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PACKING TREES INTO COMPLETE K-PARTITE GRAPH

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Abstract. In this work, we confirm a weak version of a conjecture proposed by Hong Wang. The ideal of the work comes from the tree packing conjecture made by Gyárfás and Lehel. Bollobás confirms the tree packing conjecture for many small tree, who showed that one can pack $T_1, T_2, \ldots, T_{n/\sqrt{2}}$ into K_n and that a better bound would follow from a famous conjecture of Erdős. In a similar direction, Hobbs, Bourgeois and Kasiraj made the following conjecture: Any sequence of trees $T_1, T_2, \ldots, T_n$, with T_i having order i, can be packed into $K_{n-1,\lceil n/2\rceil}$. Further Hobbs, Bourgeois and Kasiraj [3] proved that any two trees can be packed into a complete bipartite graph $K_{n-1,\lceil n/2\rceil}$. Motivated by the result, Hong Wang propose the conjecture: For each k-partite tree $T(\mathbb{X})$ of order n, there is a restrained packing of two copies of $T(\mathbb{X})$ into a complete k-partite graph $B_{n+m}(\mathbb{Y})$, where $m = \lfloor \frac{k}{2} \rfloor$. Hong Wong [4] confirmed this conjecture for $k = 2$. In this paper, we prove a weak version of this conjecture.

1. Introduction

For graphs G and H, an embedding of G into H is an injection $\phi : V(G) \to V(H)$ such that $\phi(a)\phi(b) \in E(H)$ whenever $ab \in E(G)$. A packing of p graphs $G_1, G_2, \ldots, G_p$ into H is a p-tuple $\Phi = (\phi_1, \phi_2, \ldots, \phi_p)$ such that, for $i = 1, 2, \ldots, p$, ϕ_i is an embedding of G_i into H and the p sets $\phi_i(E(G_i))$ are mutually disjoint. Packing problems are central to combinatorics. Many classical problems can be stated as packing problems, such as Mantel's Theorem which can be formulated by saying that if G is an n-vertex graph with less than $\binom{n}{2} - \frac{n^2}{4}$ edges, then the two graphs K_3 and G can be packed into K_n. The packing problem has received a lot of attention. Many interesting results and elegant proofs of these results were obtained. For a survey, see [5,6]. Among the best known packing problems, the famous tree packing conjecture of Gyárfás and Lehel has driven a large amount of research in the area.

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Conjecture 1.1 (Gyárfás and Lehel [2]). *Given $n \in N$ and trees $T_1, \dots, T_n$ with $|T_i| = i$, the graphs $T_1, T_2, \dots, T_n$ can be packed into a complete graph K_n.*

A packing of many of the small tree from Conjecture 1.1 was obtained by Bollobás [1], who showed that one can pack $T_1, T_2, \dots, T_{n/\sqrt{2}}$ into K_n and that a better bound would follow from a famous conjecture of Erdős. In a similar direction, Hobbs, Bourgeois and Kasiraj made the following conjecture.

Conjecture 1.2 (Hobbs, Bourgeois and Kasiraj [3]). *Any sequence of trees $T_2, T_3, \dots, T_n$, with T_i having order i, can be packed into $K_{n-1,\lceil \frac{n}{2} \rceil}$.*

The conjecture has been verified for several very special classes of trees. Hobbs, Bourgeois and Kasiraj [3] proved that any two trees of order m and n with $m < n$ can be packed into a complete bipartite graph of order $n+\lceil n/2 \rceil -1$ which admits an $(n-1, \lceil n/2 \rceil)$-bipartitions. Yuster [7] proved that any sequence of trees $T_1, \dots, T_s$, $s < \sqrt{5/8}n$ can be packed into $K_{n-1,\frac{n}{2}}$. Motivated by these results, Hong Wang propose the following conjecture. The value of this conjecture is to extend complete 2-partite graph to complete k-partite graph which has less edges than the K_n in other conjectures.

Conjecture 1.3. *For each k-partite tree $T(\mathbb{X})$ of order n, there is a restrained packing of two copies of $T(\mathbb{X})$ into a complete k-partite graph $B_{n+m}(\mathbb{Y})$, where $m = \lfloor \frac{k}{2} \rfloor$.*

Remark 1.4. In this conjecture, $\lfloor \frac{k}{2} \rfloor$ can not be reduced. A simple example is a *path* with k vertices.

This conjecture is true for $k = 2$ (see Theorem 1.5).

