcup_{j=1}^{N_k} [e_j, e_j + t_k] \cup \bigcup_{j=1}^{N_k - 1} [e_j + t_k, e_j + t_k + p_j^k].$$

For times on the intervals $[e_j, e_j + t_k]$ and $[e_j + t_k, e_j + t_k + p_j^k]$ and using (5) and (6), respectively, we get

$$\Big\| \int_0^{s_k} \varphi(X_r(y))\,dr - s_k\, u \Big\| \le \quad N_k t_k \Big(\operatorname{var}\Big(\varphi, \frac{\epsilon}{2^k}\Big) + \delta_k \Big) + (N_k - 1) 2K_k \|\varphi\|_\infty$$

as desired. □

The next lemma proves that Birkhoff averages of points in $\mathcal{R}$ oscillate between the vectors u and v.

Lemma 4.4. *For every $k \ge 1$ the following hold*:

(i) *If k is odd and $z \in L_k(q)$, then*

$$\frac{1}{l_k} \Big\| \int_0^{l_k} \varphi(X_r(z))\,dr - \ u \Big\| \to 0 \ \text{as } k \to \infty;$$

(ii) *If k is even and $z \in L_k(q)$, then*

$$\frac{1}{l_k} \Big\| \int_0^{l_k} \varphi(X_r(z))\,dr - \ v \Big\| \to 0 \ \text{as } k \to \infty.$$

Proof. (i) Fix $k \ge 0$ odd. Let $z = z(z_0, y(\underline{x_k})) \in L_k(q)$, $y = y(\underline{x_k})$ and $l_k = l_{k-1} + p_0^k + s_k$ be given as on the definition of $L_k(q)$. Then,

$$d(X_{l_{k-1}+p_0^k+t}(z), X_t(y)) \le \frac{\epsilon}{2^k}$$

for every $t \in [0, s_k]$.

Hence,

$$\begin{aligned}
\Big\| \int_0^{l_k} \varphi(X_r(z))\,dr - l_k\, u \Big\| &\le \Big\| \int_0^{l_{k-1}+p_0^k} \varphi(X_r(z))\,dr - (l_{k-1}+p_0^k)\; u \Big\| \\
&\quad + \Big\| \int_{l_{k-1}+p_0^k}^{l_k} \varphi(X_r(z))\,dr - s_k\; u \Big\| \\
&\le \Big\| \int_0^{s_k} \varphi(X_{l_{k-1}+p_0^k+r}(z))\,dr - s_k\; u \Big\| \\
&\quad + 2(l_{k-1}+p_0^k)\|\varphi\|_\infty \\
&\le \Big\| \int_0^{s_k} \varphi(X_{l_{k-1}+p_0^k+r}(z))\,dr - \int_0^{s_k} \varphi(X_r(y))\,dr \Big\| \\
&\quad + \Big\| \int_0^{s_k} \varphi(X_r(y))\,dr - s_k\; u \Big\| \\
&\quad + 2(l_{k-1}+p_0^k)\|\varphi\|_\infty .
\end{aligned}$$

Dividing the previous expression by l_k and appealing Lemma 4.3 we obtain that

$$\begin{aligned}
\frac{1}{l_k}\Big\| \int_0^{l_k} \varphi(X_r(z))\,dr - \; u \Big\| &\le \frac{s_k}{l_k} \operatorname{var}\Big(\varphi, \frac{\epsilon}{2^k}\Big) + \frac{2(l_{k-1}+p_0^k)\|\varphi\|_\infty}{l_k} \\
&\quad + \frac{N_k t_k \Big(\operatorname{var}(\varphi, \frac{\epsilon}{2^k}) + \delta_k \Big) + 2(N_k - 1)K_k\|\varphi\|_\infty}{l_k}
\end{aligned}$$

which, by (3) and (1), converges to zero as $k \to \infty$ because $\frac{s_k}{l_k} \le 1$, $\frac{N_k t_k}{l_k} \le 1$ and

$$\frac{l_{k-1}+p_0^k}{l_k} \le \frac{N_0\, t_0 + \sum_{j=1}^k N_j(t_j + K_j) + K_k}{l_k} \to 0$$

as $k \to \infty$. This proof the item (i). Since the proof of item (ii) is completely analogous we shall omit it. □

