East Asian Math. J.
Vol. 38 (2022), No. 1, pp. 001–012
http://dx.doi.org/10.7858/eamj.2022.001

ORE EXTENSIONS OVER σ-RIGID RINGS

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Abstract. Let R be a ring with an endomorphism σ and a σ-derivation δ. R is called $(\sigma,\ \delta)$-*Baer* (resp. $(\sigma,\ \delta)$-*quasi-Baer*, $(\sigma,\ \delta)$-*p.q.-Baer*, $(\sigma,\ \delta)$-*p.p.*) if the right annihilator of every right $(\sigma,\ \delta)$-set (resp., $(\sigma,\ \delta)$-ideal, principal $(\sigma,\ \delta)$-ideal, $(\sigma,\ \delta)$-element) of R is generated by an idempotent of R. In this paper, for a given Ore extension $A = R[x;\sigma,\delta]$ of R, the following properties are investigated: If R is a σ-rigid ring in which σ and δ commute, then (1) R is (σ,δ)-Baer if and only if R is (σ,δ)-quasi-Baer if and only if A is $(\bar{\sigma},\bar{\delta})$-Baer if and only if A is $(\bar{\sigma},\bar{\delta})$-quasi-Baer; (2) R is (σ,δ)-p.p. if and only if R is (σ,δ)-*p.q.*-Baer if and only if A is $(\bar{\sigma},\bar{\delta})$-*p.p.* if and only if A is $(\bar{\sigma},\bar{\delta})$-*p.q.*-Baer.

1. Introduction

Throughout this paper, R will denote an associative ring with identity and σ will be an endomorphism of R. We recall the following definition of Ore extensions for the convenience of the reading (see [9, 15, 19] for details). A map $\delta : R \to R$ is a σ-derivation, i.e.

$$\delta(a+b) = \delta(a) + \delta(b) \text{ and } \delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

for all $a, b \in R$.

A denotes the Ore extension $R[x;\sigma,\delta]$ of R, that is, A is the ring of polynomials over R in an indeterminate x with multiplication subject to the relation $xa = \sigma(a)x + \delta(a)$ for all $a \in R$. Such a ring always exists and is unique up to isomorphism (see [9] for details). Since A is a kind of ring extension of R, it is natural that $1 = 1_R$ will be a multiplicative identity for A as well. So we get that $\sigma(1) = 1$ and $\delta(1) = 0$ from the equality $x = 1x = x1 = \sigma(1)x + \delta(1)$.

When $\sigma = id_R$ (resp. $\delta = 0$), A is considered as the differential polynomial ring (resp. the skew polynomial ring) and is simply denoted by $R[x;\delta]$ (resp. $R[x;\sigma]$), where id_R means the identity map on R. Of course, when $\sigma = id_R$ and $\delta = 0$, A is considered as the polynomial ring $R[x]$. Kaplansky [13] introduced the Baer rings (i.e., rings in which the right annihilator of every nonempty

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Received July 15, 2021; Accepted September 27, 2021.
2010 *Mathematics Subject Classification.* 16E50, 16S50.
Key words and phrases. σ-rigid ring, Ore extension, $(\sigma$-$\delta)$-Baer ring.
This work was supported by 2-year Research Grant of Pusan National University.
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(pISSN 1226-6973, eISSN 2287-2833)

subset is generated by an idempotent) to abstract various properties of rings of operators on a Hilbert spaces (also refer [3]). Clark [8] introduced the quasi-Baer rings (i.e., rings in which the right annihilator of every right ideal is generated by an idempotent) which are generalizations of Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. Further works on Baer rings and quasi-Baer rings appears in [1, 5, 6, 7, 18]. The study of Baer and quasi-Baer rings has its roots in functional analysis. Birkenmeier, et al. [7] have shown that a ring R is quasi-Baer if and only if $R[x]$ is quasi-Baer if and only if $R[[x]]$ (power series ring over R) is quasi-Baer; a reduced ring R is Baer if and only if $R[x]$ is Baer if and only if $R[[x]]$ is Baer.

Birkenmeier, et al. [7] defined a *right* (or *left*) *principally quasi-Baer* (simply, called right (or left) *p.q.-Baer*) ring as a generalization of quasi-Baer ring by the rings in which the right (or left) annihilator of every right (or left) principal ideal of R is generated by an idempotent of R. R is called a *p.q.-Baer* ring if it is both right *p.q.*-Baer and left *p.q.*-Baer. Another generalization of Baer ring is a p.p.-ring. A ring R is called a *right* (resp. *left*) *p.p.*-ring if the right (resp. left) annihilator of every element of R is generated by an idempotent of R. R is called a *p.p.*-ring if it is both right and left p.p.-ring. Hong, et al. [11] have extended the results on $R[x;\sigma,\delta]$ by showing that for an σ-ring R, R is Baer (resp. p.p.) if and only if $R[x;\sigma,\delta]$ is Baer (resp. p.p.).

