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SOME COMMUTATIVE RINGS DEFINED BY MULTIPLICATION LIKE-CONDITIONS

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Abstract. In this article we investigate the transfer of multiplication-like properties to homomorphic images, direct products and amalgamated duplication of a ring along an ideal. Our aim is to provide examples of new classes of commutative rings satisfying the above-mentioned properties.

1. Introduction

All rings considered in this article are commutative with identity and all modules are unital. In general, an ideal I of a commutative ring R is called a multiplication ideal if for every ideal $J \subseteq I$ of R, there exists an ideal K of R such that $J = IK$. If every ideal of R is a multiplication ideal, we said that R is a multiplication ring and if for every maximal ideal m of R, R_m is a multiplication ring, we said that R is an almost multiplication ring. In [2, p. 761], Anderson said that every localization of a multiplication (resp., an almost multiplication) ring is still a multiplication (resp., an almost multiplication) ring. Consequently, it is clear that every multiplication ring is an almost multiplication ring. Anderson gave an example of an almost multiplication ring which is not a multiplication ring (cf. [2, p. 765]). He also proved in [2, Theorem 1], that in a quasilocal ring every multiplication ideal is principal.

Let R be a commutative ring with identity element 1 and let I be a proper ideal of R. The amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with identity element $(1, 1)$ of $R \times R$:

$$R \bowtie I := \{(r, r + i) : r \in R,\ i \in I\}.$$

This construction has been studied, in the general case, and from the different point of view of pullbacks, by M. D'Anna and M. Fontana [8]. Also, M. D'Anna and M. Fontana, in [7], have considered the case of the amalgamated duplication of a ring, in not necessarily Noetherian setting, along a multiplicative-canonical

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ideal in the sense Heinzer-Huckaba-Papick [10]. In [6], M. D'Anna has studied some properties of $R \bowtie I$, in order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand H. R. Maimani and S. Yassemi, in [17], have studied the diameter and girth of the zero-divisor graph of the ring $R \bowtie I$. For instance, see [6, 8, 17].

Let M be an R-module. Then the idealization $R \propto M$ (also called the trivial extension), introduced by Nagata in 1956 (cf. [19]) is defined as the R-module $R \oplus M$ with multiplication defined by $(r, m)(s, n) := (rs, rn + sm)$. For instance, see [9, 11, 13].

When $I^2 = 0$, the new construction $R \bowtie I$ coincides with the idealization $R \propto I$. One main difference of the amalgamated-duplication, with respect to the idealization is that the ring $R \bowtie I$ can be a reduced ring (and, in fact, it is always reduced if R is a domain).

Our aim in this paper is to investigate the possible transfer of the multiplication and almost multiplication properties to various amalgamation constructions, homomorphic images and to direct products.

2. Main results

We begin by studying the homomorphic image of multiplication (resp., almost multiplication) rings. Notice that, (1) of the following result appears without proof in [1, Lemma 3.6] and is repeated here with a proof for completeness.

Proposition 2.1. *Let R be a commutative ring and let I be an ideal of R.*

(1) *If R is a multiplication ring, then R/I is a multiplication ring.*
(2) *If R is an almost multiplication ring, then R/I is an almost multiplication ring.*

Proof. (1) Let $\bar{J} = J/I$ be an ideal of R/I and let $\bar{J'} = J'/I$ an ideal of R/I such that $\bar{J'} \subset \bar{J}$. Then clearly, $J' \subset J$ are ideals of R. Since R is a multiplication ring, there exists an ideal J'' of R such that $J' = JJ''$. So $\bar{J'} = J'/I = JJ''/I = J/IJ''/I = \bar{J}\bar{J''}$. Then $\bar{J}$ is a multiplication ideal and hence R/I is a multiplication ring.