We are now able to prove that $\mathcal{R}(x,\epsilon) \subset I_\varphi \cap \Delta$ for every small $\epsilon > 0$ and every $q \in D$. Let $x \in \mathcal{R}(x,\epsilon)$, then there is an even integer number k such that $x \in \widetilde{B}_{l_k}(z, \epsilon/2^k)$ for some $z \in L_k(q)$. Note that $z = z(z_0, y(\underline{x_k}))$ satisfies $d(X_t(z), X_t(x)) < \frac{\epsilon}{2^k}$ for all $t \in [0, l_k]$, by definition (4). On the other hand, since that $z_0 \in L_{k-1}$ we obtain that $d(X_t(z), X_t(z_0)) < \frac{\epsilon}{2^k}$ for all $t \in [0, l_{k-1}]$. Therefore, $d(X_t(x), X_t(z_0)) < \frac{\epsilon}{2^{k-1}}$, for all $t \in [0, l_{k-1} - 1]$. It follows that

$$\begin{aligned}
&\Big\| \int_0^{l_{k-1}} \varphi(X_r(x))\,dr - l_{k-1}u \Big\| \\
\le &\Big\| \int_0^{l_{k-1}} \varphi(X_r(x))\,dr - \int_0^{l_{k-1}} \varphi(X_r(z_0))\,dr \Big\|
\end{aligned}$$

$$+\Big\|\int_0^{l_{k-1}} \varphi(X_r(z_0))\,dr - l_{k-1}u\Big\|$$
$$\le l_{k-1}\,\mathrm{var}\Big(\varphi,\frac{\epsilon}{2^k}\Big) + \Big\|\int_0^{l_{k-1}} \varphi(X_r(z_0))\,dr - l_{k-1}u\Big\|.$$

Using Lemma 4.4 we get that

$$\Big\|\frac{1}{l_{k-1}}\int_0^{l_{k-1}} \varphi(X_r(x))\,dr - u\Big\| \to 0 \text{ as } k\to\infty,$$

therefore

$$\Big\|\frac{1}{l_{k-1}}\int_0^{l_{k-1}} \varphi(X_r(x))\,dr - u\Big\| \to 0 \text{ as } k\to\infty.$$

In the similar way, if k is odd, we can proves that $\Big\|\frac{1}{l_k}\int_0^{l_k} \varphi(X_r(x))\,dr - v\Big\| \to 0$ as $k\to\infty$. This proves that $x\in I_\varphi\cap\Delta$ and therefore $\mathcal{R}(q,\epsilon)\subset I_\varphi\cap\Delta$ and consequently $\mathcal{R}\subset I_\varphi\cap\Delta$.

4.2. Proof of Theorem 2.2

We need some preparatory notions and results. Let $(X_t)_t$ be a suspension flow defined over a continuous map $f: M\to M$ and $\Phi: M\to\mathbb{R}^d$ be a continuous observable. The set of points with historic behaviour with respect to Φ is defined by

$$\widehat{I}_\Phi = \Big\{(x,s)\in M_r : \lim_{T\to\infty}\frac{1}{T}\int_0^T \Phi(X_t(x,s))\,dt \text{ does not exist and } 0\le s\le r(x)\Big\}.$$

The following lemma gives another description for the set of points with historic behavior for suspension flows.

Lemma 4.5. *Let M be a compact metric space, $f: M\to M$ be an homeomorphism and $r: M\to(0,\infty)$ be a continuous roof function bounded away from zero. Suppose that $(X_t)_t$ is the suspension flow over f with roof function r. If $\Phi: M_r\to\mathbb{R}^d$ is a continuous observable and $\varphi(x)=\int_0^{r(x)}\Phi(x,t)\,dt$, then*

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \Phi(X_t(x,s))dt = \lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\varphi(f^i(x))}{\sum_{i=0}^{n-1} r(f^i(x))}.$$

Proof. Let $(X_t)_t$, f and Φ be as in the hypothesis. Given $(x,s)\in M_r$ and $T>0$. There is $n\in\mathbb{N}$ such that $T=r^n(x)+a$ and $0\le a<r(f^n(x))$. Note that r satisfies $r^n(x)=\sum_{i=0}^{n-1} r\circ f^i(x)$. Thus, since r is bounded away from zero

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \Phi(X_t(x,s))\,dt = \lim_{n\to+\infty}\frac{1}{r^n(x)+a}\int_0^{r^n(x)+a} \Phi(X_t(x,s))\,dt$$
$$= \lim_{n\to+\infty}\frac{1}{r^n(x)}\int_0^{r^n(x)} \Phi(X_t(x,s))\,dt$$

$$= \lim_{n\to+\infty} \frac{1}{r^n(x)} \sum_{i=0}^{n-1} \int_0^{r(f^j(x))} \Phi(X_t(f^i(x),t))\,dt$$
$$= \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \varphi(f^i(x))}{\sum_{i=0}^{n-1} r(f^i(x))}.$$

This proves the lemma. □

The previous lemma ensures that there is a relationship between $\widehat{I}_\Phi$ and the set

$$\widehat{I}_{\varphi,r} = \left\{ x \in M : \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \varphi(f^i(x))}{\sum_{i=0}^{n-1} r(f^i(x))} \text{ does not exist} \right\}.$$

The following result, whose proof follows through an argument of [12], shows that the set of points with historic behavior is non-empty of a suspension flow over a homeomrphisms with gluing orbit property is large from the topological viewpoint. A similar result was proved by Barreira, Li and Valls in [3] under weak specification property.