A subset S of a ring R is called a (σ, δ)-*set* if S is a (σ, δ)-stable set, i.e., $\sigma(S) \subseteq S$ and $\delta(S) \subseteq S$. In particular, if a singleton set $S = \{a\}$ of R is (σ, δ)-set, i.e., $\sigma(a) = a$ and $\delta(a) = a$, then a is called a (σ,δ)-element of R. A left (right, two-sided) ideal I of R is called a left (right, two-sided) (σ, δ)-*ideal* if I is a (σ, δ)-set. By analog, we can define a (σ, δ)-*Baer* ring (resp. (σ, δ)-*quasi-Baer*-ring) by the ring in which the right annihilator of every right (σ, δ)-set (resp. (σ, δ)-ideal) is generated by an idempotent. Jordan [12] has shown that if R is a right noetherian Jacobson ring and δ is a derivation on R then $R[x;\delta]$ is a right noetherian Jacobson ring by considering δ-ideal of R. Lam, et al. [14] have investigated the primeness and semiprimeness of Ore extensions over a ring R by considering (σ, δ)-ideal of R.

We also define a *right* (or *left*) (σ, δ)-*p.q.-Baer* ring (resp. *right* (or *left*) (σ, δ)-*p.p.*-ring) by the ring in which the right (or left) annihilator of every right (or left) principal (σ, δ)-ideal (resp. (σ, δ)-element) is generated by an idempotent. R is called a (σ, δ)-*p.q.-Baer* ring (resp. (σ, δ)-*p.p.*-ring) if it is both right (σ, δ)-*p.q.*-Baer and left (σ, δ)-*p.q.*-Baer (resp. right (σ, δ)-*p.p.* and left (σ, δ)-*p.p.*).

In this paper, we denote the right (resp. left) annihilator of a subset S of a ring R by $r_R(S) = \{a \in R \mid Sa = 0\}$ (resp. $\ell_R(S) = \{a \in R \mid aS = 0\}$). We recall that R is a σ-*rigid* (resp. *reduced*) ring if for some endomorphism σ of R, $a\sigma(a) = 0$ (resp. $a^2 = 0$) implies that $a = 0$ for each $a \in R$. Now we can observe the following implications:

(1) Every Baer ring is a (σ, δ)-Baer ring, and it is true for quasi-Baer rings, right (or left) *p.q.*-Baer rings, and right (or left) *p.p.*-Baer rings, respectively.

(2) Every (σ, δ)-Baer ring is a(σ, δ)-quasi-Baer ring.

(3) Every (σ, δ)-quasi-Baer ring is a (σ, δ)-*p.q.*-Baer ring.

All the implications are strict by the following examples.

Example 1. We refer to [11, Example 9]. Let $\mathbb{Z}$ be the ring of integers, and consider the ring $\mathbb{Z} \oplus \mathbb{Z}$ with the usual addition and multiplication. Then the subring $R = \{(a, b) \in \mathbb{Z}\oplus\mathbb{Z} \mid a \equiv b \, (mod\, 2) \}$ of $\mathbb{Z}\oplus\mathbb{Z}$ is a commutative reduced ring which has only two idempotents $(0, 0)$ and $(1, 1)$. Observe that R is not p.p. (and then R is not Baer, not *p.q.*-Baer, not quasi-Baer). Let $\sigma : R \to R$ be a map defined by $\sigma(a, b) = (b, a)$ for all $(a, b) \in R$. Then σ is an endomorphism of R. Clearly, R is not σ-rigid. Note that all σ-sets of R are $S \oplus S$ for some subset S of $\mathbb{Z}$. Let $T = S\oplus S$. If $T = (0)$, then $r_R(T) = R = (1, 1)R$. If $T \neq (0)$, then $r_R(T) = (0) = (0, 0)R$. Hence R is σ-Baer, and then R is σ-quasi-Baer and σ-p.p.. Also, R is (σ, δ)-quasi-Baer, (σ, δ)-quasi-Baer, and σ-p.p. for every derivation δ on R.

Recall that a ring R with a σ-derivation δ is called σ-rigid (resp. reduced) if $a\sigma(a) = 0$ (resp. $a^2 = 0$) for $a \in R$ implies $a = 0$. Note that σ-rigid rings are reduced, and this implication is proper as can be seen by $\sigma(a, b) = (b, a)$ in $\mathbb{Z}\oplus\mathbb{Z}$. In section 2, we will show that if R is a σ-rigid ring with a σ-derivation δ, then (1) R is (σ, δ)-Baer if and only if R is (σ, δ)-quasi-Baer; R is (σ, δ)-*p.q.*-Baer if and only if R is (σ, δ)-p.p..

