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OPTIMAL CONTROL FOR SELF-ORGANIZING TARGET DETECTION MODEL IN THE 1D CASE

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Abstract. This paper is concerned with the optimal control problem associated to the self-organizing target detection model in 1D domains. That is, we show the global existence of weak solution and the existence of optimal control.

1. Introduction

In this paper we consider the following optimal control problem:

$$(\mathbf{P}) \qquad \text{minimize} \quad J(u,v)$$

with the cost functional $J(u,v)$ of the form

$$J(u,v) = \int_0^T \|y(u,v) - y_d\|^2_{H^1(I)} dt + \gamma \int_0^T \left(\|u\|^2_{H^1(I)} + \|v\|^2_{H^1(I)}\right) dt,$$

where $y = y(u,v)$, $\rho = \rho(u,v)$ and $w = w(u,v)$ are governed by

$$\begin{aligned}
&\frac{\partial y}{\partial t} = a_1 \frac{\partial^2 y}{\partial x^2} - \frac{\partial}{\partial x}\Big[y\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big)\Big] \qquad \text{in } I \times (0,T],\\
&\frac{\partial \rho}{\partial t} = a_2 \frac{\partial^2 \rho}{\partial x^2} + g_1 T(x) y - d\rho + u \qquad \text{in } I \times (0,T],\\
&\frac{\partial w}{\partial t} = a_3 \frac{\partial^2 w}{\partial x^2} + g_2 y - hw + v \qquad \text{in } I \times (0,T],\\
&\frac{\partial y}{\partial x} = \frac{\partial \rho}{\partial x} = \frac{\partial w}{\partial x} = 0 \qquad \text{on } \partial I \times (0,T],\\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x), \quad w(x,0) = w_0(x) \qquad \text{in } I.
\end{aligned} \tag{1.1}$$

Here, $I = (0, L)$ is a bounded interval in $\mathbb{R}$. $y = y(x,t)$ is the density of bioparticles in I at time t. $\rho = \rho(x,t)$ is the concentration of chemical attractants

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in I at time t. $w = w(x,t)$ is the concentration of chemical repellents in I at time t. $-\frac{\partial}{\partial x}\left[y\left(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\right)\right]$ indicates that bioparticles are affected by chemical attractants and chemical repellents. $\chi_1(\rho)$ and $\chi_2(w)$ are the sensitivity functions of biopatricles to chemical attractants and chemical repellents. $g_1 T(x) y$ indicates that the bioparticles produce chemical attractant when they find the target $T(x)$. $g_2 y$ indicates that bioparticles release chemical repellents. $-d\rho$ is decay rate of chemical attractants. $-hw$ is decay rate of chemical repellents. u and v are the control functions.

We assume that $\chi_1(\rho)$ and $\chi_2(w)$ are real smooth functions satisfying

$$\sup_{0\le\rho<\infty}\left|\frac{d^i\chi_1(\rho)}{d\rho^i}\right| < \infty \text{ and } \sup_{0\le w<\infty}\left|\frac{d^i\chi_2(w)}{dw^i}\right| < \infty \text{ for } i = 1, 2, \tag{1.2}$$

and $T(x)$ satisfies

$$0 \le T(x) \le 1 \text{ and } T(x) \in H^1(I). \tag{1.3}$$

The model (1.1) was introduced by Okaie et al. [3] to develop a mathematical model of mobile bionanosensor networks for target tracking. (1.1) was influenced by Keller-Segel equations [2]. In [1], author showed that the global existence of the $\mathcal{C}^1$ solution for (1.1) and the existence of exponential attractors in 1D domains. In [5], Ryu and Yagi studied the optimal control problem governed by Keller-Segel equations. In this paper we consider the optimal control problem for (1.1) in 1D domains. That is to say, we show the global existence of weak solution and the existence of the optimal control.

2. Local existence of weak solution

In this section, we will show local existence of weak solution to model (1.1). Let $A_1 = -a_1\frac{\partial^2}{\partial x^2} + 1$ and $A_2 = -a_2\frac{\partial^2}{\partial x^2} + d$ and $A_3 = -a_3\frac{\partial^2}{\partial x^2} + h$ with the same domain $\mathcal{D}(A_i) = H^2_n(I) = \{z \in H^2(I); \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(L) = 0\}$ $(i = 1, 2, 3)$.