Theorem 4.6. *Let M be a compact metric space, $f : M \to M$ be a homeomorphism with the gluing orbit property, $\psi : M \to \mathbb{R}^d$ be continuous observable, φ as in the lemma above and $r : M \to (0,\infty)$ be a continuous roof function. Then, $\widehat{I}_{\varphi,r}$ is either empty or a Baire residual subset and carries full topological pressure on M. In other words, if $\widehat{I}_{\varphi,r}$ is not empty, then $\widehat{I}_{\varphi,r}$ is a Baire residual subset of M and $P_{\widehat{I}_{\varphi,r}}(f,\psi) = P(f,\psi)$.*

The proof of the previous theorem is similar to the proof of Theorem E in [12]. First, the proof that $\widehat{I}_{\varphi,r}$ is either empty or a Baire residual subset of M follow the replacement the family of sets $P(w,\delta,n)$, in the proof of Theorem D in [12], by

$$\left\{ x \in \ M : \left\| \frac{\sum_{i=0}^{n-1} \varphi(f^i(x))}{\sum_{i=0}^{n-1} r(f^i(x))} - w \right\| < \delta \right\}$$

and to use Hopf Ergodic Theorem for quotients of Birkhoff averages we obtain that $\widehat{I}_{\varphi,r}$ is either empty or a Baire residual. Analogously, the proof that $\widehat{I}_{\varphi,r}$ is either empty or carries full topological entropy follow the replacement

$$\sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ and } \int \varphi(x)\,d\mu_i$$

in the proof of Theorem E of [12], by

$$\frac{\sum_{i=0}^{n-1} \varphi(f^i(x))}{\sum_{i=0}^{n-1} r(f^i(x))} \text{ and } \frac{\int \varphi(x)\,d\mu_i}{\int r(x)\,d\mu_i}$$

respectively and to use Hopf Ergodic Theorem for quotients of Birkhoff average again. We also need of the following auxiliary result.

Theorem 4.7 ([18, Theorem 5.8]). *Let (M, d) be a compact metric space, $f : M \to M$ be a homeomorphism and $r : M \to (0, \infty)$ be a continuous roof function. Let $(X_t)_t$ be a suspension flow over f acting in M_r. For an arbitrary Borel set $Z \subset M$, define $Z_r := \{(z, s) : z \in Z, 0 \le s < r(s)\}$. If $\beta = \sup\{t > 0\colon P_Z(f, -tr) > 0\}$, then $h_{Z_r}((X_t)_t) \ge \beta$.*

Finally, we are in position to prove Theorem 2.2.

Proof of Theorem 2.2. Lemma 4.5 allow us to obtain results for $\widehat{I}_\Phi$ from a corresponding results about the set $\widehat{I}_{\varphi,r}$, using that $\widehat{I}_\Phi = \bigcup_{t\in\mathbb{R}}\{X_t(x, 0)\colon x \in \widehat{I}_{\varphi,r}\}$. If $\widehat{I}_{\varphi,r}$ is empty we are done, so we suppose that $\widehat{I}_{\varphi,r}$ is not empty. Theorem 4.6 implies that $\widehat{I}_{\varphi,r}$ is a Baire residual subset of M, and so $M\setminus\widehat{I}_{\varphi,r} = \cup_i F_i$ is contained in a countable union of closed sets F_i with empty interior. Then we observe that

$$M_r \setminus \widehat{I}_\Phi = \big(\bigcup_i \big\{X_t(x, 0)\colon x \in F_i,\ t \in [0, r(x)]\big\}\big)/\approx$$

is meager as well, and so $\widehat{I}_\Phi$ is a Baire generic subset of M_r.

We now consider the topological entropy of $\widehat{I}_\Phi$. It is well known that $\beta = h_{top}((X_t)_t)$ is the unique number so that $P(f, -tr) = 0$. Theorem 4.6 implies that $P(f, -tr) = P_{\widehat{I}_{\varphi,r}}(f, -tr)$ for all $t \in \mathbb{R}$. Hence we conclude $h_{top}((X_t)_t)$ is the unique zero of the pressure function $t \mapsto P_{\widehat{I}_{\varphi,r}}(f, -tr)$. This, together with Theorem 4.7 and the previous relation between $\widehat{I}_{\varphi,r}$ and $\widehat{I}_\Phi$, implies that $h_{\widehat{I}_\Phi}((X_t)_t) \ge h_{top}((X_t)_t)$. As the other inequality always holds we conclude that $h_{\widehat{I}_\Phi}((X_t)_t) = h_{top}((X_t)_t)$. This proves Theorem 2.2. □

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