Let R be a ring with a σ-derivation δ and let $A = R[x; \sigma, \delta]$. Consider maps $\bar{\sigma}, \bar{\delta} : A \to A$ defined by $\bar{\sigma}(f) = \sum_{j=0}^{n} \sigma(a_j)x^j$, $\bar{\delta}(f) = \sum_{j=0}^{n} \delta(a_j)x^j$ for all $f = \sum_{j=0}^{n} a_j x^j \in A$, respectively. Then $\bar{\sigma}$ (resp. $\bar{\delta}$) is a well-defined one which extends σ (resp. δ). Observe that even though σ is an endomorphism of R, but $\bar{\sigma}$ is not necessarily an endomorphism of A.

In 2012, Bergen and Grzeszczuk [4] considered q-skew σ-derivation δ for an algebra R over a field F where $0 \neq q \in F$ defined by $\delta(\sigma(r)) = q(\sigma(\delta(r))$ for all $r \in R$. In particular, they proved that for a q-skew σ-derivation δ of R where $1 + q + \cdots + q^{n-1} \neq 0$ for all positive integers n, if σ has a locally finite order and $\delta \neq 0$, then $q = 1$ (equivalently, σ and δ commute) and F has characteristic 0. In section 3, by considering a ring with a σ-derivation δ in which σ and δ commute (denoted by $\sigma\delta = \delta\sigma$), we will show that (1) $\bar{\sigma}$ is the unique extended endomorphism of σ and $\bar{\delta}$ is the unique extended $\bar{\sigma}$-derivation; (2) R is (σ, δ)-Baer if and only if R is (σ, δ)-quasi-Baer if and only if A is $(\bar{\sigma}, \bar{\delta})$-Baer if and only if A is $(\bar{\sigma}, \bar{\delta})$-quasi-Baer; (3) R is (σ, δ)-*p.p.* if and only if R is (σ, δ)-*p.q.*-Baer if and only if A is $(\bar{\sigma}, \bar{\delta})$-*p.p.* if and only if A is $(\bar{\sigma}, \bar{\delta})$-*p.q.*-Baer.

2. Preliminaries

Following [7], an idempotent $e \in R$ is *left* (resp. *right*) *semicentral* in R if $re = ere$ (resp. $er = ere$) for all $r \in R$. Equivalently, an idempotent $e \in R$

is left (resp. right) semicentral in R if eR (resp. Re) is an ideal of R. Due to Bell [2], a ring R is called to satisfy the *Insertion-of-Factors-Property* (simply, *an IFP ring*) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Narbonne [16] and Shin [20] used the terms *semicommutative* and *SI* for the IFP, respectively. The class of IFP rings contains commutative rings and reduced rings. There exist many IFP ring but not reduced as can be seen by the ring of integers modulo m^k for $m, k \geq 2$. A ring is usually called *Abelian* if every idempotent is central. A simple computation yields that every IFP ring is Abelian.

Theorem 2.1. *Let R be an IFP ring with an endomorphism σ and a σ-derivation δ. Then*
(1) *R is (σ, δ)-Baer if and only if R is (σ, δ)-quasi-Baer.*
(2) *R is σ-Baer if and only if R is σ-quasi-Baer.*

Proof. (1) It suffices to show the sufficiency. Suppose that R is (σ, δ)-quasi-Baer. Let S be a (σ, δ)-set of R. Consider the right ideal SR of R generated by S. Since S be a (σ, δ)-set, SR is a right (σ, δ)-ideal of R. Since R is (σ, δ)-quasi-Baer, $r_R(SR) = eR$ for some idempotent $e \in R$. We will show that $r_R(S) = r_R(SR)$. Clearly, $r_R(SR) \subseteq r_R(S)$. Let $a \in r_R(S)$. Then $Sa = 0$. If R is IFP, then $Sa = 0$ implies $SRa = 0$. This yields $r_R(SR) \supseteq r_R(S)$, entailing $r_R(SR) = r_R(S)$. Thus R is (σ, δ)-Baer.

(2) follows (1), by letting $\delta = 0$. □

Every σ-rigid ring is reduced (hence IFP), and so we get the following by Theorem 2.1.

