J. Korean Math. Soc. **59** (2022), No. 2, pp. 299–309
https://doi.org/10.4134/JKMS.j210123
pISSN: 0304-9914 / eISSN: 2234-3008

GREEN'S ADDITIVE COMPLEMENT PROBLEM FOR k-TH POWERS

Yuchen Ding and Li-Yuan Wang

Abstract. Let $k \geqslant 2$ be an integer, $S^k = \{1^k, 2^k, 3^k, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ be an additive complement of S^k, which means all sufficiently large integers can be written as the sum of an element of S^k and an element of B. In this paper we prove that

$$\limsup_{n\to\infty} \frac{\Gamma\left(2-\frac{1}{k}\right)^{\frac{k}{k-1}} \Gamma\left(1+\frac{1}{k}\right)^{\frac{k}{k-1}} n^{\frac{k}{k-1}} - b_n}{n} \geqslant \frac{k}{2(k-1)} \frac{\Gamma\left(2-\frac{1}{k}\right)^2}{\Gamma\left(2-\frac{2}{k}\right)},$$

where $\Gamma(\cdot)$ is Euler's Gamma function.

1. Introduction

Additive complements are popular topics in combinatorial number theory. For any two infinite sequences of nonnegative integers A and B, if their sum, which is defined to be

$$A + B := \left\{a + b \mid a \in A, b \in B\right\}$$

contains all sufficiently large integers, then we say that they are additive complements and B is an additive complement of A. For any set $D \subseteq \mathbb{N}$, let $D(x)$ be the number of elements of D not exceeding x. Let $S = \{1^2, 2^2, 3^2, \ldots\}$ be the set of all squares. Given a positive integer N, let $\mathscr{B}_N$ be a subset of $\{0, 1, 2, \ldots, N\}$ such that every integer n between 1 and N can be written as the sum of two elements of S and $\mathscr{B}_N$ (in this case we say that $\mathscr{B}_N$ is an additive complement of S up to N). Let θ_N be the least cardinality of the above $\mathscr{B}_N$. On the one hand, it is not difficult to see that $\left\{0, 1, 2, 3, \ldots, 2[\sqrt{N}] + 1\right\}$

Received February 20, 2021; Accepted December 31, 2021.
2020 *Mathematics Subject Classification.* Primary 11B13, 11B75.
Key words and phrases. Additive complement, Gamma function.
The first author was supported by the Natural Science Foundation of Jiangsu Province of China (No. BK20210784). He was also supported by the foundations of the projects "Jiangsu Provincial Double–Innovation Doctor Program" (No. JSSCBS20211023) and "Golden Phenix of the Green City–Yang Zhou" to excellent PhD (No. YZLYJF2020PHD051). The second author was supported by the Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (Grant No. 21KJB110001).

is such an additive complement of S up to N. So

$$\limsup_{N\to\infty} \frac{\theta_N}{\sqrt{N}} \leqslant 2.$$

On the other hand, it is obvious that $\theta_N \cdot \left[\sqrt{N}\right] \geqslant N$, which implies

$$\liminf_{N\to\infty} \frac{\theta_N}{\sqrt{N}} \geqslant 1.$$

Erdős [9] asked whether $\liminf_{N\to\infty} \frac{\theta_N}{\sqrt{N}}$ is strictly larger than 1. This question was answered affirmatively by Moser [11] who showed that

$$\liminf_{N\to\infty} \frac{\theta_N}{\sqrt{N}} \geqslant 1.06.$$

This result has been improved by many authors (see [1–4, 6, 8, 10, 12, 13]). Up to now the best known result is

$$\liminf_{N\to\infty} \frac{\theta_N}{\sqrt{N}} \geqslant \frac{4}{\pi},$$

which was obtained by Cilleruelo [6], Habsieger [10] and Balusubramanian and Ramana [3].