We set three product Hilbert spaces $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$ as

$$\mathcal{V} = H^1(I) \times H^2_n(I) \times H^2_n(I), \quad \mathcal{H} = L^2(I) \times H^1(I) \times H^1(I),$$

and

$$\mathcal{V}' = (H^1(I))' \times L^2(I) \times L^2(I).$$

Also, we set a symmetric bilinear form on $\mathcal{V} \times \mathcal{V}$:

$$a(Y, \widetilde{Y}) = a_1 \int_I \frac{dy}{dx}\frac{d\tilde{y}}{dx}dx + \int_I y\tilde{y}dx + \left(A_2\rho, A_2\tilde{\rho}\right)_{L^2(I)} + \left(A_3 w, A_3\tilde{w}\right)_{L^2(I)},$$

where $Y = \begin{pmatrix} y \\ \rho \\ w \end{pmatrix}, \widetilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \\ \tilde{w} \end{pmatrix} \in \mathcal{V}$. Obviously, the form satisfies

$$|a(Y,\widetilde{Y})| \leq M\|Y\|_{\mathcal{V}}\|\widetilde{Y}\|_{\mathcal{V}}, \quad Y,\widetilde{Y} \in \mathcal{V}, \tag{a.i}$$

$$a(Y,Y) \geq \delta\|Y\|_{\mathcal{V}}^2, \quad Y \in \mathcal{V} \tag{a.ii}$$

with constants $\delta, M > 0$.

Then, this form defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$. We consider the following semilinear problem

$$\frac{dY}{dt} + AY = F(Y) + U(t), \quad 0 < t \leq T,$$
$$Y(0) = Y_0$$

in the space $\mathcal{V}'$. Here, $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F(Y) = \begin{pmatrix} y - \frac{\partial}{\partial x}\Big[y\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big)\Big] \\ g_1 T(x) y \\ g_2 y \end{pmatrix}.$$

Y_0 is defined by $Y_0 = \begin{pmatrix} y_0 \\ \rho_0 \\ w_0 \end{pmatrix}$ and $U(t) = \begin{pmatrix} 0 \\ u(t) \\ v(t) \end{pmatrix}$.

Then, $F(\cdot)$ is continuous function from $\mathcal{V}$ to $\mathcal{V}'$ satisfying for each $\eta > 0$, there exists an increasing continuous function $\phi_\eta, \psi_\eta : [0,\infty) \to [0,\infty)$ such that

$$\|F(Y)\|_{\mathcal{V}'} \leq \eta\|Y\|_{\mathcal{V}} + \phi_\eta(\|Y\|_{\mathcal{H}}), \quad Y \in \mathcal{V} \tag{2.1}$$

and

$$\begin{aligned} \|F(\widetilde{Y}) - F(Y)\|_{\mathcal{V}'} &\leq \eta\|\widetilde{Y} - Y\|_{\mathcal{V}} \\ &+ (\|\widetilde{Y}\|_{\mathcal{V}} + \|Y\|_{\mathcal{V}} + 1)\psi_\eta(\|\widetilde{Y}\|_{\mathcal{H}} + \|Y\|_{\mathcal{H}})\|\widetilde{Y} - Y\|_{\mathcal{H}}, \quad \widetilde{Y}, Y \in \mathcal{V}. \end{aligned} \tag{2.2}$$

Indeed, since $\chi_i(\cdot)(i = 1,2)$ are smooth function, we obtain that

$$\|\chi_i(\rho)\|_{H^1(I)} \leq p(\|\rho\|_{H^1(I)}), \quad \rho \in H^1(I), \tag{2.3}$$

$$\|\chi_i(\rho)\|_{H^2(I)} \leq p(\|\rho\|_{H^1(I)})(\|\rho\|_{H^2(I)} + 1) \quad \rho \in H^2(I) \tag{2.4}$$

and

$$\begin{aligned} &\|\chi_i(\rho_1) - \chi_i(\rho_2)\|_{H^1(I)} \\ &\quad \leq p(\|\rho_1\|_{H^1(I)} + \|\rho_2\|_{H^1(I)})\|\rho_1 - \rho_2\|_{H^1(I)}, \quad \rho_1, \rho_2 \in H^1(I), \end{aligned} \tag{2.5}$$

where $p(\cdot)$ is some continuous increasing function(see [4], [6]).

By using (2.3), we obtain

$$\begin{aligned}\Big\| \frac{\partial}{\partial x}\Big[y\frac{\partial}{\partial x}\chi_1(\rho)\Big]\Big\|_{(H^1(1))'} &\le C\|y\|_{L^\infty(I)}\|\chi_i(\rho)\|_{H^1(I)} \\ &\le C\|y\|^{1/2}_{L^2(I)}\|y\|^{1/2}_{H^1(I)}p(\|\rho\|_{H^1(I)}) \le \epsilon\|y\|_{H^1(I)} + C_\epsilon\|y\|_{L^2(I)}p(\|\rho\|_{H^1(I)})\end{aligned}$$

with an arbitary $\epsilon > 0$. And by using (1.3), we have

$$\|T(x)y\|_{L^2(I)} \le C\|T(x)\|_{L^\infty(I)}\|y\|_{L^2(I)} \le C\|y\|_{L^2(I)}.$$