Corollary 2.2. *Let R be a σ-rigid ring with a σ-derivation δ. Then*
(1) *R is (σ, δ)-Baer if and only if R is (σ, δ)-quasi-Baer.*
(2) *R is σ-Baer if and only if R is σ-quasi-Baer.*

Corollary 2.3. *Let R be a reduced ring with a derivation δ. Then*
(1) *R is δ-Baer if and only if R is δ-quasi-Baer.*
(2) *R is Baer if and only if R is quasi-Baer.*

Proof. (1) follows Theorem 2.1, by letting $\sigma = 1$. (2) follows (1), by letting $\delta = 0$. □

Lemma 2.4. *Let R be a σ-rigid ring with a σ-derivation δ. Then the following statements are equivalent:*
(1) *R is right (σ, δ)-p.p.;*
(2) *R is (σ, δ)-p.p.;*
(3) *R is right (σ, δ)-p.q.-Baer;*
(4) *R is (σ, δ)-p.q.-Baer;*
(5) *For every (σ, δ)-element $a \in R$ and every positive integer n, $r_R(a^n R) = eR$ for some idempotent $e \in R$.*

Proof. Since R is σ-rigid, $r_R(a) = \ell_R(a) = r_R(aR) = \ell_R(Ra) = r_R(a^nR)$ for every (σ, δ)-element $a \in R$ and every positive integer n. Hence we have the result.

□

Corollary 2.5. *Let R be a σ-rigid ring. Then the following statements are equivalent:*
(1) *R is a right σ-p.p.-ring;*
(2) *R is a σ-p.p.-ring;*
(3) *R is a right σ-p.q.-Baer ring;*
(4) *R is a σ-p.q.-Baer ring;*
(5) *For every σ-element $a \in R$ and every positive integer n, $r_R(a^nR) = eR$ for some idempotent $e \in R$.*

Proof. The proof follows Lemma 2.4, by letting $\delta = 0$. □

Corollary 2.6. *Let R be a reduced ring. Then the following statements are equivalent:*
(1) *R is a right p.p.-ring;*
(2) *R is a p.p.-ring;*
(3) *R is a right p.q.-Baer ring;*
(4) *R is a p.q.-Baer ring;*
(5) *For every (σ, δ)-element $a \in R$ and every positive integer n, $r_R(a^nR) = eR$ for some idempotent $e \in R$.*

Proof. It follows from Lemma 2.4 by letting $\sigma = 1$ and $\delta = 0$. □

Lemma 2.7. *Let R be a ring with a σ-derivation δ. Then we have the followings:*
(1) *If I is a right (σ, δ)-ideal of R, then RI is a (σ, δ)-ideal of R;*
(2) *If I is a left (σ, δ)-ideal of R, then IR is a (σ, δ)-ideal of R.*

Proof. (1) Let I be a right (σ, δ)-ideal of R. Clearly RI is a ideal of R. Let $t \in RI$ be arbitrary. Then $t = \sum_{i=1}^{n} a_ib_i$ for some $a_i \in R, b_i \in I$ and some positive integer n. Since I is a (σ, δ)-set of R, $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$. For each i, $\delta(a_ib_i) = \sigma(a_i)\delta(b_i) + \delta(a_i)b_i \in RI$, and $\sigma(a_ib_i) = \sigma(a_i)\sigma(b_i) \in RI$. Thus $\sigma(t) = \sum_{i=1}^{n} \sigma(a_ib_i) \in RI$ and $\delta(t) = \sum_{i=1}^{n} \delta(a_ib_i) \in RI$. Hence RI is a (σ, δ)-ideal of R.

(2) follows the similar argument as given in the proof of (1). □

In [11], Hong, et al. obtained the following Lemma:

Lemma 2.8. *Let R be a σ-rigid ring with σ-derivation δ and $a, b \in R$. Then we have the followings.*
(1) *If $ab = 0$, then $a\sigma^n(b) = \sigma^n(a)b = 0$ for every positive integer n.*
(2) *If $ab = 0$, then $a\delta^n(b) = \delta^n(a)b = 0$ for every positive integer n.*
(3) *If $a\sigma^k(b) = \sigma^k(a)b = 0$ for some positive integer k, then $ab = 0$.*

3. Main results

Let R be a ring with a σ-derivation δ and let $A = R[x;\sigma,\delta]$. Consider maps $\bar{\sigma}, \bar{\delta} : A \to A$ defined by $\bar{\sigma}(f) = \sum_{j=0}^{n} \sigma(a_j)x^j$, $\bar{\delta}(f) = \sum_{j=0}^{n} \delta(a_j)x^j$ for $f = \sum_{j=0}^{n} a_j x^j \in A$ respectively. Then $\bar{\sigma}$ (resp. $\bar{\delta}$) is a well-defined one which extends σ (resp. δ). Observe that even though σ is an endomorphism of R, but $\bar{\sigma}$ is not necessarily an endomorphism of A. In case that $\sigma\delta = \delta\sigma$, we have the following.