The above results can be generalized to k-th powers, where $k \geqslant 2$ is an integer. Let $S^k = \{1^k, 2^k, 3^k, \ldots\}$. Given any positive integer N, let $\mathscr{B}_N^k \subseteq \{0, 1, 2, 3, \ldots, N\}$ be an additive complement of S^k up to N and θ_N^k be the least cardinal number of such $\mathscr{B}_N^k$. It (see for example [6]) has been proved that

$$\liminf_{N\to\infty} \frac{\theta_N^k}{N^{\frac{k-1}{k}}} \geqslant \frac{1}{\Gamma(2-\frac{1}{k})\Gamma(1+\frac{1}{k})},$$

where $\Gamma(\cdot)$ is Euler's Gamma function.

Let's define

$$R(n) := \#\big\{(l, b) \mid n = l^2 + b,\ l \in \mathbb{N},\ b \in B\big\}.$$

Then it is clear that $R(n) \geqslant 1$ for sufficiently large n if B is an additive complement of the squares. Ben Green noticed that if $B = \{b_n\}_{n=1}^{\infty}$ satisfies $b_n = \frac{\pi^2}{16} n^2 + o(n^2)$, i.e.,

$$B(N) = \frac{4}{\pi}\sqrt{N} + o(\sqrt{N}),$$

then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} R(n) = 1.$$

He [5] was curious that whether there exists an additive complement $B = \{b_n\}_{n=1}^{\infty}$ of S such that $b_n = \frac{\pi^2}{16} n^2 + o(n^2)$. Chen and Fang [5] investigated this problem and proved that if $B = \{b_n\}_{n=1}^{\infty}$ is an additive complement of S, then

$$\limsup_{n\to\infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n^{1/2}\log n} \geqslant \sqrt{\frac{2}{\pi}}\frac{1}{\log 4}.$$

They further conjectured that

$$\limsup_{n\to\infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n^{1/2}\log n} = +\infty.$$

Recently, this conjecture was confirmed by the first author [7]. In fact, he proved the following stronger result:

$$\limsup_{n\to\infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n} \geqslant \frac{\pi}{4}. \tag{1.1}$$

Based on the above rich literature, we propose to study the additive complement of the k-th powers in Ben Green's manner, where $k \geqslant 2$ is an integer. Inspired by Ben Green's observation, we noticed that if $B = \{b_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ with

$$b_n = (1 + o(1))\Gamma\left(2 - \frac{1}{k}\right)^{\frac{k}{k-1}} \Gamma\left(1 + \frac{1}{k}\right)^{\frac{k}{k-1}} n^{\frac{k}{k-1}},$$

i.e.,

$$B(N) = (1 + o(1))\Gamma\left(2 - \frac{1}{k}\right)^{-1} \Gamma\left(1 + \frac{1}{k}\right)^{-1} N^{\frac{k-1}{k}},$$

then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} R_k(n) = 1,$$

where

$$R_k(n) := \#\big\{(l, b) \mid n = l^k + b,\ l \in \mathbb{N},\ b \in B\big\}.$$

This led us to study what can we say about the set B if it is an additive complement of S^k. As a generalization of (1.1), in this paper we obtain the following theorem, the proof of which will be given in the next section.

Theorem 1.1. *Given any positive integer $k \geqslant 2$, if $B = \{b_1, b_2, b_3, \ldots\}$ is an additive complement of S^k, then*

$$\limsup_{n\to\infty} \frac{\Gamma\left(2 - \frac{1}{k}\right)^{\frac{k}{k-1}} \Gamma\left(1 + \frac{1}{k}\right)^{\frac{k}{k-1}} n^{\frac{k}{k-1}} - b_n}{n} \geqslant \frac{k}{2(k-1)} \frac{\Gamma\left(2 - \frac{1}{k}\right)^2}{\Gamma\left(2 - \frac{2}{k}\right)}, \tag{1.2}$$

where $\Gamma(\cdot)$ is Euler's Gamma function.