Therefore, (2.1) is satisfied. Similarily, by (1.3), (2.4) and (2.5) we obtain

$$\begin{aligned}&\Big\| \frac{\partial}{\partial x}\Big[y_1\frac{\partial}{\partial x}\chi_1(\rho_1) - y_2\frac{\partial}{\partial x}\chi_1(\rho_2)\Big]\Big\|_{(H^1(1))'} \\ &\le C\|y_1 - y_2\|_{L^2(I)}\|\chi_i(\rho_1)\|_{H^2(I)} + C\|y_2\|_{H^1(I)}\|\chi_i(\rho_1) - \chi_i(\rho_2)\|_{H^1(I)} \\ &\le C\|y_1 - y_2\|_{L^2(I)}p(\|\rho_1\|_{H^1(I)})(\|\rho_1\|_{H^2(I)} + 1) \\ &\quad + \|y_2\|_{H^1(I)}p(\|\rho_1\|_{H^1(I)} + \|\rho_2\|_{H^1(I)})\|\rho_1 - \rho_2\|_{H^1(I)}\end{aligned}$$

and

$$\|T(x)(y_1 - y_2)\|_{L^2(I)} \le C\|T(x)\|_{L^\infty(I)}\|y_1 - y_2\|_{L^2(I)} \le C\|y_1 - y_2\|_{L^2(I)}.$$

Hence, the condition (2.2) is fulfilled.

Then, we obtain the following result.

Theorem 2.1. *For any $Y_0 \in \mathcal{H}$ with $y_0, \rho_0, w_0 \ge 0$ and $U \in L^2(0, T; \mathcal{V}')$ with $u, v \ge 0$, (1.1) has a unique local weak solution*

$$\begin{aligned}&0 \le y \in H^1(0, T(Y_0, U); (H^1(I))') \cap \mathcal{C}([0, T(Y_0, U)]; L^2(I)) \cap L^2(0, T(Y_0, U); H^1(I)), \\ &0 \le \rho \in H^1(0, T(Y_0, U); L^2(I)) \cap \mathcal{C}([0, T(Y_0, U)]; H^1(I)) \cap L^2(0, T(Y_0, U); H^2_n(I)), \\ &0 \le w \in H^1(0, T(Y_0, U); L^2(I)) \cap \mathcal{C}([0, T(Y_0, U)]; H^1(I)) \cap L^2(0, T(Y_0, U); H^2_n(I)),\end{aligned}$$

the number $T(Y_0, U) \in (0, T]$ is determined by the norms $\|Y_0\|_{\mathcal{H}}$, $\|U\|_{L^2(0,T;\mathcal{V}')}$.

Proof. Since (a.i), (a.ii), (2.1) and (2.2) are satisfied, we obtain from ([5, Theorem 2,1]) that for any $Y_0 \in \mathcal{H}$ and $U \in L^2(0, T; \mathcal{V}')$, (1.1) has a unique local weak solution

$$Y \in H^1(0, T(Y_0, U); \mathcal{V}') \cap \mathcal{C}([0, T(Y_0, U)]; \mathcal{H}) \cap L^2(0, T(Y_0, U); \mathcal{V}),$$

the number $T(Y_0, U) \in (0, T]$ is determined by the norms $\|Y_0\|_{\mathcal{H}}$, $\|U\|_{L^2(0,T;\mathcal{V}')}$.

The remaining part is the positivity of the solutions. Denote $y^- = \max(-y, 0)$. Multiply the first equation of (1.1) by $-y^-$ and integrate the product in I. Then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I |y^-|^2 dx + a_1 \int_I \Big|\frac{\partial y^-}{\partial x}\Big|^2 dx \\
&= \int_I \frac{\partial}{\partial x}\Big[y\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big)\Big] y^- dx \\
&= \int_I \frac{\partial y^-}{\partial x} y\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big) dx \\
&\le \Big(\int_I \Big|\frac{\partial y^-}{\partial x}\Big|^2 dx\Big)^{1/2} \Big(\int_I |y^-|^2 \Big|\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big)\Big|^2 dx\Big)^{1/2} \\
&\le \epsilon \int_I \Big|\frac{\partial y^-}{\partial x}\Big|^2 dx + C_\epsilon \int_I |y^-|^2 dx \Big\|\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big)\Big\|^2_{L^\infty(I)} \\
&\le \epsilon \int_I \Big|\frac{\partial y^-}{\partial x}\Big|^2 dx + C_\epsilon \int_I |y^-|^2 dx \Big(\Big\|\frac{\partial}{\partial x}\chi_1(\rho)\Big\|^2_{H^1(I)} + \Big\|\frac{\partial}{\partial x}\chi_2(w)\Big\|^2_{H^1(I)}\Big) \\
&\le \epsilon \int_I \Big|\frac{\partial y^-}{\partial x}\Big|^2 dx + \tilde{p}(\|\rho\|_{H^1(I)} + \|w\|_{H^1(I)})(\|\rho\|^2_{H^2(I)} + \|w\|^2_{H^2(I)} + 1)\int_I |y^-|^2 dx,
\end{aligned}$$

