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REPRESENTATIONS OF n-FOLD CYCLIC BRANCHED COVERINGS OF $(1,1)$-KNOTS UP TO 10 CROSSINGS AS DUNWOODY MANIFOLDS

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Abstract. In this paper, we discuss the relationship between doubly-pointed Heegaard diagrams of $(1,1)$-knots in lens spaces and Dunwoody 3-manifolds, and then give explicit representations of n-fold cyclic branched coverings of all $(1,1)$-knots in S^3 up to 10 crossings in Rolfsen's knot table as Dunwoody 3-manifolds.

1. Introduction

A $(1,1)$-*decomposition* of a knot K in M is a decomposition of the pair (M,K) into a union $(M,K)=(V_1,K_1)\cup(V_2,K_2)$, where $V_1\cup V_2$ is a genus one Heegaard splitting of M and K_1 and K_2 are properly embedded trivial arcs in V_1 and V_2, respectively, with $K=K_1\cup K_2$. A knot K in M is said to be $(1,1)$-*decomposable* if K admits a $(1,1)$-decomposition. If a knot in M is $(1,1)$-decomposable, it is called a $(1,1)$-*knot* in M. By definition, $(1,1)$-knots are knots in lens spaces, possibly S^3. We note that there are knots in S^3 or in a lens space which are not $(1,1)$-knots. In particular, the class of $(1,1)$-knots in S^3 is one of large and important classes of knots in S^3. For example, all 2-bridge knots and torus knots are $(1,1)$-knots [12] and iterated torus knots are also $(1,1)$-knots [2, 14]. Further, MorimotoSakumaYokota and KlimenkoSakuma classified $(1,1)$-knots that are also Montesinos knots [11, 13] and MorimotoSakumaYokota in [13] classified $(1,1)$-knots up to crossing number 10.

Let K be a $(1,1)$-knot in M with a $(1,1)$-decomposition $(M,K)=(V_1,K_1)\cup_h(V_2,K_2)$ via an orientation reversing attaching homeomorphism $h:\partial V_2\to\partial V_1$, and let m_1 and m_2 denote the standard meridian curves on V_1 and V_2, respectively. Then the quadruple $H=(\partial V_1,m_1,h(m_2),\partial K_1)$ is called a (genus one) *doubly-pointed Heegaard diagram* of K. Cutting $\partial V_1-\partial K_1$ along a meridean,

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one can represent the genus one doubly-pointed Heegaard diagram H as a planar diagram $D(a, b, c, r)$ which is completely determined by some four nonnegative integers a, b, c and r (see Figure 1). We call this diagram $D(a, b, c, r)$ a $(1, 1)$-*diagram* of the $(1, 1)$-knot K.

In [6], M. J. Dunwoody introduced a large class of closed orientable 3-manifolds admitting cyclic group presentations for their fundamental groups (so now called *Dunwoody manifold*), depending only on 6-tuples (a, b, c, n, r, s) of integers with $n > 0, a, b, c \geq 0$ and $a + b + c > 0$ satisfying certain conditions (*admissible* 6-*tuple*). This had been done by considering 3-regular planar graphs $\Gamma(a, b, c, n)$ with cyclic symmetry of order n (see Figure 4). It had been shown that all these manifolds turn out to be strongly-cyclic branched coverings of $(1, 1)$-knots in lens spaces, possibly $S^1 \times S^2$ and S^3 [4, 8]. Moreover, the explicit Dunwoody representations (Dunwoody six parameters) for all cyclic branched coverings of 2-bridge knots has been obtained in [8]. An interesting problem which naturally arises is that of finding the explicit Dunwoody representations of the cyclic branched coverings of important classes of $(1, 1)$-knots, in particular, $(1, 1)$-knot in S^3. In [1], H. Aydin, I. Gultekyn and M. Mulazzani gave an explicit representation (Dunwoody six parameter) as Dunwoody manifolds of all cyclic branched coverings of torus knots of type $(p, mp - 1)$, with $p > 1$ and $m > 0$, thus including all torus knots with bridge number ≤ 4.

On the other hand, the authors in [10] gave a list of $(1, 1)$-diagrams $D(a, b, c, r)$ of all $(1, 1)$-knots up to 10 crossings in Rolfsen's knot table. Using this list, we give in this paper a complete list of explicit Dunwoody representations $(a, b, c, n, r, s) \in \mathbb{Z}^6$ as Dunwoody manifolds of all n-fold cyclic branched coverings of S^3 branched along $(1, 1)$-knots up to 10 crossings in Rolfsen's knot table [15].

This paper is organized as follows. In Section 2, we summarize the basic terminology of $(1, 1)$-knots and $(1, 1)$-diagrams $D(a, b, c, r)$, which provide a particularly convenient way to describe genus one doubly-pointed Heegaard diagrams coupled with $(1, 1)$-knots. In Section 3, we demonstrate a relationship between $(1, 1)$-knots and n-fold cyclic branched coverings as Dunwoody manifolds by means of $(1, 1)$-diagrams $D(a, b, c, r)$. In Section 4, we list explicit Dunwoody representations of n-fold cyclic branched coverings of $(1, 1)$-knots up to 10 crossings.

2. $(1, 1)$-knots and $(1, 1)$-diagrams

In this section, we review $(1, 1)$-knots, $(1, 1)$-diagrams and related preliminaries from [10]. Let V be a handlebody of genus $g \geq 0$. A collection $\mathbf{a} = \{t_1, \ldots, t_b\}$ of mutually disjoint arcs t_i which are properly embedded in V is called a *b-string trivial arc system* in V if there is a collection $\{s_1, \ldots, s_b\}$ of arcs s_i in ∂V such that for each $i \in \{1, \ldots, b\}$, $t_i \cup s_i$ is a circle which bounds a disk d_i in V and those disks $d_1, \ldots, d_b$ are mutually disjoint. Let M be a closed oriented 3-manifold and let K be a knot in M. A (g, b)-*decomposition of* K *in* M

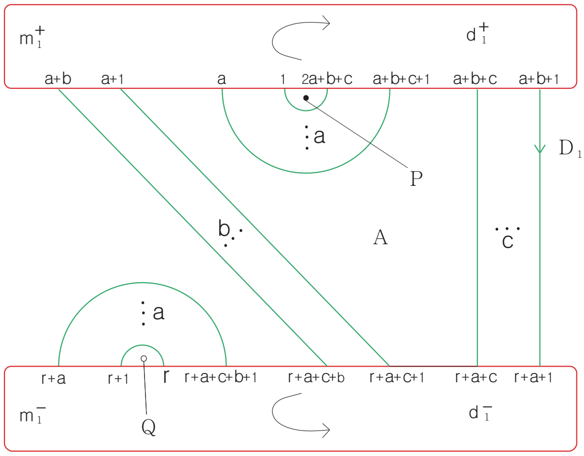


FIGURE 1. $D(a,b,c,r)$

is a decomposition of the pair (M,K) into a union $(M,K) = (V_1,\mathbf{a}_1) \cup (V_2,\mathbf{a}_2)$ such that $V_1 \cup V_2$ is a genus g Heegaard splitting of M and $K = \mathbf{a}_1 \cup \mathbf{a}_2$ intersects the Heegaard surface $S_g(= \partial V_1 = \partial V_2)$ transversally, where $\mathbf{a}_i (i = 1,2)$ is a b-string trivial arc system in V_i. A knot K in M is said to be *(g,b)-decomposable* or in a *genus g b-bridge position* with respect to a Heegaard surface S_g if K admits a (g,b)-decomposition in M. The *genus g-bridge number* of a knot K in M is the smallest integer m for which K is (g,m)-decomposable. If the genus g-bridge number of a knot K in M is b, then we say that K is a *genus g b-bridge knot* in M or simply a *(g,b)-knot* in M. For more details, see [7]. In this paper, we are only concern with $(1,1)$-knots.

Now we are going to review the way of describing $(1,1)$-knots discussed in [5]. For any given 4-tuple $\sigma = (a,b,c,r)$ of integers with $a,b,c \geq 0, a+b+c > 0$ and $r \in \mathbb{Z}_d = \{0,1,\ldots,d-1\}$ $(d = 2a+b+c)$, let $D(a,b,c,r)$ be a planar diagram (a 3-regular planar graph) in $\mathbb{R}^2$ defined as follows: Let m_1^+ and m_1^- be two disjoint circles in $\mathbb{R}^2$ with the clockwise and counterclockwise orientation, respectively. Consider d vertices on the circles m_1^+ and m_1^- labeled by the integers from 1 to d in accordance with the orientation (see Figure 1). The vertices on m_1^+ labeled $1,\ldots,a$ are connected to the vertices on m_1^+ labeled $2a+b+c,\ldots,a+b+c+1$ by a parallel arcs, the vertices labeled $a+1,\ldots,a+b$ in m_1^+ are connected to the vertices labeled $r+a+c+1,\ldots,r+a+c+b$ in m_1^- by b parallel arcs, the vertices labeled $a+b+1,\ldots,a+b+c$ in m_1^+ are connected to the vertices labeled $r+a+1,\ldots,r+a+c$ in m_1^- by c parallel arcs, and the vertices labeled $r+1,\ldots,r+a$ in m_1^- are connected to the vertices labeled $r,\ldots,r+a+c+b+1$ in m_1^- by a parallel arcs so that the all d arcs are mutually disjoint and r is the label of the vertex of the innermost bigon adjacent to m_1^- as shown in Figure 1, where the labels of the vertices on m_1^- are taken modulo d.

Let P be a point chosen in the interior of the bigon with two vertices on m_1^+ labeled 1 and $2a+b+c$ and let Q be a point in the interior of the bigon with two vertices on m_1^- labeled r and $r+1$ (see Figure 1). Consider one-point compactification of $\mathbb{R}^2$ that leads a 2-cell embedding of $D(a,b,c,r)$ in the 2-sphere S^2. Cutting S^2 along two circles m_1^+ and m_1^- and then removing the interior of the discs d_1^+ and d_1^- bounded by m_1^+ and m_1^- respectively, we obtain a 2-sphere S^2 with 2 holes. Gluing the circle m_1^+ with the circle m_1^- so that equally labeled vertices are identified together gives rise to a torus T and the d arcs are pairwise connected through their endpoints which produces a system $\mathcal{D}(\sigma)=\{D_1,\ldots,D_{k_\sigma}\}$ of mutually disjoint simple closed curves on the resulting torus T for some integer $k_\sigma \geq 1$. Let m_1 denote the simple closed curve on the torus T corresponding to the identified m_1^+ and m_1^-, which is a meridian on T. We denote the quadruple $(T, m_1, \{D_1,\ldots,D_{k_\sigma}\}, \{P,Q\})$ by $H(a,b,c,r,\{P,Q\})$.