Remark 1.2. Note that if $k = 2$, then

$$\Gamma\left(2 - \frac{1}{k}\right)^{\frac{k}{k-1}} \Gamma\left(1 + \frac{1}{k}\right)^{\frac{k}{k-1}} = \left(\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\right)^4 = \frac{\pi^2}{16}$$

and

$$\frac{k}{2(k-1)}\frac{\Gamma\left(2-\frac{1}{k}\right)^2}{\Gamma\left(2-\frac{2}{k}\right)}=\frac{\Gamma\left(\frac{3}{2}\right)^2}{\Gamma(1)}=\left(\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\right)^2=\frac{\pi}{4}.$$

Hence for $k=2$, the inequality (1.2) in the theorem becomes exactly (1.1). That means Theorem 1.1 is a natural generalization of the former result of the first author [7].

Next we give a simple example for the case $k=3$.

Example 1.3. If $k=3$, then one can calculate that

$$\frac{k}{2(k-1)}\frac{\Gamma\left(2-\frac{1}{k}\right)^2}{\Gamma\left(2-\frac{2}{k}\right)}=\frac{3}{4}\cdot\frac{\Gamma\left(\frac{5}{3}\right)^2}{\Gamma\left(\frac{4}{3}\right)}\approx 0.684463.$$

Thus by our theorem, if $B=\{b_1,b_2,b_3,\ldots\}$ is an additive complement of $\{1^3,2^3,3^3,\ldots\}$, then

$$\limsup_{n\to\infty}\frac{\Gamma\left(\frac{5}{3}\right)^{\frac{3}{2}}\Gamma\left(\frac{4}{3}\right)^{\frac{3}{2}}n^{\frac{3}{2}}-b_n}{n}\geqslant 0.684463.$$

We conclude this section by posing the following conjecture.

Conjecture 1.4. *Given any positive integer $k\geqslant 2$, if $B=\{b_1,b_2,b_3,\ldots\}$ is an additive complement of S^k, then*

$$\limsup_{n\to\infty}\frac{\Gamma\left(2-\frac{1}{k}\right)^{\frac{k}{k-1}}\Gamma\left(1+\frac{1}{k}\right)^{\frac{k}{k-1}}n^{\frac{k}{k-1}}-b_n}{n}=+\infty.$$

2. The proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proofs of the theorem need to be divided into two cases $k=2$ and $k>2$. If $k=2$, the theorem is reduced to

$$\limsup_{n\to\infty}\frac{\frac{\pi^2}{16}n^2-b_n}{n}\geqslant\frac{\pi}{4}.$$

This is exactly (1.1), which has already been proved in [7]. We simply omit the proof here.

From now on, let's assume $k>2$. For this case, the requirement of $k>2$ originates from the meaninglessness of $\Gamma(0)$ which shall be encountered in the derivation of (2.13) if $k=2$ (see the proof below). In fact, the proof of $k>2$ is a refinement of the former one with the help of Γ function. Perhaps the most important ingredient for $k>2$ is the conversion from the bound of b_n to that of $B(n)$ by using the famous Newton's binomial theorem. And as we can see in [7] the conversion for $k=2$ is somewhat trivial.

Before the beginning of our proof, we first state some facts on gamma function $\Gamma(\cdot)$ and beta function $B(\cdot\,,\cdot)$. These two functions are defined as follows:

$$\Gamma(s)=\int_0^{+\infty}x^{s-1}e^{-x}dx,\quad s>0,$$

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad p>0,\ q>0.$$

Recall that

$$\Gamma(s+1)=s\Gamma(s)$$

for any $s>0$, by which we have $\Gamma(n+1)=n!$ for any $n\in\mathbb{N}$. We also have the following relation between these two functions:

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (p>0,q>0).$$

Now we begin to prove the case for $k>2$. Let

$$a_k=\Gamma\left(2-\frac{1}{k}\right)^{\frac{k}{k-1}}\Gamma\left(1+\frac{1}{k}\right)^{\frac{k}{k-1}}$$

and β_k be the right side of (1.2), i.e.,

$$\beta_k=\frac{k}{2(k-1)}\frac{\Gamma\left(2-\frac{1}{k}\right)^2}{\Gamma\left(2-\frac{2}{k}\right)}.$$

Assume that the theorem doesn't hold. Then there exists a real number α_k such that

$$\limsup_{n\to\infty}\frac{a_k n^{\frac{k}{k-1}}-b_n}{n}=\alpha_k<\beta_k.$$

Let $\delta_k=(\beta_k-\alpha_k)/2$. Then there exists an integer n_1 such that

$$\frac{a_k n^{\frac{k}{k-1}}-b_n}{n}<\beta_k-\delta_k,$$

i.e.,

$$a_k\cdot n^{\frac{k}{k-1}}-(\beta_k-\delta_k)n<b_n \tag{2.3}$$

for all $n\geqslant n_1$.

We next show that there exists an integer n_2 such that

$$n<a_k\cdot(c_k\cdot n^{\frac{k-1}{k}}+g_k\cdot n^{\frac{k-2}{k}})^{\frac{k}{k-1}}-(\beta_k-\delta_k)(c_k\cdot n^{\frac{k-1}{k}}+g_k\cdot n^{\frac{k-2}{k}})$$

for all $n>n_2$, where

$$c_k=\frac{1}{\Gamma(2-\frac{1}{k})\Gamma(1+\frac{1}{k})}\text{ and }g_k=\frac{k-1}{k}\left(\beta_k-\frac{\delta_k}{2}\right)c_k^2.$$

In fact, by the Newton's binomial theorem we have

$$\begin{aligned}a_k\cdot(c_k\cdot n^{\frac{k-1}{k}}+g_k\cdot n^{\frac{k-2}{k}})^{\frac{k}{k-1}}&=a_k(c_k\cdot n^{\frac{k-1}{k}})^{\frac{k}{k-1}}\left(1+\frac{g_k}{c_k}\cdot n^{-\frac{1}{k}}\right)^{\frac{k}{k-1}}\\&=n\left(1+\frac{g_k}{c_k}\cdot n^{-\frac{1}{k}}\right)^{\frac{k}{k-1}}\end{aligned}$$

$$= n\left(1 + \frac{k}{k-1}\cdot\frac{g_k}{c_k}\cdot n^{-\frac{1}{k}} + O(n^{-2/k})\right)$$
$$= n + \frac{k}{k-1}\cdot\frac{g_k}{c_k}\cdot n^{1-\frac{1}{k}} + O(n^{\frac{k-2}{k}})$$
$$= n + \left(\beta_k - \frac{\delta_k}{2}\right) c_k \cdot n^{1-\frac{1}{k}} + O(n^{\frac{k-2}{k}})$$

(note that $a_k \cdot c_k^{\frac{k}{k-1}} = 1$).

Hence there exists an integer n_2 such that for all $n > n_2$ we have

$$a_k \cdot (c_k \cdot n^{\frac{k-1}{k}} + g_k \cdot n^{\frac{k-2}{k}})^{\frac{k}{k-1}} > n + (\beta_k - \delta_k)\, c_k \cdot n^{1-\frac{1}{k}} + (\beta_k - \delta_k)\, g_k \cdot n^{\frac{k-2}{k}},$$

i.e.,

$$n < a_k \cdot (c_k \cdot n^{\frac{k-1}{k}} + g_k \cdot n^{\frac{k-2}{k}})^{\frac{k}{k-1}} - (\beta_k - \delta_k)(c_k \cdot n^{\frac{k-1}{k}} + g_k \cdot n^{\frac{k-2}{k}}). \tag{2.4}$$

Now let's define

$$f(\lambda) := a_k \cdot \lambda^{\frac{k}{k-1}} - (\beta_k - \delta_k)\lambda. \tag{2.5}$$