where $\tilde{p}(\cdot)$ is some continuous increasing function. If we take $\epsilon = \frac{a_1}{2}$, then we obtain

$$\frac{d}{dt}\int_I |y^-|^2 dx \le \tilde{p}(\|\rho\|_{H^1(I)} + \|w\|_{H^1(I)})(\|\rho\|^2_{H^2(I)} + \|w\|^2_{H^2(I)} + 1)\int_I |y^-|^2 dx.$$

By using Gronwall's lemma, we have

$$\int_I |y^-|^2 dx \le \|y^-(0)\|^2_{L^2(I)} e^{\int_0^S \tilde{p}(\|\rho\|_{H^1(I)} + \|w\|_{H^1(I)})(\|\rho\|^2_{H^2(I)} + \|w\|^2_{H^2(I)} + 1)dt}.$$

Since $y^-(0) = 0$, we obtain $y \ge 0$. Similarily, multiply the second equation of (1.1) by $-\rho^-$ and integrate the product in I. Then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I |\rho^-|^2 dx + a_1 \int_I \Big|\frac{\partial \rho^-}{\partial x}\Big|^2 dx \\
&\qquad = -\int_I g_1 T(x) y \rho^- dx + d\int_I |\rho^-|^2 dx - \int_I u\rho^- dx.
\end{aligned}$$

Since $y \ge 0$, $T(x) \ge 0$ and $u \ge 0$, we obtain

$$\frac{d}{dt}\int_I |\rho^-|^2 dx \le d\int_I |\rho^-|^2 dx.$$

By using Gronwall's lemma, we have

$$\int_I |\rho^-|^2 dx \le C\|\rho^-(0)\|^2_{L^2(I)}.$$

Since $\rho^-(0) = 0$, we obtain $\rho \ge 0$. By using similar method for w. we can obtain $w \ge 0$. $\square$

3. Global existence of weak solution

In this section we will prove the global existence of weak solution to (1.1).

Theorem 3.1. *For any* $(y_0, \rho_0, w_0) \in L^2(I) \times H^1(I) \times H^1(I)$ *with* $y_0, \rho_0, w_0 \geq 0$, *and* $0 \leq u, v \in L^2(0,T;H^1(I))$, *(1.1) has a unique global weak solution*

$$0 \leq y \in H^1(0,T;(H^1(I))') \cap \mathcal{C}([0,T];L^2(I)) \cap L^2(0,T;H^1(I)),$$
$$0 \leq \rho \in H^1(0,T;L^2(I)) \cap \mathcal{C}([0,T];H^1(I)) \cap L^2(0,T;H^2_n(I)),$$
$$0 \leq w \in H^1(0,T;L^2(I)) \cap \mathcal{C}([0,T];H^1(I)) \cap L^2(0,T;H^2_n(I)).$$

Proof. Let y, ρ and w be any nonnegative weak solution as in Theorem 2.1 on an interval $[0,S]$.

Step 1. Integrate the first equation of (1.1). Then, we have

$$\frac{d}{dt}\|y\|_{L^1(I)} = 0.$$

That is,

$$\|y(t)\|_{L^1(I)} = \|y_0\|_{L^1(I)}, \quad 0 \leq t \leq S.$$

Step 2. Consider the following linear equation:

$$\frac{d\rho}{dt} + A_2\rho = g_1 T(x)y + u, \quad 0 < t \leq S,$$
$$\rho(0) = \rho_0$$

in the space $(H^1(I))'$. Here, A_2 is a positive definite self-adjoint operator from $H^1(I)$ to $(H^1(I))'$. e^{-tA_2} is an analytic semigroup generated by A_2 on $H^1(I)$ with the estimate $\|e^{-tA_2}\|_{\mathcal{L}((H^1(I))')} \leq Ce^{-dt}$, $0 \leq t < \infty$(see [6]). Then, ρ is represented by

$$\rho(t) = e^{-tA_2}\rho_0 + g_1\int_0^t e^{-(t-s)A_2}T(x)y(s)ds + \int_0^t e^{-(t-s)A_2}u(s)ds$$

and

$$A_2\rho(t) = e^{-tA_2}A_2\rho_0 + g_1\int_0^t A_2^{\frac{7}{8}}e^{-\frac{(t-s)}{2}A_2}e^{-\frac{(t-s)}{2}A_2}A_2^{\frac{1}{8}}T(x)y(s)ds$$
$$+ \int_0^t e^{-(t-s)A_2}A_2u(s)ds.$$