Theorem 1.5 ([4]). *Let $S(U_0, U_1)$ and $T(V_0, V_1)$ be two trees of order n with $|U_i| = |V_i|$ $(i = 0, 1)$. Then there exists a complete bipartite graph $B_{n+1}(X_0, X_1)$ such that there is a packing of $S(U_0, U_1)$ and $T(V_0, V_1)$ in $B_{n+1}(X_0, X_1)$.*

In the paper, we prove a weak version of this conjecture.

Theorem 1.6. *For each k-partite tree $T(\mathbb{X})$ of order n with partition $\mathbb{X} = (X_1, X_2, \dots, X_k)$, there is a restrained packing of two copies of $T(\mathbb{X})$ into a complete k-partite graph $B_{n+k-1}(\mathbb{Y})$, where $\mathbb{Y} = (Y_1, Y_2, \dots, Y_k)$ and $|X_i| \leq |Y_i|$, $i = 1, 2, \dots, k$.*

2. Preliminaries

For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A *neighbor* of a vertex v is a vertex adjacent to v in G. $N_G(v)$ denotes the set of neighbors of a vertex v in G. The *degree* of v, denoted by $\deg_G(v)$, is $|N_G(v)|$. Given a subset A of $V(G)$, $N_G(v, A)$ is $N_G(v) \cap A$ for the vertex $v \in V(G)$, and $\deg_G(v, A)$ is the order of $N_G(v, A)$. When the context is clear, the subscript G is omitted. Furthermore, an *end vertex* is a vertex of degree 1 and a *non-end vertex* is a vertex of degree > 1. A *node* is a vertex adjacent to an end vertex. A *supernode* is a node x of G such that, with one exception, every neighbor of x is an end vertex. For $n > 1$, the complete bipartite graph $K_{1,n-1}$ is called a *star*.

A k-partite graph G with the partition $\mathbb{X} = (X_1, X_2, \ldots, X_k)$ is denoted as $G(X_1, X_2, \ldots, X_k)$ or $G(\mathbb{X})$. In this case, it is said that G *admits* the partition $\mathbb{X}$ and $|\mathbb{X}| = k$. If G admits two distinct partitions $\mathbb{X}$ and $\mathbb{X}'$, then the notion that $G(\mathbb{X}) \neq G(\mathbb{X}')$ is adopted here. If G and H admit the k-partitions $\mathbb{X}$ and $\mathbb{Y}$, respectively, and ϕ is an embedding of G into H such that $\phi(X_i) \subset Y_i$, then ϕ is restrained and this is denoted as $\phi : G(\mathbb{X}) \to H(\mathbb{Y})$. A packing Φ of $G_1(X_1), G_2(X_2), \ldots, G_p(X_p)$ into $H(\mathbb{Y})$ is restrained if each embedding of Φ is restrained. A k-partite tree is a k-partite graph without cycles. Let $T(\mathbb{X})$ denote the k-partite tree with $\mathbb{X} = (X_1, X_2, \ldots, X_k)$ as its k-partition and $B_n(\mathbb{Y})$ denote the complete k-partite graph of order n with $\mathbb{Y} = (Y_1, Y_2, \ldots, Y_k)$ as its k-partition.

In the following lemmas, we assume that $T(\mathbb{X})$ is a counter-example of Theorem 1.6 of minimum order n with partition $\mathbb{X} = (X_1, X_2, \ldots, X_k)$.

Lemma 2.1. *The end vertices of $T(\mathbb{X})$ are adjacent to the same node if they are in the same partite.*

Proof. Assume that $v_1 \in X_1$ and $v_2 \in X_1$ are arbitrary two end vertices not adjacent to the same node. Suppose v_1 is adjacent to a node w_1 and v_2 is adjacent to a node w_2, where $w_1 \neq w_2$.