Definition 2.1. *A 4-tuple $\sigma=(a,b,c,r)$ is said to be admissible if $k_\sigma = 1$, i.e., the set $\mathcal{D}(\sigma)$ contains only one simple closed curve D_1 and the torus T is still connected after cutting it along D_1.*

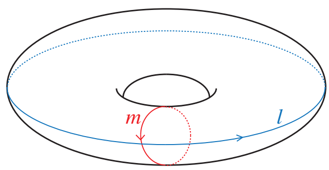


FIGURE 2. A preferred longitude ℓ and meridian m of $V = D^2 \times S^1$.

Let $V = D^2 \times S^1$ be the standard solid torus and let l and m be a preferred longitude and meridian of V (see Figure 2), respectively. Note that l and m generate the fundamental group $\pi_1(\partial V) = \mathbb{Z} \oplus \mathbb{Z}$. If a simple closed curve on ∂V represents the homotopy class $p[l]+q[m] \in \pi_1(\partial V)$, we call such a curve a (p,q)-*curve*. The 3-manifold obtained by gluing two standard solid tori V_1 and V_2 along their boundaries via the homeomorphism $h : \partial V_2 \to \partial V_1$ that takes a meridian m in ∂V_2 to a (p,q)-curve in ∂V_1 is called a *lens space* of type (p,q) and denoted by $L(p,q)$. It is well known that two lens spaces $L(p,q)$ and $L(p',q')$ are homeomorphic if and only if $p=p'$ and $q \equiv \pm q'^{\pm 1} \pmod p$. It is noted that the lens space $L(1,0)$ is homeomorphic to S^3 and more generally the lens space $L(p,q)$ is homeomorphic to S^3 if and only if $p = \pm 1$.

For any given admissible 4-tuple $\sigma=(a,b,c,r)$, we can construct a $(1,1)$-knot in a lens space $L(p,q)$ from the planar diagram $D(a,b,c,r)$ as follows: Since $\sigma=(a,b,c,r)$ is admissible, $H(a,b,c,r,\{P,Q\}) = (T, m_1, D_1, \{P,Q\})$ gives rise to a genus one doubly-pointed Heegaard diagram of a closed orientable 3-manifold, denoted by $M(a,b,c,r)$, which is indeed a lens space $L(p,q)$, possibly S^3. Let V_1 and V_2 be two solid tori with $T = \partial V_1 = \partial V_2$ and let $K_1 \subset V_1$ and $K_2 \subset V_2$ be properly embedded trivial arcs with $\partial K_1 = \partial K_2 = \{P, Q\}$. Then the Heegaard diagram $H(a,b,c,r,\{P,Q\}) = (T, m_1, D_1, \{P,Q\})$ yields a $(1,1)$-knot $K = K_1 \cup K_2$ in $M(a,b,c,r)$ with a $(1,1)$-decomposition:

$$M(a,b,c,r) = (V_1, K_1) \cup_h (V_2, K_2), \tag{2.1}$$

where $h : (\partial V_2, \partial K_2) \longrightarrow (\partial V_1, \partial K_1)$ is an orientation-reversing homeomorphism that takes a meridian m_2 in ∂V_2 to the simple closed curve D_1 on $T = \partial V_1$ (see Figure 3).

Definition 2.2. *The* $(1,1)$*-knot* $K = K_1 \cup K_2$ *in* $M(a,b,c,r)$ *is called the* $(1,1)$*-knot associated to* $D(a,b,c,r)$ *and denoted by* $K(a,b,c,r)$*. And, we call* $D(a,b,c,r)$ *a* $(1,1)$*-diagram of* $K(a,b,c,r)$*.*

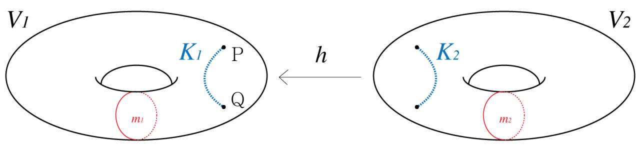


FIGURE 3. A $(1,1)$-decomposition of a knot $K = K_1 \cup K_2$ in $M(a,b,c,r)$.

We now describe how to obtain a $(1,1)$-diagram $D(a,b,c,r)$ from a $(1,1)$-decomposition of a $(1,1)$-knot in S^3. Let K be a $(1,1)$-knot in S^3. Then K admits a $(1,1)$-decomposition

$$(S^3, K) = (V_1, K_1) \cup_h (V_2, K_2),$$

where V_1 and V_2 are solid tori, $K_1 \subset V_1$ and $K_2 \subset V_2$ are properly embedded trivial arcs, and $h : (\partial V_2, \partial K_2) \longrightarrow (\partial V_1, \partial K_1)$ is an attaching homeomorphism. Let $T = \partial V_1$ be the Heegaard surface of genus one (torus) of this $(1,1)$-decomposition and let $\{P,Q\} = T \cap K = \partial K_1 = \partial K_2$. Taking a meridian disk d_i in each solid torus V_i so that d_i is disjoint from K_i, we have a 4-tuple $(T, m_1, h(m_2), \{P,Q\})$ such that $(T, m_1, h(m_2))$ is a genus one Heegaard diagram of S^3 and $\{P,Q\}$ is the set of two endpoints of each trivial arc K_i, where $m_i = \partial d_i$. If m_1 and $h(m_2)$ meet efficiently in $T - \{P,Q\}$, i.e., any bigon in T obtained from the intersection of m_1 and $h(m_2)$ contains either P or Q (indeed, any bigon arc which does not contain a point of ∂K_1 can be removed up to isotopy), then $(T, m_1, h(m_2), \{P,Q\})$ is said to be *minimal.* Hayashi [9] showed that such a minimal diagram $(T, m_1, h(m_2), \{P,Q\})$ is uniquely determined by the $(1,1)$-decomposition up to its homeomorphic type. Cutting the solid torus V_1 along d_1, we obtain a solid cylinder C_1 with two cutting disks, say, d_1^+ and d_1^-. Let

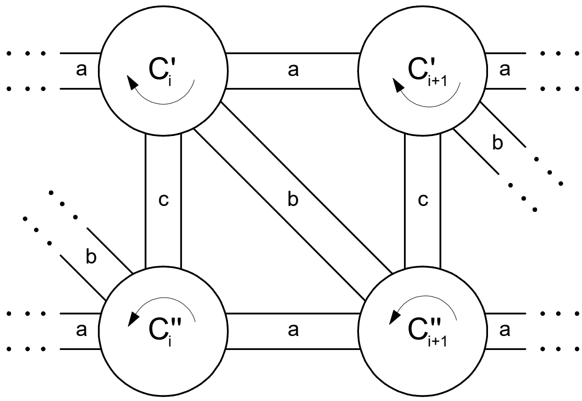


FIGURE 4. The graph $\Gamma(a, b, c, n)$.

m_1^+ and m_1^- denote the boundary circles of two cutting disks d_1^+ and d_1^-, respectively. By applying isotopies if necessary, we obtain a $(1, 1)$-diagram $D(a, b, c, r)$ as shown in Figure 1 consisting of the annulus $A_1 = \overline{\partial C_1 - (d_1^+ \cup d_1^-)}$ and a set $\mathfrak{A}_1$ of $2a + b + c$ arcs coming from $h(m_2)$ governed by some nonnegative integers a, b, c and r such that $a + b + c > 0, r \in \mathbb{Z}_d$ $(d = 2a + b + c)$ and the 4-tuple $\sigma = (a, b, c, r)$ is admissible. It should be noted that the $(1, 1)$-diagram $D(a, b, c, r)$ of a $(1, 1)$-knot K in S^3 is not uniquely determined. For more details, we refer to [8, 10].

3. $(1, 1)$-knots and Dunwoody manifolds

In [6], M. J. Dunwoody constructed a large class of closed orientable 3-manifolds with the fundamental groups admitting cyclic group presentation (now called *Dunwoody manifold*), which is totally represented by 6-tuples (a, b, c, n, r, s) of integers with $n > 0, a, b, c \geq 0$ and $a + b + c > 0$ satisfying certain conditions, called *admissible*. This construction had been done by considering 3-regular planar graphs $\Gamma(a, b, c, n)$ as shown in Figure 1 with cyclic symmetry of order n. In this section, we first give a sketch of the construction of Dunwoody manifolds following [8] with changing of some notations for our convenience, and then we demonstrate a relationship between $(1, 1)$-knots and n-fold cyclic branched coverings as Dunwoody manifolds by means of $(1, 1)$-diagrams $D(a, b, c, r)$.

Let a, b, c, n be integers such that $n > 0, a, b, c \geq 0$ and $a + b + c > 0$. Let $\Gamma = \Gamma(a, b, c, n)$ be the planar 3-regular graph in $\mathbb{R}^2$ depicted in Figure 4. It contains n upper cycles $C'_1, \ldots, C'_n$ and n lower cycles $C''_1, \ldots, C''_n$, and each cycle has $d = 2a + b + c$ trivalent vertices. For each $i = 1, \ldots, n$, the cycle C'_i (resp. C''_i) is connected to the cycle C'_{i+1} (resp. C''_{i+1}) by a parallel arcs, connected to the cycle C''_i by c parallel arcs, and connected to the cycle C''_{i+1} by b parallel arcs, where we assume $n+1=1$.