Then, we have

$$\|A_2\rho(t)\|_{(H^1(I))'} \leq C\Big[e^{-dt}\|A_2\rho_0\|_{(H^1(I))'}$$
$$+ \int_0^t (t-s)^{-\frac{7}{8}}e^{-\frac{d}{2}(t-s)}\|T(x)y(s)\|_{L^1(I)}ds + \int_0^t e^{-d(t-s)}\|A_2u(s)\|_{(H^1(I))'}ds\Big].$$

Therefore, we obtain from $\|T(x)y(s)\|_{L^1(I)} \le \|T(x)\|_{L^\infty(I)}\|y(s)\|_{L^1(I)} \le \|y_0\|_{L^1(I)}$ and Hölder inequality that

$$\begin{aligned}&\|\rho(t)\|_{H^1(I)}\\ &\quad\le C\Big[e^{-dt}\|\rho_0\|_{H^1(I)} + \|y_0\|_{L^1(I)} + \|u(t)\|_{L^2(0,T;H^1(I))}\Big], \quad 0\le t\le S. \end{aligned}\tag{3.1}$$

By using similar method, we have

$$\begin{aligned}&\|w(t)\|_{H^1(I)}\\ &\quad\le C\Big[e^{-dt}\|w_0\|_{H^1(I)} + \|y_0\|_{L^1(I)} + \|v(t)\|_{L^2(0,T;H^1(I))}\Big], \quad 0\le t\le S. \end{aligned}\tag{3.2}$$

Step 3. Multifly the second equation of (1.1) by ρ and integrate the product in I. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_I \rho^2dx + a_2\int_I\Big|\frac{\partial\rho}{\partial x}\Big|^2dx + d\int_I\rho^2dx &= g_1\int_I T(x)y\rho dx + \int_I u\rho dx\\ &\le C(g_1\|T(x)\|_{L^\infty(I)}\int_I y^2dx + \int_I u^2dx) + \frac{d}{2}\int_I\rho^2dx.\end{aligned}$$

Multifly the second equation of (1.1) by $\frac{\partial^2\rho}{\partial x^2}$ and integrate the product in I. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_I\Big|\frac{\partial\rho}{\partial x}\Big|^2dx + a_2\int_I\Big|\frac{\partial^2\rho}{\partial x^2}\Big|^2dx + d\int_I\Big|\frac{\partial\rho}{\partial x}\Big|^2dx &= g_1\int_I T(x)y\frac{\partial^2\rho}{\partial x^2}dx + \int_I u\frac{\partial^2\rho}{\partial x^2}dx\\ &\le C(g_1\|T(x)\|_{L^\infty(I)}\int_I y^2dx + \int_I u^2dx) + \frac{a_2}{2}\int_I\Big|\frac{\partial^2\rho}{\partial x^2}\Big|^2dx.\end{aligned}$$

If we take $k_1 = \min\{a_2, d\}$. we obtain

$$\frac{d}{dt}\|\rho(t)\|^2_{H^1(I)} + k_1\|\rho(t)\|^2_{H^1(I)} + k_1\int_I\Big|\frac{\partial^2\rho}{\partial x^2}\Big|^2dx \le C(\|y\|^2_{L^2(I)} + \|u\|^2_{H^1(I)}). \tag{3.3}$$

Multifly the third equation of (1.1) by w and integrate the product in I. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_I w^2dx + a_3\int_I\Big|\frac{\partial w}{\partial x}\Big|^2dx + h\int_I w^2dx &= g_2\int_I ywdx + \int_I vwdx\\ &\le C(\int_I y^2dx + \int_I v^2dx) + \frac{h}{2}\int_I w^2dx.\end{aligned}$$

Multifly the third equation of (1.1) by $\frac{\partial^2 w}{\partial x^2}$ and integrate the product in I. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_I\Big|\frac{\partial w}{\partial x}\Big|^2dx + a_3\int_I\Big|\frac{\partial^2 w}{\partial x^2}\Big|^2dx + h\int_I\Big|\frac{\partial w}{\partial x}\Big|^2dx &= g_2\int_I y\frac{\partial^2 w}{\partial x^2}dx + \int_I v\frac{\partial^2 w}{\partial x^2}dx\\ &\le C(\int_I y^2dx + \int_I v^2dx) + \frac{a_3}{2}\int_I\Big|\frac{\partial^2 w}{\partial x^2}\Big|^2dx.\end{aligned}$$

If we take $k_2 = \min\{a_3, h\}$. we obtain

$$\frac{d}{dt}\|w(t)\|^2_{H^1(I)} + k_2\|w(t)\|^2_{H^1(I)} + k_2\int_I \Big|\frac{\partial^2 w}{\partial x^2}\Big|^2 dx \le C(\|y\|^2_{L^2(I)} + \|v\|^2_{H^1(I)}). \tag{3.4}$$

Step 4. We denote the notation

$$p_1(Y,U) = p\Big(\|\rho\|_{H^1(I)} + \|w\|_{H^1(I)} + \|y\|_{L^1(I)} + \|u\|_{L^2(0,T;H^1(I))} + \|v\|_{L^2(0,T;H^1(I))}\Big),$$

where $p(\cdot)$ is some continuous increasing function.