By the minimality of $T(\mathbb{X})$, there is a restrained packing $\Phi = (\phi_1, \phi_2)$ of $T(\mathbb{X}) - \{v_1, v_2\}$ into some $B_{n'+k-1}(\mathbb{Y}')$, where $n' = n-2$, $\mathbb{Y}' = (Y_1', Y_2', \ldots, Y_k')$, $|X_1| - 2 \leq |Y_1'|$ and $|X_i| \leq |Y_i'|$, $i = 2, \ldots, k$. Now, add two vertices y_1 and y_2 to Y_1'. If $\phi_2(w_1) = \phi_1(w_1)$ or $\phi_2(w_2) = \phi_1(w_2)$, then Φ can be extended to $T(\mathbb{X})$ so that $\phi_2(v_1) = \phi_1(v_2) = y_2$ and $\phi_2(v_2) = \phi_1(v_1) = y_1$, i.e., Φ is a restrained packing of $T(\mathbb{X})$ into the $B_{n+k-1}(\mathbb{Y})$, where $\mathbb{Y} = (Y_1' \cup \{y_1, y_2\}, Y_2', \ldots, Y_k')$, a contradiction. So, $\phi_2(w_1) \neq \phi_1(w_1)$ and $\phi_2(w_2) \neq \phi_1(w_2)$. Thus Φ can be extended to $T(\mathbb{X})$ so that $\phi_2(v_1) = \phi_1(v_1) = y_1$ and $\phi_2(v_2) = \phi_1(v_2) = y_2$, i.e., Φ is a restrained packing of $T(\mathbb{X})$ into the $B_{n+k-1}(\mathbb{Y})$, where $\mathbb{Y} = (Y_1' \cup \{y_1, y_2\}, Y_2', \ldots, Y_k')$, a contradiction.

Therefore, the end vertices of $T(\mathbb{X})$ are adjacent to the same node if they are in the same partite. □

Lemma 2.2. *If w is a node of $T(\mathbb{X})$ adjacent to end vertices in a partite X, then $|X|$ is odd and $deg(w, X) = \frac{|X|+1}{2}$.*

Proof. Let w be a node of $T(\mathbb{X})$. Without loss of generality, we may assume that $w \in X_2$ and let $v \in X_1$ be an end vertex adjacent to w. By the minimality of $T(\mathbb{X})$, there is a restrained packing $\Phi = (\phi_1, \phi_2)$ of $T(\mathbb{X}) - \{v\}$ into some $B_{n'+k-1}(\mathbb{Y}')$, where $n' = n-1$, $\mathbb{Y}' = (Y_1', Y_2', \ldots, Y_k')$, $|X_1| - 1 \leq |Y_1'|$ and $|X_i| \leq |Y_i'|$, $i = 2, \ldots, k$. It may be assumed that $\phi_1(w) = \phi_2(w)$, for otherwise, by adding a vertex y to Y_1', Φ can be extended to $T(\mathbb{X})$ such that $\phi_1(v) = \phi_2(v) = y$ and a restrained packing of two copies of $T(\mathbb{X})$ into the $B_{n+k-1}(\mathbb{Y})$ is obtained, where $\mathbb{Y} = (Y_1' \cup \{y\}, Y_2', \ldots, Y_k')$, a contradiction. Therefore $\phi_1(N(w, X_1 \setminus \{v\})) \cap \phi_2(N(w, X_1 \setminus \{v\})) = \emptyset$. Now, we claim that $Y_1' = \phi_1(N(w, X_1 \setminus \{v\})) \cup \phi_2(N(w, X_1 \setminus \{v\}))$. Suppose to contradict that there exists $r \in Y_1'$ such that $r \notin \phi_1(N(w, X_1 \setminus \{v\})) \cup \phi_2(N(w, X_1 \setminus \{v\}))$. If $r \notin \phi_2(X_1 \setminus \{v\})$, then adding a vertex y to Y_1' we extend the restrained packing Φ to $T(\mathbb{X})$ such that $\phi_1(v) = y$, $\phi_2(v) = r$. Thus, we get a restrained packing Φ of $T(\mathbb{X})$ into the $B_{n+k-1}(\mathbb{Y})$, where $\mathbb{Y} = (Y_1' \cup \{y\}, Y_2', \ldots, Y_k')$, a contradiction. So there exists $u \in X_1 \setminus \{v\}$ such that $\phi_2(u) = r$. But by adding a vertex y to Y_1', the restrained packing Φ can be extended to $T(\mathbb{X})$ such that $\phi_1(v) = y$, $\phi_2(u) = y$, $\phi_2(v) = r$, and so a restrained packing of two copies of $T(\mathbb{X})$ into the $B_{n+k-1}(\mathbb{Y})$ is obtained, where $\mathbb{Y} = (Y_1' \cup \{y\}, Y_2', \ldots, Y_k')$, a contradiction. Thus $Y_1' = \phi_1(N(w, X_1 \setminus \{v\})) \cup \phi_2(N(w, X_1 \setminus \{v\}))$ which implies that $|Y_1'|$ is even.