Let $\mathcal{C}' = \{C'_1, \ldots, C'_n\}$ and $\mathcal{C}'' = \{C''_1, \ldots, C''_n\}$. Let A' (resp. A'') be the set of the arcs of Γ belonging to a cycle of $\mathcal{C}'$ (resp. $\mathcal{C}''$) and let A be the set of the other arcs of the graph Γ. The one-point compactification of the plane $\mathbb{R}^2$ gives to a 2-cell embedding of Γ in the 2-sphere S^2. It is straightforward that the graph Γ in S^2 is invariant under a rotation $\rho_n : S^2 \to S^2$ of S^2 by $\frac{2\pi}{n}$-radians along a suitable axis intersecting S^2 in two points not belonging to the graph Γ. Without loss of generality, we may assume that $\rho_n(C'_i) = C'_{i+1}$ and $\rho_n(C''_i) = C''_{i+1}$ for each $i = 1, \ldots, n$ (we assume $n+1=1$). By cutting the sphere S^2 along all C'_i and C''_i and by removing the interior of the corresponding discs, we obtain a 2-sphere S^2 with $2n$ holes.

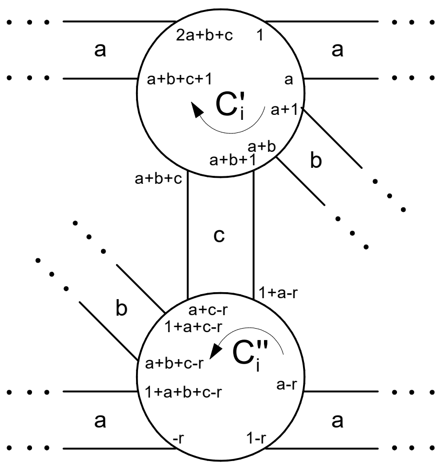


FIGURE 5. Labelling of the vertices of C'_i and C''_i $(1 \le i \le n)$.

Now let r and s be two new integers taken mod d and mod n, respectively. We choose a clockwise (resp. counterclockwise) orientation to the cycles in $\mathcal{C}'$ (resp. in $\mathcal{C}''$) and label their vertices from 1 to d in accordance with these orientations (see Figure 5) in order to satisfy the following two conditions:

(i) The vertex 1 of each $C'_i \in \mathcal{C}'$ is the endpoint of the first arc of A connecting C'_i to C'_{i+1} as shown in Figure 5.
(ii) The vertex $1-r$ (mod d) of each $C''_i \in \mathcal{C}''$ is the endpoint of the first arc of A connecting C''_i to C''_{i+1} as shown in Figure 5.

For each $i = 1, \ldots, n$, we glue the cycle C'_i with the cycle C''_{i-s} so that the equally labelled vertices are identified together, where $i - s \in \{1, \ldots, n\}$ taken mod n. Then we obtain an orientable surface Σ_n of genus n in which the nd

arcs belonging to A are pairwise connected through their endpoints and give rise to m cycles $D_1, \ldots, D_m$ for some integer $m \geq 1$, where $d = 2a + b + c$. It is direct by construction that the cut of Σ_n along the n cycles $C_i := C'_i = C''_{i-s}$ (indeed, a meridean on Σ_n) does not disconnect the surface Σ_n. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ and $\mathcal{D} = \{D_1, \ldots, D_m\}$.

Definition 3.1. *A 6-tuples $(a, b, c, n, r, s) \in \mathbb{Z}^6$ such that $n > 0, a, b, c \geq 0$ and $a + b + c > 0$ is said to be admissible if it satisfies the following conditions:*

(1) *The set $\mathcal{D}$ contains exactly n cycles.*
(2) *The surface Σ_n is not disconnected by cutting along the cycles in $\mathcal{D}$.*

Let $\sigma = (a, b, c, n, r, s)$ be an admissible 6-tuple. From Definition 3.1, we have

$$\mathcal{C} = \{C_1, \ldots, C_n\} \text{ and } \mathcal{D} = \{D_1, \ldots, D_n\}. \tag{3.2}$$

Further, the cut along the cycles of $\mathcal{D}$ does not disconnect the surface Σ_n. This implies that the triple $(\Sigma_n, \mathcal{C}, \mathcal{D})$ forms a genus n Heegaard diagram of a closed orientable 3-manifold, which is completely determined by the admissible 6-tuple $\sigma = (a, b, c, n, r, s)$ and so is denoted by $M(a, b, c, n, r, s)$.

Definition 3.2. *Let $\sigma = (a, b, c, n, r, s)$ be an admissible 6-tuple. The closed orientable 3-manifold $M(a, b, c, n, r, s)$ with a genus n Heegaard diagram $(\Sigma_n, \mathcal{C}, \mathcal{D})$ is called a Dunwoody manifold associated to σ.*

Throughout this paper, the genus n Heegaard diagram $(\Sigma_n, \mathcal{C}, \mathcal{D})$ of the Dunwoody manifold $M(a, b, c, n, r, s)$ will be denoted by $H(a, b, c, n, r, s)$. That is,

$$H(a, b, c, n, r, s) = (\Sigma_n, \mathcal{C}, \mathcal{D}). \tag{3.3}$$

It is noted that the graph $\Gamma(a, b, c, n)$ in Figure 4 with the labels of vertices as illustrated in Figure 5 can be regarded as an *open Heegaard diagram* obtained from $H(a, b, c, n, r, s)$ by cutting along the meridean curves $C_1, \ldots, C_n$ on Σ_n. We will denote the open Heegaard diagram by $D(a, b, c, n, r, s)$.

Let $\sigma = (a, b, c, n, r, s)$ be an admissible 6-tuple and let $\Gamma'(a, b, c, n)$ denote the regular 4-valent graph embedded in the Heegaard surface Σ_n corresponding to $\Gamma(a, b, c, n)$. Observe that the vertices of $\Gamma'(a, b, c, n)$ are the intersection points of $\cup_{i=1}^{n} C_i$ and $\cup_{j=1}^{n} D_j$ in (3.2). Hence they inherit the labelling of the corresponding glued vertices of Γ. Since the gluing of the cycles of $\mathcal{C}'$ and $\mathcal{C}''$ is invariant under the rotation ρ_n, the cyclic group $G_n =< \rho_n >$ generated by ρ_n naturally induces a cyclic group action of order n on Σ_n such that the quotient $\Sigma_1 = \Sigma_n / G_n$ is homeomorphic to a torus. Furthermore, the labelling of the vertices of $\Gamma'(a, b, c, n)$ is invariant under the rotation ρ_n and $\rho_n(C_i) = C_{i+1}$ because $\rho_n(C'_i) = C'_{i+1}$ and $\rho_n(C''_i) = C''_{i+1}$ (assume $n + 1 = 1$) for each $i = 1, \ldots, n$. The following Lemma shows that this last property also hold for the cycles of $\mathcal{D}$.

Lemma 3.3. [8, Lemma 1]

(1) *Let* $\sigma = (a,b,c,n,r,s)$ *be an admissible* 6*-tuple. Then* ρ_n *induces a cyclic permutation on the curves of* $\mathcal{D}$*. Thus, if* D *is a cycle of* $\mathcal{D}$*, then* $\mathcal{D} = \{\rho_n^{k-1}(D) | k = 1,\dots,n\}$*.*
(2) *If* (a,b,c,n,r,s) *is admissible, then* $(a,b,c,1,r,0)$ *is also admissible and the Heegaard diagram* $H(a,b,c,1,r,0)$ *is the quotient of* $H(a,b,c,n,r,s)$ *by* G_n*.*

Let $\sigma = (a,b,c,n,r,s)$ be an admissible 6-tuple with $n \geq 1$ and let v be the vertex belonging to the cycle C_1 of the Heegaard diagram $H(a,b,c,n,r,s)$ and labelled by $a+b+1$. We denote by D_1 the curve in $\mathcal{D}$ containing v and by v' the vertex of C'_1 in the graph $\Gamma(a,b,c,n)$ corresponding to v. Orient the arc $e' \in A$ of $\Gamma(a,b,c,n)$ containing v' so that v' is its first endpoint and then orient the curve D_1 in accordance with the orientation of this arc e'. Now, let $D_k = \rho_n^{k-1}(D_1)$ for each $k = 1,\dots,n$. By the rotation ρ_n, the orientation on D_1 induces the orientation also on the curves D_k $(k = 1,\dots,n)$. Moreover, these orientation on the cycles of $\mathcal{D}$ induce an orientation on the arcs of the graph $\Gamma(a,b,c,n)$ belonging to A. By orienting the arcs of $\mathcal{C}'$ and $\mathcal{C}''$ in accordance with the fixed orientations of the cycles C'_i and C''_i, the graph $\Gamma(a,b,c,n)$ becomes an oriented graph whose orientation is invariant under G_n. This orientation is called the *canonical orientation* of $\Gamma(a,b,c,n)$.

Let Δ be the set of the first d arcs of D_1, following the canonical orientation, starting from the arc coming out from the vertex v' of C'_1 labelled $a+b+1$. Let $\tilde{p}'_\sigma$ (resp. $\tilde{p}''_\sigma$) denote by the number of the arcs in Δ oriented from a cycle of $\mathcal{C}'$ to a cycle of $\mathcal{C}''$ (resp. oriented from a cycle of $\mathcal{C}''$ to a cycle of $\mathcal{C}'$). Similarly, let $\tilde{q}'_\sigma$ (resp. $\tilde{q}''_\sigma$) denote by the number of the arcs in Δ oriented from either a cycle $\mathcal{C}'_i$ to a cycle $\mathcal{C}'_{i+1}(\text{mod } n)$ or a cycle $\mathcal{C}''_i$ to a cycle $\mathcal{C}''_{i+1}(\text{mod } n)$ (resp. oriented from either a cycle $\mathcal{C}'_{i+1}(\text{mod } n)$ to a cycle $\mathcal{C}'_i$ or a cycle $\mathcal{C}''_{i+1}(\text{mod } n)$ to a cycle $\mathcal{C}''_i$) for some $i \in \{1,\dots,n\}$. We define two integers $\tilde{p}_\sigma$ and $\tilde{q}_\sigma$ by

$$\tilde{p}_\sigma = \tilde{p}'_\sigma - \tilde{p}''_\sigma \text{ and } \tilde{q}_\sigma = \tilde{q}'_\sigma - \tilde{q}''_\sigma. \tag{3.4}$$

It is noted that $\tilde{p}_\sigma$ has the same parity of $b+c$ and $\tilde{q}_\sigma$ has the same parity of $2a+b$ and hence of b. It is evident that $\tilde{p}_\sigma$ and $\tilde{q}_\sigma$ depend only on the integers a,b,c and r.