(2) Assume that R is an almost multiplication ring. Let M be a maximal ideal of R/I. Then there is a maximal ideal m of R containing I such that $M = m/I$. Therefore, it is clear that $(R/I)_M \cong R_m/I_m$. Since R is an almost multiplication ring, R_m is a multiplication ring. Therefore by (1), R_m/I_m is a multiplication ring and so $(R/I)_M$ is a multiplication ring. Thus R/I is an almost multiplication ring. $\square$

Next, we study the transfer of the multiplication and almost multiplication properties to direct products. The following theorem is exactly [1, Lemma 3.5] without proof and is repeated here with a proof for completeness and for his usefulness in this paper.

Theorem 2.2. *Let* $(R_i)_{i=1,\dots,n}$ *be a family of rings. Then* $\prod_{i=1}^n R_i$ *is a multiplication ring if and only if* R_i *is a multiplication ring for each* $i = 1, \dots, n$.

Proof. Assume that $\prod_{i=1}^n R_i$ is a multiplication ring. Let $p_i : \prod_{i=1}^n R_i \to R_i$ be the natural projection of $\prod_{i=1}^n R_i$ on R_i, for all $1 \le i \le n$. It is clear that p_i is a surjective ring homomorphism. Then, $R_i \cong (\prod_{i=1}^n R_i)/(ker(p_i))$. Since $\prod_{i=1}^n R_i$ is a multiplication ring, then by Proposition 2.1(1),

$$(\prod_{i=1}^n R_i)/(ker(p_i))$$

is a multiplication ring. Then, R_i is a multiplication ring for each $i = 1, \dots, n$. Conversely, assume that each R_i is a multiplication ring for all $1 \le i \le n$. We claim that $\prod_{i=1}^n R_i$ is a multiplication ring. It suffices to prove that the result is true for $n = 2$. Let R_1 and R_2 be two multiplication rings. We recall from [3], that every ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for some ideals I_1 and I_2 of R_1 and R_2, respectively. Let $I_1 \times I_2$ be an ideal of $R_1 \times R_2$ and let $I'_1 \times I'_2$ an ideal of $R_1 \times R_2$ such that $I'_1 \times I'_2 \subseteq I_1 \times I_2$. Then, $I'_1 \subseteq I_1$ and $I'_2 \subseteq I_2$, therefore there exist an ideal I''_1 of R_1 and an ideal I''_2 of R_2 such that $I'_1 = I_1 I''_1$ and $I'_2 = I_2 I''_2$. Then, $I'_1 \times I'_2 = I_1 I''_1 \times I_2 I''_2 = (I_1 \times I_2)(I''_1 \times I''_2)$. Then $I_1 \times I_2$ is a multiplication ideal. Thus $R_1 \times R_2$ is a multiplication ring. □

Theorem 2.3. *Let* $(R_i)_{i=1,\dots,n}$ *be a family of rings. Then* $\prod_{i=1}^n R_i$ *is an almost multiplication ring if and only if* R_i *is an almost multiplication ring for each* $i = 1, \dots, n$.

To prove Theorem 2.3, we need the following lemma:

Lemma 2.4. *Let* (A, B) *be a pair of commutative ring, let* m *and* q *be two maximal ideals of* A *and* B*, respectively. Put* $M = m \times B$ *and* $Q = A \times q$ *two maximal ideals of* $A \times B$*. Then*:

$$f : (A \times B)_M \to A_m \qquad \frac{(a,b)}{(s,k)} \mapsto \frac{a}{s}$$

and

$$g : (A \times B)_Q \to B_q \qquad \frac{(a',b')}{(s',k')} \mapsto \frac{b'}{k'}$$

are two ring isomorphisms.