Note that $|Y_1'| > |X_1 \setminus \{v\}|$ or $|Y_1'| = |X_1 \setminus \{v\}|$. If $|Y_1'| > |X_1 \setminus \{v\}|$, then $\phi_1(X_1 \setminus \{v\}) \subset Y_1'$ and $\phi_2(X_1 \setminus \{v\}) \subset Y_1'$, since both ϕ_1 and ϕ_2 are injections. $\phi_1(X_1 \setminus \{v\}) \subset Y_1'$ implies that there exists $a \in Y_1'$ such that $a \notin \phi_1(X_1 \setminus \{v\})$, and $\phi_2(X_1 \setminus \{v\}) \subset Y_1'$ implies that there exists $b \in Y_1'$ such that $b \notin \phi_2(X_1 \setminus \{v\})$. We claim $a \neq b$, for otherwise, from $a \notin \phi_1(X_1 \setminus \{v\})$ we can infer $a \in \phi_2(X_1 \setminus \{v\})$ since $Y_1' = \phi_1(N(w, X_1 \setminus \{v\})) \cup \phi_2(N(w, X_1 \setminus \{v\}))$, a contradiction. So, if $a \notin \phi_1(X_1 \setminus \{v\})$ and $b \notin \phi_2(X_1 \setminus \{v\})$, then $a \in \phi_2(X_1 \setminus \{v\})$ and $b \in \phi_1(X_1 \setminus \{v\})$. Now adding a vertex z to Y_2' of $B_{n'+k-1}(\mathbb{Y}')$ we extend the Φ to $T(\mathbb{X})$ such that $\phi_2(w) = z$, $\phi_1(v) = a$, $\phi_2(v) = b$, and a restrained packing of two copies of $T(\mathbb{X})$ into the $B_{n+k-1}(\mathbb{Y})$ is obtained, where $\mathbb{Y} = (Y_1', Y_2' \cup \{z\}, \ldots, Y_k')$, a contradiction. Therefore $|Y_1'| = |X_1 \setminus \{v\}|$ which implies X_1 must be odd and $\deg(w, X_1) = \frac{|Y_1'|}{2} + 1 = \frac{|X_1|+1}{2}$. $\square$

3. Proof of Theorem 1.6

We use induction on k. Theorem 1.6 holds for $k = 2$ by Theorem 1.5. Suppose that Theorem 1.6 holds for x-partite tree, where $x < k$. Let $T(\mathbb{X})$ be a k-partite tree of order n with partition $\mathbb{X} = (X_1, X_2, \ldots, X_k)$.

We assume that $T(\mathbb{X})$ is a counter-example of Theorem 1.6 of minimum order n with partition $\mathbb{X} = (X_1, X_2, \ldots, X_k)$. Then $T(\mathbb{X})$ can not be the k-partite tree with $|X_i| = 1$, $i = 1, 2, \ldots, k$, since Theorem 1.6 holds for such a k-partite tree clearly.

If $T(\mathbb{X})$ has exactly one node w, then $T(\mathbb{X})$ is a star. So $\deg(w, X_i) = |X_i| \geq 2$ for some partite set X_i of $T(\mathbb{X})$, which contradict to Lemma 2.2. Hence $T(\mathbb{X})$ has at least two nodes. By observing a longest path of T, there exist at least two supernodes in $T(\mathbb{X})$. Let w be a supernode of $T(\mathbb{X})$ and u be the only one non-end vertex adjacent to w. Without loss of generality, we may assume that $w \in X_1$. Let $V_i = N(w, X_i)$, $i = 2, \ldots, k$. Then there is at least one of V_i, $i = 2, \ldots, k$, which is non-empty. Without loss of generality, we may assume that $V_2 \neq \emptyset$. So all the end vertices of X_2 are in the set V_2 by Lemma 2.1. Let

$W = X_2 \setminus V_2$. Then $X_2 = V_2 \cup W$ and $|V_2| = |W| + 1$ by Lemma 2.2. So we may assume that $V_2 = \{v_1, v_2, \ldots, v_t\}$ and $W = \{v_{t+1}, v_{t+2}, \ldots, v_{2t-1}\}$.

Consider the graph $T'(\mathbb{X}') = T(\mathbb{X}) - (X_2 \cup \{w\})$, where $\mathbb{X}' = (X'_1, X'_3, \ldots, X'_k)$ with $X'_1 = X_1 \setminus \{w\}$, $X'_3 = X_3, \ldots, X'_k = X_k$. So $T'(\mathbb{X}')$ is a $k-1$ partite tree with order n', where $n' = n - 1 - |X_2|$. By the induction hypothesis, there is a restrained packing $\Phi' = (\phi'_1, \phi'_2)$ of $T'(\mathbb{X}')$ into some $B_{n'+m'}(\mathbb{Y}')$, where $n' = n - 1 - |X_2|$, $m' = k - 2$, $\mathbb{Y}' = (Y'_1, Y'_3, \ldots, Y'_k)$ with $|X'_i| \leq |Y'_i|$, $i = 1, 3, \ldots, k$.