Obviously there is an integer n_3 such that $f(\lambda)$ is strictly increasing in $[n_3, \infty)$. Take $M = \max\{b_{n_1}, n_2, b_{n_3}\}$. Let $n > M$ be any integer. Let $B(n) = t$, i.e., t is the largest integer such that

$$b_t \leqslant n.$$

In view of (2.4) and (2.5) we have

$$b_t \leqslant n < f(c_k \cdot n^{\frac{k-1}{k}} + g_k \cdot n^{\frac{k-2}{k}}).$$

Since $n > M \geqslant \max\{b_{n_1}, b_{n_3}\}$ we have $t \geqslant \max\{n_1, n_3\}$. Recall that $t \geqslant n_1$ implies $f(t) < b_t$ from (2.3) and (2.5). Thus we have

$$f(t) < b_t \leqslant n < f(c_k \cdot n^{\frac{k-1}{k}} + g_k \cdot n^{\frac{k-2}{k}}).$$

Note that $f(\lambda)$ is strictly increasing on $[n_3, \infty)$, so we have

$$B(n) = t < c_k \cdot n^{\frac{k-1}{k}} + g_k \cdot n^{\frac{k-2}{k}} \tag{2.6}$$

for all $n > M$.

As in the introduction, let

$$R_k(n) = \#\{(l, b) \mid n = l^k + b,\ l \in \mathbb{N},\ b \in B\}$$

be the representation function of n. Then in view of (2.6) we deduce that

$$\sum_{n=1}^{N} R(n) = \sum_{n=1}^{N} \sum_{\substack{n=l^k+b \\ b\in B}} 1$$
$$= \sum_{\substack{n^k+b\leqslant N \\ b\in B}} 1$$

$$
\begin{aligned}
&= \sum_{n\leqslant N^{1/k}} \sum_{\substack{b\leqslant N-n^k\\ b\in B}} 1\\
&= \sum_{n\leqslant N^{1/k}} B(N-n^k)\\
&\leqslant \sum_{n\leqslant N^{1/k}} \Big(c_k\cdot(N-n^k)^{\frac{k-1}{k}} + g_k\cdot(N-n^k)^{\frac{k-2}{k}}\Big) + O(1)\\
&= c_k \sum_{n\leqslant N^{1/k}} (N-n^k)^{\frac{k-1}{k}} + g_k \sum_{n\leqslant N^{1/k}} (N-n^k)^{\frac{k-2}{k}} + O(1). \qquad (2.7)
\end{aligned}
$$

Now we estimate the first two parts of the above equation. Suppose that $N = K^k$ for some positive integer K. For the first part we let $g(t) = (N-t^k)^{\frac{k-1}{k}}$, then by Euler-Maclaurin formula, we have

$$
\begin{aligned}
\sum_{n\leqslant N^{1/k}} (N-n^k)^{\frac{k-1}{k}} &= \sum_{0<n\leqslant N^{1/k}} g(n)\\
&= \int_0^{N^{1/k}} g(t)dt - \Big(g(N^{1/k}) - g(0)\Big)\Big(-\frac{1}{2}\Big)\\
&\quad + \int_0^{N^{1/k}} \Big(\{t\}-\frac{1}{2}\Big)g'(t)dt\\
&= \int_0^{N^{1/k}} g(t)dt - \frac{1}{2}N^{1-\frac{1}{k}} + \int_0^{N^{1/k}} \Big(\{t\}-\frac{1}{2}\Big)g'(t)dt. \qquad (2.8)
\end{aligned}
$$