Multifly the first equation of (1.1) by y and integrate the product in I. Then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega y^2 dx + a_1\int_I \Big|\frac{\partial y}{\partial x}\Big|^2 dx + \frac{a_1}{2}\int_I |y|^2 dx \\
&\quad = \frac{a_1}{2}\int_I |y|^2 dx + \int_I y\frac{\partial y}{\partial x}\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big)dx \\
&\quad \le \frac{a_1}{2}\int_I |y|^2 dx + \frac{a_1}{2}\int_I \Big|\frac{\partial y}{\partial x}\Big|^2 dx + C\int_I y^2\Big(\frac{\partial}{\partial x}\chi_1(\rho) - \frac{\partial}{\partial x}\chi_2(w)\Big)^2 dx \\
&\quad \le \frac{a_1}{2}\int_I |y|^2 dx + \frac{a_1}{2}\int_I \Big|\frac{\partial y}{\partial x}\Big|^2 dx + C\int_I y^2\Big(\chi_1'(\rho)\frac{\partial \rho}{\partial x} - \chi_2'(w)\frac{\partial w}{\partial x}\Big)^2 dx.
\end{aligned}$$

Here, it follows from (1.2), (3.1) and (3.2) that

$$\begin{aligned}
&\int_I y^2\Big(\chi_1'(\rho)\frac{\partial \rho}{\partial x} - \chi_2'(w)\frac{\partial w}{\partial x}\Big)^2 dx \\
&\quad \le \|y\|^2_{L^4(I)}\Big(\|\chi_1'(\rho)\|^2_{L^\infty(I)}\Big\|\frac{\partial \rho}{\partial x}\Big\|^2_{L^4(I)} + \|\chi_2'(w)\|^2_{L^\infty(I)}\Big\|\frac{\partial w}{\partial x}\Big\|^2_{L^4(I)}\Big) \\
&\quad \le C\|y\|_{H^1(I)}\|y\|_{L^1(I)}\Big(\Big\|\frac{\partial \rho}{\partial x}\Big\|^{\frac{1}{2}}_{H^1(I)}\Big\|\frac{\partial \rho}{\partial x}\Big\|^{\frac{3}{2}}_{L^2(I)} + \Big\|\frac{\partial w}{\partial x}\Big\|^{\frac{1}{2}}_{H^1(I)}\Big\|\frac{\partial w}{\partial x}\Big\|^{\frac{3}{2}}_{L^2(I)}\Big) \\
&\quad \le \epsilon(\|y\|^2_{H^1(I)} + \|\rho\|^2_{H^2(I)} + \|w\|^2_{H^2(I)}) + C_\epsilon p_1(Y_0, U).
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_I y^2 dx + k_3\int_I \Big|\frac{\partial y}{\partial x}\Big|^2 dx + k_3\int_I |y|^2 dx \\
&\quad \le \frac{a_1}{2}\int_I |y|^2 dx + \epsilon(\|y\|^2_{H^1(I)} + \|\rho\|^2_{H^2(I)} + \|w\|^2_{H^2(I)}) + C_\epsilon p_1(Y_0, U),
\end{aligned} \tag{3.5}$$

where $k_3 = \min\{k_1, k_2, a_1\}$. By summing up (3.3), (3.4) anf (3.5), we obtain

$$\begin{aligned}
&\frac{d}{dt}\psi(t) + k_3\psi(t) + k_3\Big[\int_I \Big|\frac{\partial y}{\partial x}\Big|^2 dx + \int_I \Big|\frac{\partial^2 \rho}{\partial x^2}\Big|^2 dx + \int_I \Big|\frac{\partial^2 w}{\partial x^2}\Big|^2 dx\Big] \\
&\le C(\|y\|^2_{L^2(I)} + \|u\|^2_{H^1(I)} + \|v\|^2_{H^1(I)}) + \epsilon(\|y\|^2_{H^1(I)} + \|\rho\|^2_{H^2(I)} + \|w\|^2_{H^2(I)}) + C_\epsilon p_1(Y_0, U),
\end{aligned}$$

where $\psi(t) = \|y\|^2_{L^2(I)} + \|\rho\|^2_{H^1(I)} + \|w\|^2_{H^1(I)}$.