Case 1. If there is only one V_i which is non-empty, then all the end vertices adjacent to w are in the V_i and $u \in X_i$. Without loss of generality, we may assume that $i = 2$ and $u = v_1$. Now, we extend the Φ' to $T(\mathbb{X})$ as follows: add a vertex w' to Y'_1 and a set of vertices to the $B_{n'+m'}(\mathbb{Y}')$ as the partite set Y'_2 such that $|Y'_2| = |X_2| + 1$. Let $Y'_2 = \{y_1, y_2, \ldots, y_{2t}\}$. Now we define $\Phi(x) = \Phi'(x)$ for $x \in T'(\mathbb{X}')$; $\phi_1(w) = \phi_2(w) = w'$. Define $\Phi(X_2)$ as follows: $\phi_1(v_i) = y_i$, $i = 1, \ldots, 2t - 1$; $\phi_2(v_1) = y_{2t}$; $\phi_2(v_i) = y_{t+i-1}$ for $i = 2, 3, \ldots, t$ and $\phi_2(v_i) = y_{i-t+1}$ for $i = t + 1, t + 2, \ldots, 2t - 1$. Thus we extend the Φ' to $T(\mathbb{X})$ so that a restrained packing Φ of $T(\mathbb{X})$ into the $B_{n+m}(\mathbb{Y})$ is obtained, where $m = m' + 1 = k - 1$, $\mathbb{Y} = (Y'_1 \cup \{w\}, Y'_2, \ldots, Y'_k)$, a contradiction.

Case 2. Suppose there are $i \neq j$ such that $V_i \neq \emptyset$, $V_j \neq \emptyset$ and $u \notin X_j$. Without loss of generality, we may assume that $j = 2$. Then the packing Φ' can be extended to $T(\mathbb{X})$ as follows: add a set $\{w', s\}$ of two vertices to Y'_1 and a set of vertices to the $B_{n'+m'}(\mathbb{Y}')$ as the partite set Y'_2 such that $|Y'_2| = |X_2|$. Let $Y'_2 = \{y_1, y_2, \ldots, y_{2t-1}\}$. Now we define $\Phi(x) = \Phi'(x)$ for $x \in T'(\mathbb{X}')$; $\phi_1(w) = w'$; $\phi_2(w) = s$. Define $\Phi(X_2)$ as follows: $\phi_1(v_i) = y_i$ for $i = 1, 2, \ldots, 2t-1$; $\phi_2(v_1) = \phi_1(v_1) = y_1$; $\phi_2(v_i) = y_{t+i-1}$ for $i = 2, 3, \ldots, t$ and $\phi_2(v_i) = y_{i-t+1}$ for $i = t + 1, t + 2, \ldots, 2t - 1$. Thus we extend the Φ' to $T(\mathbb{X})$ so that a restrained packing Φ of $T(\mathbb{X})$ into the $B_{n+m}(\mathbb{Y})$ is obtained, where $m = m' + 1 = k - 1$, $\mathbb{Y} = (Y'_1 \cup \{w, s\}, Y'_2, \ldots, Y'_k)$, a contradiction.

The proof is completed.

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References

[1] B. Bollobás, *Some remarks on packing trees*, Discrete Math. **46** (1983), no. 2, 203–204. https://doi.org/10.1016/0012-365X(83)90254-6

[2] A. Gyárfás and J. Lehel, *Packing trees of different order into K_n*, in Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, 463–469, Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam, 1978.

[3] A. M. Hobbs, B. A. Bourgeois, and J. Kasiraj, *Packing trees in complete graphs*, Discrete Math. **67** (1987), no. 1, 27–42. https://doi.org/10.1016/0012-365X(87)90164-6

[4] H. Wang, *Packing two forests into a bipartite graph*, J. Graph Theory **23** (1996), no. 2, 209–213.

[5] M. Woźniak, *Packing of graphs—a survey*, Discrete Math. **276** (2004), 379–391.

[6] H. P. Yap, *Packing of graphs—a survey*, Discrete Math. **72** (1988), no. 1-3, 395–404. https://doi.org/10.1016/0012-365X(88)90232-4

[7] R. Yuster, *On packing trees into complete bipartite graphs*, Discrete Math. **163** (1997), no. 1-3, 325–327. https://doi.org/10.1016/S0012-365X(96)00014-3

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