Theorem 3.4. [8, Theorem 6] *Let* $\sigma = (a,b,c,n,r,s)$ *be any given admissible* 6*-tuple with* $n > 1$*. Then the Dunwoody manifold* $M(a,b,c,n,r,s)$ *is the* n*-fold cyclic covering of the manifold* $M' = M(a,b,c,1,r,0)$ *branched over the* $(1,1)$*-knot* $K(a,b,c,1,r,0)$ *only depending on the integers* a,b,c,r*. Further,* M' *is homeomorphic to:*

(1) S^3 *if* $\tilde{p}_\sigma = \pm 1$.
(2) $S^1 \times S^2$ *if* $\tilde{p}_\sigma = 0$.
(3) *A lens space* $L(\alpha,\beta)$ *with* $\alpha = |\tilde{p}_\sigma|$ *if* $|\tilde{p}_\sigma| > 1$.

Corollary 3.5. [8, Corollary 7] *Let* $\sigma_1 = (a,b,c,1,r,0)$ *be any given admissible* 6*-tuple with* $\tilde{p}_{\sigma_1} = \pm 1$ *and* $s_{\sigma_1} = -\tilde{p}_{\sigma_1}\tilde{q}_{\sigma_1}$*. Then the* 6*-tuple* $\sigma_n =$

(a,b,c,n,r,s_{σ_1}) is admissible for each integer $n>1$ and the Dunwoody manifold $M_n = M(a,b,c,n,r,s_{\sigma_1})$ is the n-fold cyclic coverings of S^3 branched over the $(1,1)$-knot $K(a,b,c,1,r,0)$, which is independent on n.

Example 3.6. *The Dunwoody manifolds $M(0,0,1,1,0,0), M(1,0,0,1,1,0)$ and $M(0,0,c,\ 1,r,0)$ with c and r coprime are homeomorphic to $S^3, S^1\times S^2$ and the lens space $L(c,r)$, respectively. Furthermore, for each coprime integers a and c with $a>0$, the Dunwoody manifold $M(a,0,c,1,a,0)$ is homeomorphic to the lens space $L(c,a)$. On the other hand, for each integer $n>1$, the Dunwoody manifolds $M(0,0,1,n,0,0), M(1,0,0,n,1,0)$ and $M(0,0,c,n,r,0)$ with c and r coprime are n-fold cyclic coverings of $S^3, S^1\times S^2$ and $L(c,r)$, respectively, branched over a trivial knot. Indeed, these Dunwoody manifolds are the connected sum of n copies of $S^3, S^1\times S^2$ and $L(c,r)$, respectively. For more details, we refer to* [6, 8].

Now we are ready to relate open Heegaard diagrams $D(a,b,c,1,r,0)$ associated to admissible 6-tuples $(a,b,c,1,r,0)$ and $(1,1)$-diagrams $D(x,y,z,w)$ as depicted in Figure 1.

Lemma 3.7. *Let $(a,b,c,1,r,0)$ be an admissible 6-tuple with $r\in\mathbb{Z}_d$ ($d = 2a+b+c$). Then the open Heegaard diagram $D(a,b,c,1,r,0)$ and the $(1,1)$-diagram $D(a,b,c,d-r)$ in Figure 1 give rise to the same genus one Heegaard diagram $H(a,b,c,1,r,0)$. Consequently, the 6-tuple $(a,b,c,1,r,0)$ admissible if and only if the 4-tuple $(a,b,c,d-r)$ is admissible.*

Proof. Let $(a,b,c,1,r,0)$ be an admissible 6-tuple with $r\in\mathbb{Z}_d$. Then it is seen from Figure 5 with $n=1$ that the open Heegaard diagram $D(a,b,c,1,r,0)$ looks like the planar 3-regular graph depicted in (i) of Figure 6. By construction, it is evident that the labels $1,\dots,d$ for the vertices of the cycle C''_1 can be taken modulo d. (The labels are not changed modulo d. This shows that the glue of C'_1 with C''_1 are not changed from the change of labels for the vertices of C''_1 modulo n.) Let $r' = -r+d$. Then $-r\equiv r' \pmod d$ and $r'\in\mathbb{Z}_d$ because $r\in\mathbb{Z}_d$. This gives that the two graphs in (i) and (ii) of Figure 6 give rise to the same genus one Heegaard diagram $H(a,b,c,1,r,0)$. Using plane isotopies, it is not difficult to transform the graph in (ii) into the graph in (iii) and then the graph in (iv) of Figure 6, which is obvioully plane isotopic to the $(1,1)$-diagram $D(a,b,c,r')$ in Figure 1. Hence we see that two graphs $D(a,b,c,1,r,0)$ and $D(a,b,c,d-r)$ give rise to the same genus one Heegaard diagram $H(a,b,c,1,r,0)$. This proves the first part of Lemma.

Now, since $D(a,b,c,1,r,0)$ and $D(a,b,c,d-r)$ give rise to the same genus one Heegaard diagram $H(a,b,c,1,r,0)$, we see that the d arcs in $D(a,b,c,1,r,0)$ (resp. $D(a,b,c,d-r)$) forms only one simple closed curve D_1 and the torus Σ_1 is still connected after cutting it along D_1. Therefore the second part is straightforward from Definitions 2.1 and 3.1. This completes the proof. □

Let $\sigma = (a,b,c,r)$ be an admissible 4-tuple and let $D(a,b,c,r)$ be the associated $(1,1)$-diagram as shown in Figure 1. We take the orientation on D_1

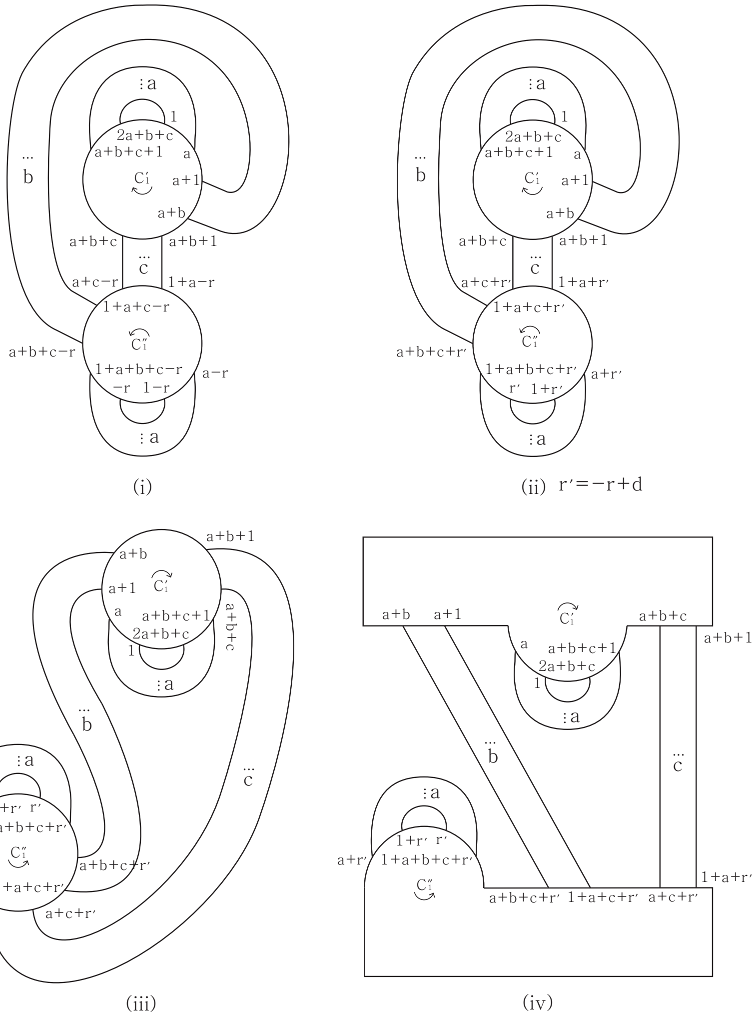


FIGURE 6. Open Heegaard diagrams of $H(a,b,c,1,r,0)$

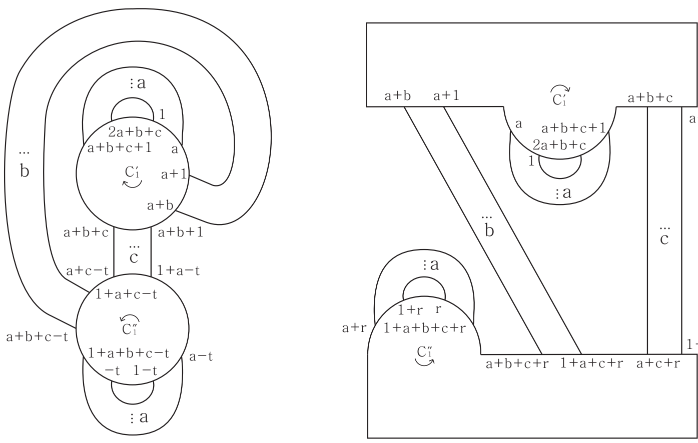
induced from the orientation on the arc going from the vertex labeled $a+b+1$ to the vertex labeled $r+a+1$ (see Figure 1). Let p'_σ (resp. p''_σ) denote by the number of the arcs with the orientation of pointing down the page from m_1^+ to m_1^- (resp. pointing up the page from m_1^- to m_1^+) in $D(a,b,c,r)$. Similarly, let q'_σ (resp. q''_σ) denote by the number of the arcs with the orientation of pointing right the page (resp. pointing left the page) in $D(a,b,c,r)$. We define two integers p_σ and q_σ by

$$p_\sigma = p'_\sigma - p''_\sigma \text{ and } q_\sigma = q'_\sigma - q''_\sigma. \tag{3.5}$$

Theorem 3.8. *Let $\sigma = (a, b, c, r)$ be an admissible 4-tuple and let $D(a, b, c, r)$ be the associated $(1, 1)$-diagram in Figure 1. Then the 3-manifold $M(a, b, c, r)$ in (2.1) is homeomorphic to:*

(1) S^3 *if* $p_\sigma = \pm 1$.
(2) $S^1 \times S^2$ *if* $p_\sigma = 0$.
(3) *Lens space* $L(p, q)$ *with* $p = |p_\sigma|$ *if* $|p_\sigma| > 1$.