Lemma 3.1. *Let R be a ring with an endomorphism σ and a σ-derivation δ, and let $A = R[x;\sigma,\delta]$. If $\sigma\delta = \delta\sigma$, then we have the followings:*

(1) $x^n a = \sum_{i=0}^{n} \binom{n}{i} \sigma^{n-i}\delta^i(a)x^{n-i}$ for all $a \in R$ and all nonnegative integer n;

(2) $xf = \bar{\sigma}(f)x + \bar{\delta}(f)$ for all $f \in A$;

(3) $\bar{\sigma}$ is an endomorphism of A;

(4) $\bar{\delta}$ is a $\bar{\sigma}$-derivation.

Proof. (1) It follows from the induction on n.

(2) Let $f = \sum_{j=0}^{n} a_j x^j \in A$. Then we have

$$\begin{aligned} xf &= \sum_{j=0}^{n} (xa_j)x^j = \sum_{j=0}^{n} \left(\sigma(a_j)x + \delta(a_j)\right)x^j \\ &= \left(\sum_{j=0+}^{n} \sigma(a_j)x^j\right) x + \sum_{j=0}^{n} \delta(a_j)x^j = \bar{\sigma}(f)x + \bar{\delta}(f). \end{aligned}$$

(3) Let $f = \sum_{j=0}^{n} a_j x^j, g = \sum_{k=0}^{m} b_k x^k \in A$. By (1), we have

$$fg = \sum_{j=0}^{n}\sum_{k=0}^{m} a_j(x^j b_k)x^k = \sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{j} \binom{j}{i} a_j\, \sigma^{j-i}\delta^i(b_k)\, x^{j-i+k}.$$

So we have

$$\begin{aligned} \bar{\sigma}(f)\bar{\sigma}(g) &= \sum_{j=0}^{n} \sigma(a_j)x^j \sum_{k=0}^{m} \sigma(b_k)x^k = \sum_{j=0}^{n}\sum_{k=0}^{m} \sigma(a_j)\, x^j \sigma(b_k)\, x^k \\ &= \sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{j} \binom{j}{i} \sigma(a_j)\, \sigma^{j-i+1}\delta^i(b_k)\, x^{j-i+k} \\ &= \sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{j} \binom{j}{i} \sigma\left(a_j\, \sigma^{j-i}\delta^i(b_k)\right) x^{j-i+k} = \bar{\sigma}(fg). \end{aligned}$$

This implies that $\bar{\sigma}$ is an endomorphism of A.

(4) Let $f=\sum_{j=0}^{n} a_j x^j, g=\sum_{k=0}^{m} b_k x^k \in A$. Then

$$\bar{\delta}(f)=\sum_{j=0}^{n}\delta(a_j)x^j,\ \bar{\delta}(g)=\sum_{j=0}^{m}\delta(b_k)x^k,$$

and so

$$\begin{aligned}
&\bar{\sigma}(f)\bar{\delta}(g)+\bar{\delta}(f)g\\
=&\sum_{j=0}^{n}\sigma(a_j)x^j\sum_{k=0}^{m}\delta(b_k)x^k+\sum_{j=0}^{n}\delta(a_j)x^j\sum_{k=0}^{m}b_kx^k\\
=&\sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{j}\binom{j}{i}\sigma(a_j)\sigma^{j-i}\delta^{i+1}(b_k)x^{j-i+k}\\
&+\sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{j}\binom{j}{i}\delta(a_j)\sigma^{j-i}\delta^{i}(b_k)x^{j-i+k}\\
=&\sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{j}\binom{j}{i}\sigma(a_j)\left(\sigma^{j-i}\delta^{i+1}(b_k)+\delta(a_j)\sigma^{j-i}\delta^{i}(b_k)\right)x^{j-i+k}\\
=&\sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{j}\binom{j}{i}\delta(a_j\sigma^{j-i}\delta^{i}(b_k))x^{j-i+k}\\
=&\bar{\delta}\left(\sum_{j=0}^{n}\sum_{k=0}^{m}a_j(x^jb_k)x^k\right)=\bar{\delta}(fg).
\end{aligned}$$

Thus $\bar{\delta}$ is a $\bar{\sigma}$-derivation of A. □

This lemma will do a basic role in the following arguments.

Lemma 3.2. *Let $A=R[x;\sigma,\delta]$ be an Ore extension over a ring R with a δ-derivation δ. If $\sigma\delta=\delta\sigma$, then $\bar{\sigma}$ is the unique extended endomorphism of σ such that $\bar{\sigma}(x)=x$, and $\bar{\delta}$ is the unique extended $\bar{\sigma}$-derivation of δ such that $\bar{\delta}(x)=0$.*

Proof. It is obvious that $\bar{\sigma}$ is the unique extended endomorphism of σ such that $\bar{\sigma}(x)=x$. Let δ^* be an extended $\bar{\sigma}$-derivation of δ such that $\delta^*(x)=0$. Then by using the induction on n, it follows that $\delta^*(x^n)=0$, and $\delta^*(ax^n)=\delta(a)x^n$ for all a in R. Let $f=a_nx^n+\cdots+a_1x+a_0$ in R. Then

$$\delta^*(f)=\delta^*\left(a_nx^n+\cdots+a_1x+a_0\right)=\delta(a_n)x^n+\cdots+\delta(a_1)x+\delta(a_0)=\bar{\delta}(f)$$