Now we integrate the first term by substitution. Letting $t^k = Nx$, then $t = N^{\frac{1}{k}}x^{\frac{1}{k}}$. Hence

$$
\begin{aligned}
\int_0^{N^{1/k}} g(t)d &= \int_0^{N^{1/k}} (N-t^k)^{1-\frac{1}{k}}dt\\
&= \frac{N}{k}\int_0^1 x^{\frac{1}{k}-1}(1-x)^{1-\frac{1}{k}}dx\\
&= \frac{N}{k}B\Big(\frac{1}{k}, 2-\frac{1}{k}\Big)\\
&= \frac{N}{k}\frac{\Gamma(\frac{1}{k})\Gamma(2-\frac{1}{k})}{\Gamma(2)}\\
&= \Gamma\Big(1+\frac{1}{k}\Big)\Gamma\Big(2-\frac{1}{k}\Big)N. \qquad (2.9)
\end{aligned}
$$

We next show that

$$
\int_0^{N^{1/k}} \Big(\{t\}-\frac{1}{2}\Big)g'(t)dt \leqslant 0 \qquad (2.10)
$$

for $N = K^k$. Since $g'(t) = (1-k)t^{k-1}(N-t^k)^{-\frac{1}{k}}$, we have

$$\int_0^{N^{1/k}} \left(\{t\} - \frac{1}{2}\right) g'(t)dt = \int_0^{N^{1/k}} \left(\{t\} - \frac{1}{2}\right)(1-k)t^{k-1}(N-t^k)^{-\frac{1}{k}}dt$$

$$= (1-k)\sum_{i=1}^{K}\int_{i-1}^{i} \left(\{t\} - \frac{1}{2}\right)t^{k-1}(N-t^k)^{-\frac{1}{k}}dt. \tag{2.11}$$

For each $i = 1, 2, \ldots, K$ we let $t = i - \frac{1}{2} + \mu$, then $\{t\} - \frac{1}{2} = \mu$. Hence

$$\int_{i-1}^{i} \left(\{t\} - \frac{1}{2}\right)t^{k-1}(N-t^k)^{-\frac{1}{k}}d\mu$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu \cdot \left(i - \frac{1}{2} + \mu\right)^{k-1}\left(N - \left(i - \frac{1}{2} + \mu\right)^k\right)^{-\frac{1}{k}} d\mu.$$

Now for any $i = 1, 2, \ldots, K$ we define

$$h_i(\mu) := \left(i - \frac{1}{2} + \mu\right)^{k-1}\left(N - \left(i - \frac{1}{2} + \mu\right)^k\right)^{-\frac{1}{k}}.$$

It is easy to see that $h_i(\mu)$ is monotonically increasing on $[-\frac{1}{2}, \frac{1}{2}]$ for any $i = 1, 2, \ldots, K$. Therefore

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mu \cdot \left(i - \frac{1}{2} + \mu\right)^{k-1}\left(N - \left(i - \frac{1}{2} + \mu\right)^k\right)^{-\frac{1}{k}} d\mu$$

$$= \int_0^{\frac{1}{2}} \mu \cdot h_i(\mu)d\mu + \int_{-\frac{1}{2}}^{0} \mu \cdot h_i(\mu)d\mu$$

$$= \int_0^{\frac{1}{2}} \mu \cdot \big[h_i(\mu) - h_i(-\mu)\big]d\mu \geqslant 0.$$

Now (2.10) follows from the above inequality and (2.11). Combining (2.8), (2.9) and (2.10) we obtain that

$$\sum_{n \leqslant N^{1/k}} (N - n^k)^{\frac{k-1}{k}} \leqslant \Gamma\left(1 + \frac{1}{k}\right)\Gamma\left(2 - \frac{1}{k}\right)N - \frac{1}{2}N^{1-\frac{1}{k}}.$$