Proof. By Lemma 3.7 and its proof, we see that the $(1, 1)$-diagram $D(a, b, c, r)$ and the open Heegaard diagram $D(a, b, c, 1, t, 0)$ give rise to the same genus one Heegaard diagram $H(a, b, c, 1, t, 0)$ (see Figure 7), where $t = d - r$, and $(a, b, c, 1, t, 0)$ is an admissible 6-tuple. Hence the 3-manifold $M(a, b, c, r)$ is homeomorphic to the Dunwoody manifold $M(a, b, c, 1, t, 0)$. Let $\sigma_1 = (a, b, c, 1, t, 0)$. It is easily seen from (3.4) and Figure 7 that $p_\sigma = \tilde{p}_{\sigma_1}$ and $q_\sigma = \tilde{q}_{\sigma_1}$. Then the result is immediate from Theorem 3.4. This completes the proof. □



(i) D(a,b,c,1,t,0), t=d−r (iv) D(a,b,c,r)

FIGURE 7. Open Heegaard diagrams of $H(a, b, c, 1, d - r, 0)$

Theorem 3.9. *Let $\sigma = (a, b, c, r)$ be any given admissible 4-tuple, $d = 2a+b+c$, and let $D(a, b, c, r)$ be the associated $(1, 1)$-diagram in Figure 1. If $p_\sigma = \pm 1$ and $s_\sigma = -p_\sigma q_\sigma$, then the 6-tuple $\sigma_n = (a, b, c, n, d - r, s_\sigma)$ is admissible for each integer $n > 1$ and the Dunwoody manifold $M(a, b, c, n, d - r, s_\sigma)$ is the n-fold cyclic covering of S^3 branched over the $(1, 1)$-knot $K(a, b, c, r)$, which is independent on n.*

Proof. Let $\sigma_1 = (a, b, c, 1, d - r, 0)$. Since $\sigma = (a, b, c, r)$ is admissible, it follows from Lemma 3.7 that σ_1 is admissible. It is easily seen from (3.4) and Figure 7 that $p_\sigma = \tilde{p}_{\sigma_1}$ and $q_\sigma = \tilde{q}_{\sigma_1}$. Now, suppose that $p_\sigma = \pm 1$ and $s_\sigma = -p_\sigma q_\sigma$.

Then $\tilde{p}_{\sigma_1} = \pm 1$ and $s_{\sigma_1} = -\tilde{p}_{\sigma_1}\tilde{q}_{\sigma_1}$. By Corollary 3.5, we see that the 6-tuple $\sigma_n = (a, b, c, n, d-r, s_{\sigma_1})$ is admissible for each integer $n > 1$ and the Dunwoody manifold $M_n = M(a, b, c, n, d-r, s_{\sigma_1})$ is the n-fold cyclic coverings of S^3 branched over the $(1,1)$-knot $K(a, b, c, 1, d-r, 0)$, which is independent on n. Further, by Lemma 3.7, we have that the $(1,1)$-diagram $D(a, b, c, r)$ and the open Heegaard diagram $D(a, b, c, 1, d-r, 0)$ give rise to the same genus one Heegaard diagram $H(a, b, c, 1, d-r, 0)$. Hence $M(a, b, c, r)$ is homeomorphic to $M(a, b, c, 1, d-r, 0)$, and thus the same $(1,1)$-decomposition. This shows that the $(1,1)$-knot $K(a, b, c, 1, d-r, 0)$ and the $(1,1)$-knot $K(a, b, c, r)$ are the same. This completes the proof. □

We end this section with some remarks. It follows from Lemma 3.3 (2) that if (a, b, c, n, r, s) is admissible, then $(a, b, c, 1, r, 0)$ is also admissible. But the converse is not true in general. By Corollary 3.5, we see that if $\sigma_1 = (a, b, c, 1, r, 0)$ is an admissible 6-tuple with $\tilde{p}_{\sigma_1} = \pm 1$, then the 6-tuple $\sigma_n = (a, b, c, n, r, s_{\sigma_1})$ is admissible for each integer $n > 1$. On the other hand, we have from Lemma 3.7 that the 6-tuple $(a, b, c, 1, r, 0)$ admissible if and only if the 4-tuple $(a, b, c, d-r)$ is admissible.

On the other hand, for given two positive integer n and k such that k divides n, if (a, b, c, r, n, s) is admissible, then (a, b, c, r, k, b) is also admissible and the Heegaard diagram $H(a, b, c, r, k, b)$ is the quotient of $H(a, b, c, r, n, b)$ by the action of a cyclic group of order $\frac{n}{k}$. Furthermore, the Dunwoody manifold $M(a, b, c, n, r, s)$ is the $\frac{n}{k}$-fold cyclic covering of the manifold $M' = M(a, b, c, k, r, s)$ branched over a $(k, 1)$-knot in M'.

4. Cyclic branched coverings of $(1,1)$-knots in S^3 up to 10 crossings

In regard to Theorem 3.9, Grasselli and Mulazzani have raised the following problem [5, 8]:

Problem C. Characterize the class $\mathcal{K}$ of all branching $(1,1)$-knots $K(a, b, c, r)$ in S^4 involved in Theorem 3.9.

Indeed, this problem is to find explicit representations (Dunwoody six parameters $(a, b, c, n, r, s) \in \mathbb{Z}^6$) as Dunwoody manifolds of n-fold cyclic branched coverings of important classes of $(1,1)$-knots, in particular, $(1,1)$-knots in S^3. Up to now, explicit representations of all n-fold cyclic branched coverings of 2-bridge knots and a certain class of torus knots have been presented in [1, 8]. In this section, we shall give an explicit representation of all n-fold cyclic branched coverings of all $(1,1)$-knots up to 10 crossings in Rolfsen's knot table [15]. We begin with collecting the previous results.

Recall that a genus 0 2-bridge knot K, i.e., a $(0,2)$-knot, is a classical 2-*bridge knot* in S^3, which are completely determined by two integers α and β such that $\gcd(\alpha, \beta) = 1, \alpha > 0$ odd and $-\alpha < \beta < \alpha$. In this case, we say that the knot K in S^3 is the 2-bridge knot of type (α, β) or K has a Schubert normal form $b(\alpha, \beta)$. It is well known that two Schubert normal forms $b(\alpha, \beta)$ and $b(\alpha', \beta')$ represent equivalent (unoriented) 2-bridge knots if and only if $\alpha = \alpha'$ and $\beta^{\pm 1} \equiv \beta' \pmod{\alpha}$. This shows that the 2-bridge knot of type (α, β) is equivalent to the 2-bridge knot of type $(\alpha, \alpha + \beta)$, and hence β can be assumed to be always even. For more details about 2-bridge knots, we refer to [3].

Proposition 4.1. [8, Theorem 8] *The 4-tuple $\sigma = (a, 0, 1, r)$ with* $\gcd(2a + 1, 2r) = 1$ *is admissible. Moreover, the 6-tuple $\sigma_n = (a, 0, 1, n, r, -q_\sigma)$ is admissible for each integer $n > 1$ and the Dunwoody 3-manifold $M(a, 0, 1, n, r, -q_\sigma)$ is the n-fold cyclic covering of S^3 branched over the 2-bridge knot with Schubert normal form $(2a + 1, 2r)$. Thus, all n-fold branched cyclic coverings of S^3 branched over 2-bridge knots are Dunwoody 3-manifolds.*

For two relatively prime integers p and q, a *torus knot of type* (p, q), denoted by $T(p, q)$, is a simple closed curve on the surface of the unknotted torus in S^3 that cuts a meridian in p points and a longitude in q points. In [1], H. Aydin, I. Gultekyn and M. Mulazzani gave an explicit representation as Dunwoody manifolds of all cyclic branched coverings of torus knots of type $(p, mp - 1)$, with $p > 1$ and $m > 0$, thus including all torus knots with bridge number ≤ 4 as follows:

Proposition 4.2. [1, Corollary 4]

(1) *For all $m > 0$ and $p > 1$, the n-fold cyclic branched covering of the torus knot $T(p, mp + 1)$ is the Dunwoody manifold $M(1, p - 2, 2mp - 2m - p + 1, n, p, p)$.*
(2) *For all $m > 1$ and $p > 1$, the n-fold cyclic branched covering of the torus knot $T(p, mp - 1)$ is the Dunwoody manifold $M(1, p - 2, 2mp - 2m - p - 1, n, -3p + 4, -p)$.*

A *Montesinos link* $K = M(b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ with r branches is a link in S^3 as shown in Fig. 8 (a). Here r, b, α_i and β_i are integers such that $r \geq 0$, $\alpha_i \geq 2$, and $gcd(\alpha_i, \beta_i) = 1$. A box β/α stands for a rational tangle of slope β/α. (See Figure 8 (b).) If we forget the chart on the boundary, a rational tangle is merely a 2-stand trivial tangle as illustrated in Figure 8 (c); we recall the image of the arc τ in Figure 8 (c) in a rational tangle the *core* of the rational tangle.