Consequently, $\delta^*=\bar{\delta}$. □

Lemma 3.3. *Let R be a ring with an endomorphism σ and a σ-derivation δ and let $A=R[x;\sigma,\delta]$. Then R is σ-rigid if and only if A is $\bar{\sigma}$-rigid. In this case, $\sigma(e)=e$ and $\delta(e)=0$ for every idempotent $e\in R$.*

Proof. Assume that R is σ-rigid and A is not $\bar{\sigma}$-rigid. Then there exists a nonzero $f \in A$ such that $f\bar{\sigma}(f) = 0$. Since R is σ-rigid, $f \notin R$. Let $f = \sum_{i=0}^{n} a_i x^i$ where $a_i \in R$, and $a_n \neq 0$. It follows that $a_n \sigma^2(a_n) = 0$ since $f\bar{\sigma}(f) = 0$. Since R is σ-rigid, $a_n^2 = 0$ by Lemma 2.8, and then $a_n = 0$ because R is reduced, a contradiction. Hence A is $\bar{\sigma}$-rigid. The converse is true by the definition of extended endomorphism $\bar{\sigma}$ of σ. Let e be an idempotent of R. In case that A is $\bar{\sigma}$-reduced (and then A is reduced), e is a central idempotent in A, and so $ex = xe = \sigma(e)x + \delta(e)$, which implies that $\sigma(e) = e$ and $\delta(e) = 0$. (Also refer [11, Proposition 5].) □

If J is an ideal of $A = R[x;\sigma,\delta]$, we denote by J_c the set of all coefficients of polynomials in J.

Lemma 3.4. *Let R be a ring with an endomorphism σ and a σ-derivation δ and let $A = R[x;\sigma,\delta]$. Assume that $\sigma\delta = \delta\sigma$. Then we have the followings:*
(1) *If J is a $\bar{\sigma}$-ideal of A, then J is a $\bar{\delta}$-ideal of A.*
(2) *If J is a $\bar{\sigma}$-ideal of A, then J_c is a (σ,δ)-ideal of R.*
(3) *If I is a (σ,δ)-ideal of R, then IA is a $\bar{\sigma}$-ideal of A.*

Proof. (1) Let J be a $\bar{\sigma}$-ideal of A. Then for every $f \in J$, $xf - \bar{\sigma}(f)x = \bar{\delta}(f) \in J$, and so J is a $\bar{\delta}$-ideal of A.

(2) Let J be a $\bar{\sigma}$-ideal of A. Clearly, J_c is an ideal of R. To show J_c is (σ,δ)-ideal of R, let $f = a_0 + a_1 x + + \cdots + a_n x^n \in J$ be arbitrary. Then $a_0, a_1, \ldots, a_n \in J_c$. Since J is a $\bar{\sigma}$-ideal of A, $xf - \bar{\sigma}(f)x \in J$. By Lemma 3.1, we get

$$xf - \bar{\sigma}(f)x = \bar{\delta}(f) = \delta(a_0) + \delta(a_1)x + \cdots + \delta(a_n)x^n \in J,$$

and so $\delta(a_0), \delta(a_1) \ldots, \delta(a_n) \in J_c$. Thus J_c is a δ-ideal of R. Since $\bar{\sigma}(f) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n \in J$, and so $\sigma(a_0), \sigma(a_1), \ldots, \sigma(a_b) \in J_c$. Thus J_c is a σ-ideal of R. Therefore J_c is (σ,δ)-ideal of R.

(3) Clearly, IA is a right ideal of A. Since I is a (σ,δ)-ideal of R, $fb \in I[x;\sigma,\delta]$ for all $b \in I$ and all $f \in A$. Thus $AI \subseteq I[x;\sigma,\delta]$. Since I is a (σ,δ)-ideal of R, $I[x;\sigma,\delta]A \subseteq IA$, and so IA is a left ideal of R. On the other hand, since I is a σ-ideal of R, $\bar{\sigma}(bf) = \sigma(b)\bar{\sigma}(f) \in IA$ for all $b \in I$ and all $f \in A$, and so IA is a $\bar{\sigma}$-ideal of A.