Thus

$$c_k \sum_{n \leqslant N^{1/k}} (N - n^k)^{\frac{k-1}{k}} \leqslant c_k\Gamma\left(1 + \frac{1}{k}\right)\Gamma\left(2 - \frac{1}{k}\right) \cdot N - \frac{1}{2}c_k N^{1-\frac{1}{k}}$$

$$= N - \frac{1}{2}c_k N^{1-\frac{1}{k}}. \tag{2.12}$$

Now we estimate the second part of (2.7). Let's define

$$\omega(t) := (N - t^k)^{\frac{k-2}{k}},$$

then

$$\omega'(t) = (2-k)t^{k-1}(N-t^k)^{-\frac{2}{k}}.$$

Similarly, by Euler-MacLaurin formula we deduce that

$$\begin{aligned}
\sum_{n\leqslant N^{1/k}} (N-n^k)^{\frac{k-2}{k}} &= \sum_{0<n\leqslant N^{1/k}} \omega(n) \\
&= \int_0^{N^{1/k}} \omega(t)dt - \Big(\omega(N^{1/k}) - \omega(0)\Big)\Big(-\frac{1}{2}\Big) \\
&\quad + \int_0^{N^{1/k}} \Big(\{t\}-\frac{1}{2}\Big)\omega'(t)dt \\
&= \frac{1}{k}\frac{\Gamma(\frac{1}{k})\Gamma(2-\frac{2}{k})}{\Gamma(2-\frac{1}{k})}N^{1-\frac{1}{k}} - \frac{1}{2}N^{1-\frac{2}{k}} \\
&\quad + \int_0^{N^{1/k}} \Big(\{t\}-\frac{1}{2}\Big)\omega'(t)dt \\
&= \frac{\Gamma(1+\frac{1}{k})\Gamma(2-\frac{2}{k})}{\Gamma(2-\frac{1}{k})}N^{1-\frac{1}{k}} - \frac{1}{2}N^{1-\frac{2}{k}} \\
&\quad + \int_0^{N^{1/k}} \Big(\{t\}-\frac{1}{2}\Big)\omega'(t)dt. \qquad (2.13)
\end{aligned}$$

By similar arguments one can verify that

$$\int_0^{N^{1/k}} \Big(\{t\}-\frac{1}{2}\Big)\omega'(t)dt \leqslant 0$$

and so we have

$$\sum_{n\leqslant N^{1/k}} (N-n^k)^{\frac{k-2}{k}} \leqslant \frac{\Gamma(1+\frac{1}{k})\Gamma(2-\frac{2}{k})}{\Gamma(2-\frac{1}{k})}N^{1-\frac{1}{k}} - \frac{1}{2}N^{1-\frac{2}{k}}.$$

Thus

$$(2.14) \quad g_k\sum_{n\leqslant N^{1/k}} (N-n^k)^{\frac{k-2}{k}} \leqslant g_k\frac{\Gamma(1+\frac{1}{k})\Gamma(2-\frac{2}{k})}{\Gamma(2-\frac{1}{k})}\cdot N^{1-\frac{1}{k}} - \frac{1}{2}g_k\cdot N^{1-\frac{2}{k}}.$$

Combining (2.7), (2.12) and (2.14) gives

(2.15)

$$\sum_{n=1}^{N} R_k(n) \leqslant N - \left(\frac{1}{2}c_k - g_k\frac{k-2}{k}\frac{\Gamma(1+\frac{1}{k})\Gamma(1-\frac{2}{k})}{\Gamma(2-\frac{1}{k})}\right)N^{1-\frac{1}{k}} + O(N^{1-\frac{2}{k}})$$

for k-th power integers $N = K^k$.

Now we show that

$$\frac{1}{2}c_k > g_k\frac{k-2}{k}\frac{\Gamma(1+\frac{1}{k})\Gamma(1-\frac{2}{k})}{\Gamma(2-\frac{1}{k})}.$$