Proposition 4.3. [3, Chapter 12]

(1) *Suppose $r = 2$. Then K is a 2-bridge link $b(p, q)$ of type (p, q), where $p = |b\alpha_1\alpha_2 - \alpha_1\beta_2 - \alpha_2\beta_1|$ and q is an integer relatively prime to p. In particular, K is a trivial knot if and only if $p = 1$.*
(2) *Suppose $r \geq 3$. Then K is not a 2-bridge link, and it is classified by the Euler number $e(k) = b - \sum_{i=1}^{r} \beta_i/\alpha_i$, and the vector $v(K) = (\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r) \in (\mathbb{Q}/\mathbb{Z})^r$ up to cyclic permutation and reversal of the order.*

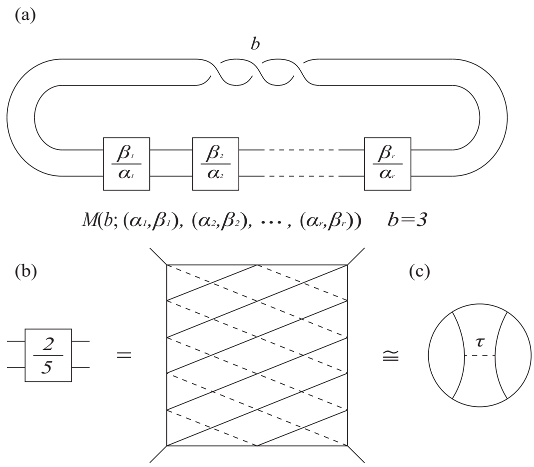
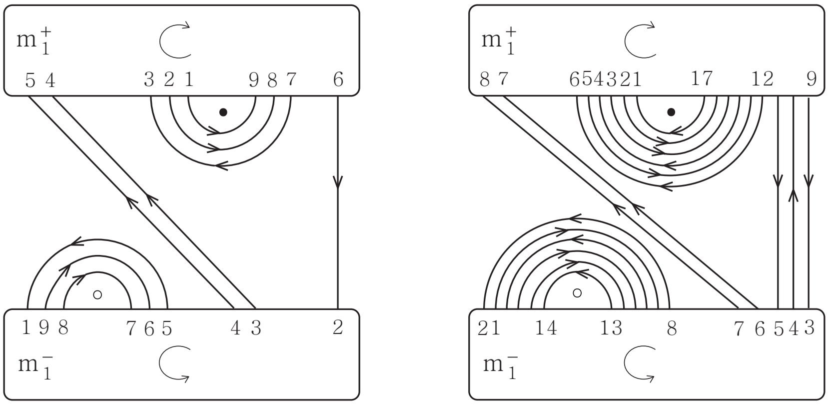


FIGURE 8. Montesinos link $M(3; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$

In [10], the authors give a table of $(1,1)$-diagrams $D(a,b,c,r)$ of all $(1,1)$-knots up to 10 crossings in Rolfsen's knot table [15]. Using this table, we give a complete list of explicit representations $(a,b,c,n,r,s) \in \mathbb{Z}^6$ as Dunwoody manifolds of all n-fold cyclic branched coverings of $(1,1)$-knots in S^3 up to 10 crossings in Rolfsen's knot table. The following theorem is the main result of this section, which give another partial answer to Problem C in addition to Propositions 4.1 and 4.2.

Theorem 4.4. *Let K be an $(1,1)$-knot in S^3 with crossing number ≤ 10 and let $\sigma = (a,b,c,r)$ be an admissible 4-tuple such that $K = K(a,b,c,r)$. If $p_\sigma = \pm 1$ and $s_\sigma = -p_\sigma q_\sigma$, then the 6-tuple $\sigma_n = (a,b,c,n,d-r,s_\sigma)$ is admissible for each integer $n > 1$ and the Dunwoody manifold $M(a,b,c,n,d-r,s_\sigma)$ is the n-fold cyclic covering of S^3 branched over the $(1,1)$-knot $K = K(a,b,c,r)$, which is independent on n. Furthermore, the associated $(1,1)$-diagram $D(a,b,c,r)$, p_σ, q_σ and the corresponding Dunwoody representation $(a,b,c,n,d-r,s_\sigma)$ are given in Tables 1, 2, 3 and 4.*

Proof. Let K be an $(1,1)$-knot in S^3 with crossing number ≤ 10. Then the associated $(1,1)$-diagram $D(a,b,c,r)$ with $K = K(a,b,c,r)$ is found in [10]. From this $(1,1)$-diagram $D(a,b,c,r)$, we can directly calculate p_σ, q_σ, $s_\sigma = -p_\sigma q_\sigma$ for $\sigma = (a,b,c,r)$ and the corresponding Dunwoody representation $(a,b,c,n,d-r,s_\sigma)$. Although the calculation is straightforward, but tedious and lengthy, so that we omit here. Instead, we illustrate the calculation with two sample calculations for two $(1,1)$-knots 9_{42} and 10_{161} in Examples 4.5 and 4.6, respectively. $\square$



(a) D(3,2,1,7) (b) D(6,2,3,13)

FIGURE 9. $(1, 1)$-diagrams of 9_{42} and 10_{136}

Example 4.5. *Consider the* $(1, 1)$*-knot* 9_{42}*, the Montesinos knot* $M(1; (2, 1), (3, 1), (5, 2))$ *which is not both* 2*-bridge knot and torus knot. From the table of* $(1, 1)$*-diagrams of* $(1, 1)$*-knots up to 10 crossings in* [10]*, we obtain* $(1, 1)$*-diagram* $D(3, 2, 1, 7)$ *of* 9_{42} *as depicted in Figure 9. Set* $\sigma = (3, 2, 1, 7)$*, admissible* 4*-tuple.*

From Figure 9(a), we have $p'_\sigma = 1, p''_\sigma = 2$ *and* $q'_\sigma = 4, q''_\sigma = 4$*. Hence it follows from (3.5) that* $p_\sigma = p'_\sigma - p''_\sigma = 1 - 2 = -1$ *and* $q_\sigma = q'_\sigma - q''_\sigma = 4 - 4 = 0$*. This gives* $s_\sigma = -p_\sigma q_\sigma = 0$*. Note that* $d - r = 9 - 7 = 2$*. Hence for each integer* $n > 1$*, the Dunwoody manifold* $M(3, 2, 1, n, 2, 0)$ *is the* n*-fold cyclic covering of* S^3 *branched over the* $(1, 1)$*-knot* 9_{42} *as listed in Table 2.*

Example 4.6. *Consider the* $(1, 1)$*-knot* 10_{136}*, the Montesinos knot* $M(1; (2, 1), (5, 2), (5, 2))$ *which is not both* 2*-bridge knot and torus knot. From the table of* $(1, 1)$*-diagrams of* $(1, 1)$*-knots in* [10]*, we have an* $(1, 1)$*-diagram* $D(6, 2, 3, 13)$ *of* 10_{136} *as depicted in Figure 9. Set* $\tau = (6, 2, 3, 13)$*, admissible* 4*-tuple.*

From Figure 9(b), we obtain $p'_\tau = 2, p''_\tau = 3$ *and* $q'_\tau = 6, q''_\tau = 8$*. Hence it follows from (3.5) that* $p_\tau = p'_\tau - p''_\tau = 2 - 3 = -1$ *and* $q_\tau = q'_\tau - q''_\tau = 6 - 8 = -2$*. This gives* $s_\tau = -p_\tau q_\tau = -2$*. Note that* $d - r = 17 - 13 = 4$*. Hence for each integer* $n > 1$*, the Dunwoody manifold* $M(6, 2, 3, n, 4, -2)$ *is the* n*-fold cyclic covering of* S^3 *branched over the* $(1, 1)$*-knot* 10_{136} *as listed in Table 4.*

K	Type	D(a,b,c,r)	p_σ	q_σ	$(a,b,c,n,d-r,s_\sigma)$
3_1	$T(2,3)=b(3,1)$	$D(1,0,1,2)$	1	2	(1,0,1,n,1,-2)
4_1	$b(5,2)$	$D(2,0,1,1)$	1	0	(2,0,1,n,4,0)
5_1	$T(2,5)=b(5,1)$	$D(2,0,1,3)$	1	4	(2,0,1,n,2,-4)
5_2	$b(7,3)$	$D(3,0,1,5)$	1	2	(3,0,1,n,2,-2)
6_1	$b(9,4)$	$D(4,0,1,2)$	1	0	(4,0,1,n,7,0)
6_2	$b(11,4)$	$D(5,0,1,2)$	1	2	(5,0,1,n,9,-2)
6_3	$b(13,5)$	$D(6,0,1,9)$	1	0	(6,0,1,n,4,0)
7_1	$T(2,7)=b(7,1)$	$D(3,0,1,4)$	1	6	(3,0,1,n,3,-6)
7_2	$b(11,5)$	$D(5,0,1,8)$	1	2	(5,0,1,n,3,-2)
7_3	$b(13,4)$	$D(6,0,1,2)$	1	-4	(6,0,1,n,11,4)
7_4	$b(15,4)$	$D(7,0,1,2)$	1	-2	(7,0,1,n,13,2)
7_5	$b(17,7)$	$D(8,0,1,12)$	1	4	(8,0,1,n,5,-4)
7_6	$b(19,7)$	$D(9,0,1,13)$	1	2	(9,0,1,n,6,-2)
7_7	$b(21,8)$	$D(10,0,1,4)$	1	0	(10,0,1,n,17,0)
8_1	$b(13,6)$	$D(6,0,1,3)$	1	0	(6,0,1,n,10,0)
8_2	$b(17,6)$	$D(8,0,1,3)$	1	4	(8,0,1,n,14,-4)
8_3	$b(17,4)$	$D(8,0,1,2)$	1	0	(8,0,1,n,15,0)
8_4	$b(19,5)$	$D(9,0,1,12)$	1	2	(9,0,1,n,7,-2)
8_5	$M(0;(2,1),(3,1),(3,1))$	$D(9,0,3,13)$	1	2	(9,0,3,n,8,-2)
8_6	$b(23,10)$	$D(11,0,1,5)$	1	2	(11,0,1,n,18,-2)
8_7	$b(23,9)$	$D(11,0,1,16)$	1	-2	(11,0,1,n,7,2)
8_8	$b(25,9)$	$D(12,0,1,17)$	1	0	(12,0,1,n,8,0)
8_9	$b(25,7)$	$D(12,0,1,16)$	1	0	(12,0,1,n,9,0)
8_{10}	$M(0;(2,1),(3,1),(3,2))$	$D(12,0,3,17)$	1	4	(12,0,3,n,10,-4)
8_{11}	$b(27,10)$	$D(13,0,1,5)$	1	2	(13,0,1,n,22,-2)
8_{12}	$b(29,12)$	$D(14,0,1,6)$	1	0	(14,0,1,n,23,0)
8_{13}	$b(29,11)$	$D(14,0,1,20)$	1	0	(14,0,1,n,9,0)
8_{14}	$b(31,12)$	$D(15,0,1,6)$	1	2	(15,0,1,n,25,-2)
8_{15}	$M(0;(2,1),(3,2),(3,2))$	$D(15,0,3,10)$	1	-2	(15,0,3,n,23,2)
8_{19}	$T(3,4)$	$D(1,2,1,3)$	1	4	(1,2,1,n,2,-4)
8_{20}	$M(1;(2,1),(3,1),(3,2))$	$D(3,0,3,5)$	1	2	(3,0,3,n,4,-2)
8_{21}	$M(1;(2,1),(3,2),(3,2))$	$D(6,0,3,4)$	1	0	(6,0,3,n,11,0)
9_1	$T(2,9)=b(9,1)$	$D(4,0,1,5)$	1	8	(4,0,1,n,4,-8)
9_2	$b(15,7)$	$D(7,0,1,11)$	1	2	(7,0,1,n,4,-2)
9_3	$b(19,6)$	$D(9,0,1,3)$	1	-6	(9,0,1,n,16,6)
9_4	$b(21,5)$	$D(10,0,1,13)$	1	4	(10,0,1,n,8,-4)
9_5	$b(23,6)$	$D(11,0,1,3)$	1	-2	(11,0,1,n,20,2)
9_6	$b(27,5)$	$D(13,0,1,16)$	1	6	(13,0,1,n,11,-6)
9_7	$b(29,13)$	$D(14,0,1,21)$	1	4	(14,0,1,n,8,-4)