□

Theorem 3.5. *Let R be a σ-rigid ring with a σ-derivation δ and let $A = R[x;\sigma,\delta]$. Assume that $\sigma\delta = \delta\sigma$. Then the followings are equivalent:*
(1) *R is (σ,δ)-quasi-Baer;*
(2) *A is $\bar{\sigma}$-quasi-Baer;*
(3) *A is $(\bar{\sigma},\bar{\delta})$-quasi-Baer.*

Proof. (1) ⇒ (2): Suppose that R is (σ,δ)-quasi-Baer. Let J be an arbitrary $\bar{\sigma}$-ideal of A. If $g \in r_A(J)$, then $fg = 0$ for all $f \in J$. Let $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{j=0}^{m} b_j x^j$. Then $a_i b_j = 0$ for all $0 \leq i \leq n, 0 \leq j \leq m$ by [11, Proposition

6]. Consider the set J_c of all coefficients of polynomials in J. Then J_c is a (σ, δ)-ideal of R. Since $a_ib_j = 0$ for all $0 \le i \le n, 0 \le j \le m$, $b_j \in r_R(J_c)$ for all $0 \le j \le m$. Since R is (σ, δ)-quasi-Baer, $r_R(J_c) = e_1R$ for some idempotent $e_1 \in R$. We will show that $r_A(J) = e_1A$. Since $a_i \in J_c$ and $a_ib_j = 0$ for all $0 \le i \le n, 0 \le j \le m$, $b_j \in r_R(J_c) = e_1R$ for all j, and so $b_j = e_1c_j$ for some $c_j \in R$ for all j. Hence $g = e_1 \sum_{j=0}^{m} c_jx^j \in e_1A$, yielding that $r_A(J) \subseteq e_1A$. To show $e_1A \subseteq r_A(J)$, let $h \in e_1A$ be arbitrary. Then $h = \sum_{k=0}^{\ell}(e_1d_k)x^k$ for some $\sum_{k=0}^{\ell} d_kx^k \in A$. Since $a_i \in J_c$ for all $0 \le i \le n$ and $e_1 \in r_R(J)$, $a_ie_1 = 0$ for all $0 \le i \le n$, and so $a_i(e_1d_k) = 0$ for all $0 \le i \le n, 0 \le k \le \ell$. Thus $fh = 0$ by [11, Proposition 6], yielding $e_1A \subseteq r_A(J)$.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Suppose that (3) holds. Let I be a (σ, δ)-ideal of R. Then $IA = I[x; \sigma, \delta]$ is a $(\bar{\sigma}, \bar{\delta})$-ideal of A by Lemma 3.4. Since A is $(\bar{\sigma}, \bar{\delta})$-quasi-Baer, $r_A(IA) = e(x)A$ for some idempotent $e(x) \in A$. We will show that $e(x) \in R$. To show this, let $e(x) = e_0 + e_1x + \cdots + e_nx^n$. If $n \ge 1$, then we have $e_n\sigma(e_n) = 0$ from $e(x)^2 = e(x)$. Since R is σ-rigid, $e_n = 0$. Continuing in this way, we get $e_n = e_{n-1} = \cdots = e_1 = 0$. Hence $e(x) = e_0 \in R$. Since $r_R(I) = R \cap r_A(IA) = R \cap e(x)A = R \cap e_0A = e_0R$, R is (σ, δ)-Baer. □

The following is shown by combining Theorem 3.5 and the results above.

Corollary 3.6. *Let R be a σ-rigid ring with a σ-derivation δ and let $A = R[x; \sigma, \delta]$. Assume that $\sigma\delta = \delta\sigma$. Then the followings are equivalent:*
(1) *R is (σ, δ)-Baer;*
(2) *R is (σ, δ)-quasi-Baer;*
(3) *A is $\bar{\sigma}$-Baer;*
(4) *A is $\bar{\sigma}$-quasi-Baer;*
(5) *A is $(\bar{\sigma}, \bar{\delta})$-Baer;*
(6) *A is $(\bar{\sigma}, \bar{\delta})$-quasi-Baer.*

Proof. (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) follows from Theorem 3.5, (1) $\Leftrightarrow$ (2) follows from Theorem 2.1. (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (6) follow from Theorem 2.1 and Lemma 3.4. □

Next we consider the case of *p.p.* rings.

Theorem 3.7. *Let R be a σ-rigid ring with a σ-derivation δ and let $A = R[x; \sigma, \delta]$. Assume that $\sigma\delta = \delta\sigma$. Then the followings are equivalent:*
(1) *R is (σ, δ)-p.p.;*
(2) *A is $\bar{\sigma}$-p.p.;*
(3) *A is $(\bar{\sigma}, \bar{\delta})$-p.p..*