If we use the estimate

$$\int_I |y|^2 dx \leq \epsilon \|y\|^2_{H^1(I)} + C_\epsilon \|y\|^2_{L^1(I)}$$

and take $\epsilon = \frac{k_3}{4}$, then we have

$$\begin{aligned}&\frac{d}{dt}\psi(t) + \frac{k_3}{2}\psi(t) + \frac{k_3}{2}\Big[\int_I \Big|\frac{\partial y}{\partial x}\Big|^2 dx + \int_I \Big|\frac{\partial^2 \rho}{\partial x^2}\Big|^2 dx + \int_I \Big|\frac{\partial^2 w}{\partial x^2}\Big|^2 dx\Big] \\ &\qquad \leq C(\|y_0\|^2_{L^1(I)} + \|u\|^2_{H^1(I)} + \|v\|^2_{H^1(I)}) + C_\epsilon p_1(Y_0, U). \quad (3.6)\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\|y(t)\|^2_{L^2(I)} \\ \leq & C\Big[e^{-\frac{k_3}{2}t}(\|y_0\|^2_{L^2(I)} + \|\rho_0\|^2_{H^1(I)} + \|w_0\|^2_{H^1(I)})\Big] \quad (3.7)\\ &+ C\Big[\|y_0\|^2_{L^1(I)} + \|u\|^2_{L^2(0,T;H^1(I))} + \|v\|^2_{L^2(0,T;H^1(I))} + p_1(Y_0, U)\Big], \quad 0 \leq t \leq S.\end{aligned}$$

Moreover, we obtain from (3.6) that

$$\begin{aligned}&\int_0^t \Big(\|y(s)\|^2_{H^1(I)} + \|\rho(s)\|^2_{H^2(I)} + \|w(s)\|^2_{H^2(I)}\Big) ds \\ &\leq C\Big(\|y_0\|^2_{L^2(I)} + \|\rho_0\|^2_{H^1(I)} + \|w_0\|^2_{H^1(I)}\Big) \quad (3.8)\\ &+ C\Big[\|y_0\|^2_{L^1(I)} + \|u\|^2_{L^2(0,T;H^1(I))} + \|v\|^2_{L^2(0,T;H^1(I))} + tp_1(Y_0, U)\Big], \quad 0 \leq t \leq S.\end{aligned}$$

Hence, for any $t_1 \in (0, S)$ with $Y(t_1) \in \mathcal{H}$. we see from (3.1), (3.2), (3.7) and (3.8) that $\|y\|_{L^2(t_1,S;H^1(I))\cap L^\infty(t_1,S;L^2(I))}$, $\|\rho\|_{L^2(t_1,S;H^2(I))\cap L^\infty(t_1,S;H^1(I))}$ and $\|w\|_{L^2(t_1,S;H^2(I))\cap L^\infty(t_1,S;H^1(I))}$ do not depend on S. As a consequence, $\|y\|_{H^1(t_1,S;(H^1(I))')}$, $\|\rho\|_{H^1(t_1,S;L^2(I))}$, $\|w\|_{H^1(t_1,S;L^2(I))}$, and $\|y\|_{\mathcal{C}([t_1,S];L^2(I))}$, $\|\rho\|_{\mathcal{C}([t_1,S];H^1(I))}$, $\|w\|_{\mathcal{C}([t_1,S];H^1(I))}$ do not depend on S. This shows that y, ρ, w can be extended as a weak solution beyond the S. By the standard argument on the extension of the weak solutions, we can then prove the desired result. □

4. Existence of optimal control

In this section, we consider the following optimal control problem

$$(\mathbf{P}) \qquad \text{minimize} \quad J(U)$$

with $J(U)$ of the form

$$J(U) = \int_0^T \|DY(U) - Y_d\|^2_{\mathcal{V}} dt + \gamma \int_0^T \|U\|^2_{\mathcal{H}} dt.$$

Here, $Y(U)$ is the weak solution of (1.1). $D \in \mathcal{L}(\mathcal{V})$ is given by $DY = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}$.

$Y_d = \begin{pmatrix} y_d \\ 0 \\ 0 \end{pmatrix} \in L^2(0,T;\mathcal{V})$ is the fixed element. γ is a positive constant. $\mathcal{U}_{ad} = \{U \in L^2(0,T;\mathcal{H}); \|U\|_{L^2(0,T;\mathcal{H})} \le C\}$.

Theorem 4.1. *There exists an optimal control $\widetilde{U} \in \mathcal{U}_{ad}$ for* (**P**) *such that* $J(\widetilde{U}) = \min_{U \in \mathcal{U}_{ad}} J(U)$.