Table 1. Dunwoody six parameter representations of n-fold cyclic branched coverings of $(1,1)$-knots in S^3 with crossings ≤ 10 as Dunwoody manifolds $(n>1)$

K	**Type**	**D(a,b,c,r)**	p_σ	q_σ	$(a, b, c, n, d-r, s_\sigma)$
9_8	$b(31, 11)$	$D(15, 0, 1, 21)$	1	2	(15,0,1,n,10,-2)
9_9	$b(31, 9)$	$D(15, 0, 1, 20)$	1	6	(15,0,1,n,11,-6)
9_{10}	$b(33, 10)$	$D(16, 0, 1, 5)$	1	-4	(16,0,1,n,28,4)
9_{11}	$b(33, 14)$	$D(16, 0, 1, 7)$	1	-4	(16,0,1,n,26,4)
9_{12}	$b(35, 13)$	$D(17, 0, 1, 24)$	1	2	(17,0,1,n,11,-2)
9_{13}	$b(37, 10)$	$D(18, 0, 1, 5)$	1	-4	(18,0,1,n,32,4)
9_{14}	$b(37, 14)$	$D(18, 0, 1, 7)$	1	0	(18,0,1,n,30,0)
9_{15}	$b(39, 16)$	$D(19, 0, 1, 8)$	1	-2	(19,0,1,n,31,2)
9_{16}	$M(-1; (2, 1), (3, 1), (3, 1))$	$D(18, 0, 3, 25)$	1	4	(18,0,3,n,14,-4)
9_{17}	$b(39, 14)$	$D(19, 0, 1, 7)$	1	2	(19,0,1,n,32,-2)
9_{18}	$b(41, 17)$	$D(20, 0, 1, 29)$	1	4	(20,0,1,n,12,-4)
9_{19}	$b(41, 16)$	$D(20, 0, 1, 8)$	1	0	(20,0,1,n,33,0)
9_{20}	$b(41, 15)$	$D(20, 0, 1, 28)$	1	4	(20,0,1,n,13,-4)
9_{21}	$b(43, 18)$	$D(21, 0, 1, 9)$	1	-2	(21,0,1,n,34,2)
9_{22}	$M(0; (2, 1), (3, 1), (5, 3))$	$D(20, 0, 3, 8)$	1	0	(20,0,3,n,35,0)
9_{23}	$b(45, 19)$	$D(22, 0, 1, 32)$	1	4	(22,0,1,n,13,-4)
9_{24}	$M(-1; (2, 1), (3, 1), (3, 2))$	$D(21, 0, 3, 14)$	1	-2	(21,0,3,n,31,2)
9_{25}	$M(0; (2, 1), (3, 2), (5, 2))$	$D(22, 0, 3, 37)$	1	0	(22,0,3,n,10,0)
9_{26}	$b(47, 18)$	$D(23, 0, 1, 9)$	1	-2	(23,0,1,n,38,2)
9_{27}	$b(49, 19)$	$D(24, 0, 1, 34)$	1	0	(24,0,1,n,15,0)
9_{28}	$M(-1; (2, 1), (3, 2), (3, 2))$	$D(24, 0, 3, 16)$	1	0	(24,0,3,n,35,0)
9_{30}	$M(0; (2, 1), (3, 2), (5, 3))$	$D(25, 0, 3, 10)$	1	2	(25,0,3,n,43,-2)
9_{31}	$b(55, 21)$	$D(27, 0, 1, 38)$	1	2	(27,0,1,n,17,-2)
9_{36}	$M(0; (2, 1), (3, 1), (5, 2))$	$D(17, 0, 3, 29)$	1	2	(17,0,3,n,8,-2)
9_{42}	$M(1; (2, 1), (3, 1), (5, 2))$	$D(3, 2, 1, 7)$	-1	0	(3,2,1,n,2,0)
9_{43}	$M(1; (2, 1), (3, 1), (5, 3))$	$D(5, 0, 3, 2)$	1	2	(5,0,3,n,11,-2)
9_{44}	$M(1; (2, 1), (3, 2), (5, 2))$	$D(7, 0, 3, 13)$	1	2	(7,0,3,n,4,-2)
9_{45}	$M(1; (2, 1), (3, 2), (5, 3))$	$D(10, 0, 3, 4)$	1	0	(10,0,3,n,19,0)
10_1	$b(17, 8)$	$D(8, 0, 1, 4)$	1	0	(8,0,1,n,13,0)
10_2	$b(23, 8)$	$D(11, 0, 1, 4)$	1	6	(11,0,1,n,19,-6)
10_3	$b(25, 6)$	$D(12, 0, 1, 3)$	1	0	(12,0,1,n,22,0)
10_4	$b(27, 7)$	$D(13, 0, 1, 17)$	1	2	(13,0,1,n,10,-2)
10_5	$b(33, 13)$	$D(16, 0, 1, 23)$	1	-4	(16,0,1,n,10,4)
10_6	$b(37, 16)$	$D(18, 0, 1, 8)$	1	4	(18,0,1,n,29,-4)
10_7	$b(43, 16)$	$D(21, 0, 1, 8)$	1	2	(21,0,1,n,35,-2)
10_8	$b(29, 6)$	$D(14, 0, 1, 3)$	1	2	(14,0,1,n,26,-2)
10_9	$b(39, 11)$	$D(19, 0, 1, 25)$	1	-2	(19,0,1,n,14,2)
10_{10}	$b(45, 17)$	$D(22, 0, 1, 31)$	1	0	(22,0,1,n,14,0)

Table 2. Dunwoody six parameter representations of n-fold cyclic branched coverings of $(1,1)$-knots in S^3 with crossings ≤ 10 as Dunwoody manifolds ($n > 1$)

K	**Type**	**D(a,b,c,r)**	p_σ	q_σ	$(a,b,c,n,d-r,s_\sigma)$
10_{11}	$b(43,13)$	$D(21,0,1,28)$	1	2	(21,0,1,n,15,-2)
10_{12}	$b(47,17)$	$D(23,0,1,32)$	1	-2	(23,0,1,n,15,2)
10_{13}	$b(53,22)$	$D(26,0,1,11)$	1	0	(26,0,1,n,42,0)
10_{14}	$b(57,22)$	$D(28,0,1,11)$	1	4	(28,0,1,n,46,-4)
10_{15}	$b(43,19)$	$D(21,0,1,31)$	1	-2	(21,0,1,n,12,2)
10_{16}	$b(47,14)$	$D(23,0,1,7)$	1	-2	(23,0,1,n,20,2)
10_{17}	$b(41,9)$	$D(20,0,1,25)$	1	0	(20,0,1,n,16,0)
10_{18}	$b(55,23)$	$D(27,0,1,39)$	1	2	(27,0,1,n,16,-2)
10_{19}	$b(51,14)$	$D(25,0,1,7)$	1	2	(25,0,1,n,44,-2)
10_{20}	$b(35,16)$	$D(17,0,1,8)$	1	2	(17,0,1,n,27,-2)
10_{21}	$b(45,16)$	$D(22,0,1,8)$	1	4	(22,0,1,n,37,-4)
10_{22}	$b(49,13)$	$D(24,0,1,31)$	1	0	(24,0,1,n,18,0)
10_{23}	$b(59,23)$	$D(29,0,1,41)$	1	-2	(29,0,1,n,18,2)
10_{24}	$b(55,24)$	$D(27,0,1,12)$	1	2	(27,0,1,n,43,-2)
10_{25}	$b(65,24)$	$D(32,0,1,12)$	1	4	(32,0,1,n,53,-4)
10_{26}	$b(61,17)$	$D(30,0,1,39)$	1	0	(30,0,1,n,22,0)
10_{27}	$b(71,27)$	$D(35,0,1,49)$	1	-2	(35,0,1,n,22,2)
10_{28}	$b(53,19)$	$D(26,0,1,36)$	1	0	(26,0,1,n,17,0)
10_{29}	$b(63,26)$	$D(31,0,1,13)$	1	2	(31,0,1,n,50,-2)
10_{30}	$b(67,26)$	$D(33,0,1,13)$	1	2	(33,0,1,n,54,-2)
10_{31}	$b(57,25)$	$D(28,0,1,41)$	1	0	(28,0,1,n,16,0)
10_{32}	$b(69,29)$	$D(34,0,1,49)$	1	0	(34,0,1,n,20,0)
10_{33}	$b(65,18)$	$D(32,0,1,9)$	1	0	(32,0,1,n,56,0)
10_{34}	$b(37,13)$	$D(18,0,1,25)$	1	0	(18,0,1,n,12,0)
10_{35}	$b(49,20)$	$D(24,0,1,10)$	1	0	(24,0,1,n,39,0)
10_{36}	$b(51,20)$	$D(25,0,1,10)$	1	2	(25,0,1,n,41,-2)
10_{37}	$b(53,23)$	$D(26,0,1,38)$	1	0	(26,0,1,n,15,0)
10_{38}	$b(59,25)$	$D(29,0,1,42)$	1	2	(29,0,1,n,17,-2)
10_{39}	$b(61,22)$	$D(30,0,1,11)$	1	4	(30,0,1,n,50,-4)
10_{40}	$b(75,29)$	$D(37,0,1,52)$	1	-2	(37,0,1,n,23,2)
10_{41}	$b(71,26)$	$D(35,0,1,13)$	1	2	(35,0,1,n,58,-2)
10_{42}	$b(81,31)$	$D(40,0,1,56)$	1	0	(40,0,1,n,25,0)
10_{43}	$b(73,27)$	$D(36,0,1,50)$	1	0	(36,0,1,n,23,0)
10_{44}	$b(79,30)$	$D(39,0,1,15)$	1	2	(39,0,1,n,64,-2)
10_{45}	$b(89,34)$	$D(44,0,1,17)$	1	0	(44,0,1,n,72,0)
10_{46}	$M(0;(2,1),(3,1),(5,1))$	$D(14,0,3,18)$	1	0	(14,0,3,n,13,0)
10_{47}	$M(0;(2,1),(3,2),(5,1))$	$D(19,0,3,24)$	1	6	(19,0,3,n,17,-6)
10_{48}	$M(0;(2,1),(3,1),(5,4))$	$D(23,0,3,19)$	1	-2	(23,0,3,n,30,2)