Proof. (1) $\Rightarrow$ (2): Suppose that R is a (σ, δ)-*p.p.*. Let $f = \sum_{i=0}^{n} a_ix^i$ be an arbitrary $\bar{\sigma}$-element of A. Then $\sigma(a_i) = a_i$ for each $i = 0, \cdots, n$, i.e., each a_i is σ-element of R. Since R is (σ, δ)-*p.p.*, there exists some idempotent $e_i \in R$ such that $r_R(a_i) = e_iR$ for each i. Let $e = e_1 \cdot e_2 \cdots e_n$. Then $e^2 = e \in R$, and then $eR = \cap_{i=0}^{n} r_R(a_i)$. By Lemma 3.3, $\sigma(e) = e$ and $\delta(e) = 0$, and thus $fe = \sum_{i=0}^{n} a_i x^i = 0$. Hence $eA \subseteq r_A(f)$. Let $g = \sum_{j=0}^{m} b_j x^j \in r_A(f)$ be arbitrary. Then $fg = 0$, and so $a_i b_j = 0$ for all $0 \leq i \leq n, 0 \leq j \leq m$ by [[11], Proposition 6]. Thus $b_j \in \cap_{i=0}^{n} r_R(a_i) = eR$ for all j, which implies that $g \in eA$, and thus $r_A(f) \subseteq eA$. So we have $eA = r_A(f)$. Therefore, A is $\bar{\sigma}$-*p.p.*.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Suppose that A is $(\bar{\sigma}, \bar{\delta})$-*p.p.*. Let a be an arbitrary (σ, δ)-element of R, i.e., $\sigma(a) = a$ and $\delta(a) = a$. Note $\bar{\sigma}(a) = \sigma(a)$ and $\bar{\delta}(a) = \delta(a)$, i.e., a is $(\bar{\sigma}, \bar{\delta})$-element of A. Since A is $(\bar{\sigma}, \bar{\delta})$-*p.p.*, $r_A(a) = eA$ for some idempotent $e \in A$. By the similar argument given in the proof of Theorem 3.5, $e \in R$. Since $r_R(a) = R \cap r_A(a) = R \cap eA = eR$, R is (σ, δ)-*p.p.*. □

The following is shown by combining Theorem 3.7 and the results above.

Corollary 3.8. *Let R be a σ-rigid ring with σ-derivation δ and let $A = R[x; \sigma, \delta]$. Assume that $\sigma\delta = \delta\sigma$. Then the followings are equivalent:*
(1) *R is (σ, δ)-p.p.;*
(2) *R is (σ, δ)-p.q.-Baer;*
(3) *A is $\bar{\sigma}$-p.p.;*
(4) *A is $\bar{\sigma}$-p.q.-Baer;*
(5) *A is $(\bar{\sigma}, \bar{\delta})$-p.p.;*
(6) *A is $(\bar{\sigma}, \bar{\delta})$-p.q.-Baer.*

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) follows from Theorem 3.7. Meanwhile (1) $\Leftrightarrow$ (2) and (5) $\Leftrightarrow$ (6) follow from Lemma 2.4 and Lemma 3.3. (3) $\Leftrightarrow$ (4) follows from Corollary 2.5 and Lemma 3.3. □

Let R be a ring with an endomorphism σ. The set $\{x_j\}_{j \geq 0}$ is easily seen to be a left Ore subset of $R[x; \sigma]$, so that one can localize $R[x; \sigma]$ and form an extension ring that is usually called the *skew Laurent polynomial* ring, written by $R[x, x^{-1}; \sigma]$. Note that every element of $R[x, x^{-1}; \sigma]$ is a finite sum of elements of the form $x^{-j} r x^i$, where $r \in R$ and i, j are nonnegative integers. A ring R is σ-rigid if and only if $R[x, x^{-1}; \sigma]$ is reduced by [17, Theorem 3].

Let $B = R[x, x^{-1}; \sigma]$. Consider maps $\bar{\sigma} : B \to B$ defined by $\bar{\sigma}(f) = \sum_{\text{finite}} x^{-j} \sigma(r) x^i$ for $f = \sum_{\text{finite}} x^{-j} r x^i \in B$. Then $\bar{\sigma}$ is a well-defined one which extends σ to B.

Now we can obtain the following, by applying the arguments related to Corollaries 3.6,3.8, and the results in [17]. The following is similar to Corollary 3.6.

Proposition 3.9. *Let R be a σ-rigid ring and $B = R[x, x^{-1}; \sigma]$. Then the following conditions are equivalent:*
(1) *R is σ-Baer;*
(2) *R is σ-quasi-Baer;*
(3) *B is $\bar{\sigma}$-Baer;*
(4) *B is $\bar{\sigma}$-quasi-Baer.*

The following is similar to Corollary 3.8.

Proposition 3.10. *Let R be a σ-rigid ring and $B = R[x, x^{-1}; \sigma]$. Then the following conditions are equivalent:*
(1) *R is σ-p.p.;*
(2) *R is σ-p.q.-Baer;*
(3) *B is $\bar{\sigma}$-p.p.;*
(4) *B is $\bar{\sigma}$-p.q.-Baer.*

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