Proof. Let $\{U_n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(U_n) = \min_{U\in\mathcal{U}_{ad}} J(U).$$

Since $\{U_n\}$ is bounded in $L^2(0,T;\mathcal{H})$, we infer that $U_n \to \widetilde{U}$ weakly in $L^2(0,T;\mathcal{H})$. For simplicity, we will write Y_n instead of the solution $Y(U_n)$ of (1.1) with respect to U_n. Using the similar estimates of Y_n as in the proof of Theorem 3.1, we see that

$$Y_n \to \widetilde{Y} \text{ weakly in } L^2(0,T;\mathcal{V}),$$
$$\frac{dY_n}{dt} \to \frac{d\widetilde{Y}}{dt} \text{ weakly in } H^1(0,T;\mathcal{V}').$$

Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, we can obtain that

$$Y_n \to \widetilde{Y} \text{ strongly in } L^2(0,T;\mathcal{H}). \tag{4.1}$$

Now, we will show that $\widetilde{Y}$ is a solution to (1.1) with respect to $\widetilde{U}$. Indeed, by a direct calculation, we have

$$\begin{aligned}
&\int_0^T \Big\| \frac{\partial}{\partial x}\Big[y_n \frac{\partial}{\partial x}\chi_1(\rho_n) - \tilde{y}\frac{\partial}{\partial x}\chi_1(\tilde{\rho})\Big]\Big\|_{(H^1(1))'} dt \\
&\le C\Big[\int_0^T \|y_n - \tilde{y}\|_{L^2(I)}\|\chi_i(\rho_n)\|_{H^2(I)}dt + C\int_0^T \|\tilde{y}\|_{H^1(I)}\|\chi_i(\rho_n) - \chi_i(\tilde{\rho})\|_{H^1(I)}dt\Big] \\
&\le C\Big[\int_0^T \|y_n - \tilde{y}\|_{L^2(I)} p(\|\rho_n\|_{H^1(I)})(\|\rho_n\|_{H^2(I)} + 1)dt \\
&\quad + \int_0^T \|\tilde{y}\|_{H^1(I)} p(\|\rho_n\|_{H^1(I)} + \|\tilde{\rho}\|_{H^1(I)})\|\rho_n - \tilde{\rho}\|_{H^1(I)}dt\Big] \\
&\le C\Big[\|y_n - \tilde{y}\|_{L^2(0,T;L^2(I))} p(\|\rho_n\|_{H^1(I)})(\|\rho_n\|_{L^2(0,T;H^2(I))} + 1) \\
&\quad + \|\tilde{y}\|_{L^2(0,T;H^1(I))} p(\|\rho_n\|_{H^1(I)} + \|\tilde{\rho}\|_{H^1(I)})\|\rho_n - \tilde{\rho}\|_{L^2(0,T;H^1(I))}\Big].
\end{aligned}$$

For $\psi_1 \in \mathcal{C}([0,T];H^1(I))$, it follows from (4.1) that

$$\int_0^T \Big\langle \frac{\partial}{\partial x}\Big[y_n \frac{\partial}{\partial x}\chi_1(\rho_n) - \tilde{y}\frac{\partial}{\partial x}\chi_1(\tilde{\rho})\Big], \psi_1\Big\rangle_{(H^1(I))',H^1(I)} dt \to 0$$

as $n \to \infty$. By using similar estimate we have

$$\int_0^T \Big\langle \frac{\partial}{\partial x}\Big[y_n \frac{\partial}{\partial x}\chi_2(w_n) - \tilde{y}\frac{\partial}{\partial x}\chi_2(\tilde{w})\Big], \psi_1 \Big\rangle_{(H^1(I))', H^1(I)} dt \to 0$$

as $n \to \infty$. Furthermore, for $\psi_2 \in \mathcal{C}([0,T]; H^2(I))$ it follows from (4.1) that

$$\int_0^T < T(x)(y_n - \tilde{y}), \psi_2 >_{L^2(I), H^2(I)} dt \le C\|y_n - \tilde{y}\|_{L^2(0,T;L^2(I))} \to 0$$

as $n \to \infty$. Therefore, by the uniqueness, $\widetilde{Y}$ is a weak solution to (1.1) with respect to $\widetilde{U}$(i.e., $\widetilde{Y} = Y(\widetilde{U})$). Since $Y_n - Y_d$ is weakly convergent to $\widetilde{Y} - Y_d$ in $L^2(0,T;\mathcal{V})$, we have

$$\min_{U \in \mathcal{U}_{ad}} J(U) \le J(\widetilde{U}) \le \liminf_{n\to\infty} J(U_n) = \min_{U \in \mathcal{U}_{ad}} J(U).$$

Hence, $J(\widetilde{U}) = \min_{U \in \mathcal{U}_{ad}} J(U)$. □

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