Table 3. Dunwoody six parameter representations of n-fold cyclic branched coverings of $(1,1)$-knots in S^3 with crossings ≤ 10 as Dunwoody manifolds $(n>1)$

K	**Type**	**D(a,b,c,r)**	p_σ	q_σ	$(a, b, c, n, d-r, s_\sigma)$
10_{49}	$M(0;(2,1),(3,2),(5,4))$	$D(28,0,3,23)$	1	-4	(28,0,3,n,36,4)
10_{50}	$M(0;(2,1),(3,1),(7,3))$	$D(25,0,3,45)$	1	2	(25,0,3,n,8,-2)
10_{51}	$M(0;(2,1),(3,2),(7,3))$	$D(32,0,3,57)$	1	4	(32,0,3,n,10,-4)
10_{52}	$M(0;(2,1),(3,1),(7,4))$	$D(28,0,3,8)$	1	0	(28,0,3,n,51,0)
10_{53}	$M(0;(2,1),(3,2),(7,4))$	$D(35,0,3,10)$	1	-2	(35,0,3,n,63,2)
10_{54}	$M(0;(2,1),(3,1),(7,2))$	$D(22,0,3,13)$	1	0	(22,0,3,n,34,0)
10_{55}	$M(0;(2,1),(3,2),(7,2))$	$D(29,0,3,17)$	1	-2	(29,0,3,n,44,2)
10_{56}	$M(0;(2,1),(3,1),(7,5))$	$D(31,0,3,46)$	1	2	(31,0,3,n,19,-2)
10_{57}	$M(0;(2,1),(3,2),(7,5))$	$D(38,0,3,56)$	1	4	(38,0,3,n,23,-4)
10_{58}	$M(0;(2,1),(5,2),(5,2))$	$D(30,0,5,51)$	1	0	(30,0,5,n,14,0)
10_{59}	$M(0;(2,1),(5,2),(5,3))$	$D(35,0,5,59)$	1	2	(35,0,5,n,16,-2)
10_{60}	$M(0;(2,1),(5,3),(5,3))$	$D(40,0,5,16)$	1	0	(40,0,5,n,69,0)
10_{70}	$M(-1;(2,1),(3,1),(5,2))$	$D(32,0,3,53)$	1	0	(32,0,3,n,14,0)
10_{71}	$M(-1;(2,1),(3,2),(5,2))$	$D(37,0,3,61)$	1	2	(37,0,3,n,16,-2)
10_{72}	$M(-1;(2,1),(3,1),(5,3))$	$D(35,0,3,14)$	1	2	(35,0,3,n,59,-2)
10_{73}	$M(-1;(2,1),(3,2),(5,3))$	$D(40,0,3,16)$	1	0	(40,0,3,n,67,0)
10_{76}	$M(-2;(2,1),(3,1),(3,1))$	$D(27,0,3,37)$	1	2	(27,0,3,n,20,-2)
10_{77}	$M(-2;(2,1),(3,1),(3,2))$	$D(30,0,3,41)$	1	4	(30,0,3,n,22,-4)
10_{78}	$M(-2;(2,1),(3,2),(3,2))$	$D(33,0,3,22)$	1	-2	(33,0,3,n,47,2)
10_{124}	$T(3,5)$	$D(2,2,1,4)$	-1	-6	(2,2,1,n,3,-6)
10_{125}	$M(1;(2,1),(3,2),(5,1))$	$D(4,0,3,6)$	1	4	(4,0,3,n,5,-4)
10_{126}	$M(1;(2,1),(3,1),(5,4))$	$D(8,0,3,7)$	1	-4	(8,0,3,n,12,4)
10_{127}	$M(1;(2,1),(3,2),(5,4))$	$D(12,0,5,8)$	1	0	(12,0,5,n,21,0)
10_{128}	$M(1;(2,1),(3,1),(7,3))$	$D(5,2,1,11)$	-1	-4	(5,2,1,n,2,-4)
10_{129}	$M(1;(2,1),(3,2),(7,3))$	$D(11,0,3,21)$	1	2	(11,0,3,n,4,-2)
10_{130}	$M(1;(2,1),(3,1),(7,4))$	$D(7,0,3,2)$	1	-2	(7,0,3,n,15,2)
10_{131}	$M(1;(2,1),(3,2),(7,4))$	$D(14,0,3,4)$	1	0	(14,0,3,n,27,0)
10_{132}	$M(1;(2,1),(3,1),(7,2))$	$D(4,2,1,3)$	-1	2	(4,2,1,n,8,2)
10_{133}	$M(1;(2,1),(3,2),(7,2))$	$D(8,0,3,5)$	1	0	(8,0,3,n,14,0)
10_{134}	$M(1;(2,1),(3,1),(7,5))$	$D(10,0,3,16)$	1	4	(10,0,3,n,7,-4)
10_{135}	$M(1;(2,1),(3,2),(7,5))$	$D(17,0,3,26)$	1	2	(17,0,3,n,11,-2)
10_{136}	$M(1;(2,1),(5,2),(5,2))$	$D(6,2,3,13)$	-1	-2	(6,2,3,n,4,-2)
10_{137}	$M(1;(2,1),(5,2),(5,3))$	$D(10,0,5,19)$	1	0	(10,0,5,n,6,0)
10_{138}	$M(1;(2,1),(5,3),(5,3))$	$D(15,0,5,6)$	1	2	(15,0,5,n,29,-2)
10_{139}	$M(1;(3,1),(3,1),(4,1))$	$D(3,1,4,5)$	1	3	(3,1,4,n,6,3)
10_{145}	$M(1;(3,1),(3,1),(5,2))$	$D(3,6,1,7)$	-1	0	(3,6,1,n,6,0)
10_{161}	$[3:-20:-20]$	$D(4,4,1,7)$	1	0	(4,4,1,n,6,0)

Table 4. Dunwoody six parameter representations of n-fold cyclic branched coverings of $(1,1)$-knots in S^3 with crossings ≤ 10 as Dunwoody manifolds $(n > 1)$

References

[1] H. Aydin, I. Gültekyn and M. Mulazzani, *Torus knots and Dunwoody manifolds*, Siberian Math. J. **45** (2004), no. 1, 1–6.

[2] S. A. Bleiler, *Two-generator cable knots are tunnel one*, Proc. Amer. Math. Soc. **122** (1994), no. 4, 1285–1287.

[3] G. Burde and H. Zieschang, *Knots*, de Gruyter Studies in Mathematics **5**. Walter de Gruyter, Berlin, 1985.

[4] A. Cattabriga and M. Mulazzani, *All strongly-cyclic branched coverings of* $(1,1)$*-knots are Dunwoody manifolds*, J. London Math. Soc. **70** (2004), no. 2, 512–528.

[5] H. R. Cho, S. Y. Lee and H.-J. Song, *Derivations of Schubert normal forms of 2-bridge knots from* $(1,1)$*-diagrams*, J. Knot Theory Ramifications **28** (2019), no. 8, 1950048 (29 pages).

[6] M. J. Dunwoody, *Cyclic presentations and 3-manifolds*, in Proc. Inter. Conf., Groups-Korea 94. Walter de Gruyter, Berlin-New York (1995), 47–55.

[7] H. Doll, *A genralized bridge number for links in* 3*-manifolds*, Math. Ann. **294** (1992), no. 1, 701–717.

[8] L. Grasselli and M. Mulazzani, *Genus one 1-bridge knots and Dunwoody manifolds*, Forum Math. **13** (2001), no. 3, 379–397.

[9] C. Hayashi, *1-genus 1-bridge splittings for knots*, Osaka J. Math. **41** (2004), no. 2, 371–426.

[10] G. Kim and S. Y. Lee, *Doubly-pointed Heegaard diagrams of* $(1,1)$*-knots up to* 10 *crossings and* $(1,1)$*-pretzel knots*, preprint (2021).

[11] E. Klimenko and M. Sakuma, *Two-generator discrete subgroups of Isom(H2) containing orientation-reversing elements*, Geom. Dedicata **72** (1998), 247–282.

[12] K. Morimoto and M. Sakuma, *On unknotting tunnels for knots*, Math. Ann. **289** (1991), no. 1, 143–167.

[13] K. Morimoto, M. Sakuma and Y. Yokota, *Identifying tunnel number one knots*, J. of the Mathematical Society of Japan **48** (1996), no. 4, 667–688.

[14] Mario Eudave Muoz, *On nonsimple* 3*-manifolds and* 2*-handle addition*, Topology Appl. **55** (1994), no. 2, 131–152.

[15] D. Rolfsen, *Knots and links*, Mathematics Lecture Series **7**, Publish or Perish, Inc., Berkeley, 1976.

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