In fact, noticing that

$$c_k = \frac{1}{\Gamma(2-\frac{1}{k})\Gamma(1+\frac{1}{k})}\ , \quad g_k = \frac{k-1}{k}\left(\beta_k - \frac{\delta_k}{2}\right)c_k^2$$

and

$$\beta_k = \frac{k}{2(k-1)}\frac{\Gamma\left(2-\frac{1}{k}\right)^2}{\Gamma\left(2-\frac{2}{k}\right)},$$

we obtain that

$$\begin{aligned} g_k \frac{k-2}{k}\frac{\Gamma(1+\frac{1}{k})\Gamma(1-\frac{2}{k})}{\Gamma(2-\frac{1}{k})} &= \frac{k-2}{k}\frac{k-1}{k}\left(\beta_k - \frac{1}{2}\delta_k\right)c_k^2\frac{\Gamma(1+\frac{1}{k})\Gamma(1-\frac{2}{k})}{\Gamma(2-\frac{1}{k})} \\ &< \frac{k-2}{k}\frac{k-1}{k}\beta_k c_k^2\frac{\Gamma(1+\frac{1}{k})\Gamma(1-\frac{2}{k})}{\Gamma(2-\frac{1}{k})} \\ &= \frac{1}{2}c_k. \end{aligned}$$

On the other hand, B is an additive complement of S^k, which means that all sufficiently large integer can be represented as the sum of two elements of B and S^k. So there exists an integer $n_4 > 0$ such that $R_k(n) \geqslant 1$ for all $n > n_4$, which implies

$$\sum_{n=1}^{N} R_k(n) \geqslant N - n_4$$

for all $N > n_4$. This contradicts with (2.15) for sufficiently large $N = K^k$, which completes the proof our theorem.

References

[1] H. L. Abbott, *On an additive completion of sets of integers*, J. Number Theory **17** (1983), no. 2, 135–143. https://doi.org/10.1016/0022-314X(83)90015-X

[2] R. Balasubramanian, *On the additive completion of squares*, J. Number Theory **29** (1988), no. 1, 10–12. https://doi.org/10.1016/0022-314X(88)90089-3

[3] R. Balasubramanian and D. S. Ramana, *Additive complements of the squares*, C. R. Math. Acad. Sci. Soc. R. Can. **23** (2001), no. 1, 6–11.

[4] R. Balasubramanian and K. Soundararajan, *On the additive completion of squares. II*, J. Number Theory **40** (1992), no. 2, 127–129. https://doi.org/10.1016/0022-314X(92)90034-M

[5] Y.-G. Chen and J.-H. Fang, *Additive complements of the squares*, J. Number Theory **180** (2017), 410–422. https://doi.org/10.1016/j.jnt.2017.04.016

[6] J. Cilleruelo, *The additive completion of kth-powers*, J. Number Theory **44** (1993), no. 3, 237–243. https://doi.org/10.1006/jnth.1993.1049

[7] Y. Ding, *Green's problem on additive complements of the squares*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 8, 897–900. https://doi.org/10.5802/crmath.107

[8] R. Donagi and M. Herzog, *On the additive completion of polynomial sets of integers*, J. Number Theory **3** (1971), 150–154. https://doi.org/10.1016/0022-314X(71)90031-X

[9] P. Erdös, *Problems and results in additive number theory*, in Colloque sur la Théorie des Nombres, Bruxelles, 1955, 127–137, Georges Thone, Liège, 1956.

[10] L. Habsieger, *On the additive completion of polynomial sets*, J. Number Theory **51** (1995), no. 1, 130–135. https://doi.org/10.1006/jnth.1995.1039

[11] L. Moser, *On the additive completion of sets of integers*, in Proc. Sympos. Pure Math., Vol. VIII, 175–180, Amer. Math. Soc., Providence, RI, 1965.

[12] D. S. Ramana, *Some topics in analytic number theory*, PhD thesis, University of Madras, May 2000.

[13] D. S. Ramana, *A report on additive complements of the squares*, in Number theory and discrete mathematics (Chandigarh, 2000), 161–167, Trends Math, Birkhäuser, Basel, 2002.

Yuchen Ding
School of Mathematical Science
Yangzhou University
Yangzhou 225002, P. R. China
Email address: ycding@yzu.edu.cn

Li-Yuan Wang
School of Physical and Mathematical Sciences
Nanjing Tech University
Nanjing 211816, P. R. China
Email address: wly@smail.nju.edu.cn