Proof. Firstly, notice that, for all $\frac{(a,b)}{(s,k)} \in (A \times B)_M$ and for all $\frac{(a',b')}{(s',k')} \in (A \times B)_Q$ with $M = m \times B$ and $Q = A \times q$, it is easy to see that $\frac{a}{s} \in A_m$ and $\frac{b'}{k'} \in B_q$. Let $x = \frac{(a,b)}{(s,k)} = \frac{(c,d)}{(l,r)} = y \in (A \times B)_M$. Then, there is $(t_1, t_2) \in (A \times B)\backslash M$ such that $(t_1 al, t_2 br) = (t_1 cs, t_2 dk)$, and so $t_1 \in A\backslash m$ and $t_1 al = t_1 cs$, i.e., $\frac{a}{s} = \frac{c}{l}$. Then, $f(x) = f(y)$ and hence f is well defined. For the same reasoning, we deduced easily that g is well defined. It is also easy to check that f and g are two ring homomorphisms. Moreover, if $x = \frac{(a,b)}{(s,k)} \in ker(f)$, then $\frac{a}{s} = 0$.

Therefore, there is $t \in A\backslash m$ such that $ta = 0$ and then $(t, 0) \in (A \times B)\backslash M$ and $(t, 0)(a, b) = 0$. Hence $x = 0$. Then f is injective. For the same reasoning, it is easy to deduced that g is also injective. Clearly, f and g are surjective by construction. Thus f and g are two ring isomorphisms. □

Proof of Theorem 2.3. Assume that $\prod_{i=1}^{n} R_i$ is an almost multiplication ring. Let p_i previously be defined in the proof of Theorem 2.2. We have seen that $R_i \cong (\prod_{i=1}^{n} R_i)/(ker(p_i))$ for each $i = 1, \dots, n$. Then, the conclusion follows easily from Proposition 2.1(2). Conversely, assume that R_i is an almost multiplication ring for each $i = 1, \dots, n$. We claim that $\prod_{i=1}^{n} R_i$ is an almost multiplication ring. It suffices to show that the result is true for $n = 2$. Let R_1 and R_2 be two almost multiplication rings. Then $(R_1)_m$ and $(R_2)_q$ are both multiplication rings for all maximal ideals m and q of R_1 and R_2, respectively. We recall from [3], that every maximal ideal of $R_1 \times R_2$ has the form $M := m \times R_2$ or $Q := R_1 \times q$ for some maximal ideals m and q of R_1 and R_2, respectively. Therefore, the conclusion follows easily from Lemma 2.4. □

The following enriches the literature with a non multiplication almost multiplication ring issued from a direct product.

Example 2.5. Let R_1 be a non Noetherian von Neumann regular ring and R_2 be a von Neumann regular ring and let $R := R_1[X] \times R_2[X]$. Then:

(1) R is an almost multiplication ring.
(2) R is not a multiplication ring.

Proof. (1) Since R_1 and R_2 are both von Neumann regular rings, $R_1[X]$ and $R_2[X]$ are both almost multiplication rings by [2, Theorem 5]. Then, R is an almost multiplication ring by Theorem 2.3.

(2) By [2, p. 765], $R_1[X]$ is not a multiplication ring since R_1 is a non Noetherian von Neumann regular ring. Then, R is not a multiplication ring by Theorem 2.2. □

Next, we give a lemma which will be very useful in the rest of this paper. (We recall, that, a commutative ring R with identity 1 is arithmetical if every finitely generated ideal of R is locally principal [12].)

Lemma 2.6. *Let R be a ring. Consider the following conditions*:

(1) *R is a multiplication ring.*
(2) *R is an almost multiplication ring.*
(3) *R is an arithmetical ring.*

Then: (1) ⇒ (2) ⇒ (3). *Moreover, if R is Noetherian, then the three conditions are equivalent.*

Proof. (1) ⇒ (2): Clear.

(2) ⇒ (3): Assume that R is an almost multiplication ring. Let I be a finitely generated ideal of R and let m be a maximal ideal of R. Then, I_m is a multiplication ideal of R_m (since R is an almost multiplication ring). Therefore

I_m is principal by [2, Theorem 1]. Then I is locally principal and hence R is arithmetical.

Moreover, assume that R is a Noetherian arithmetical ring. Let I be an ideal of R. Then I is finitely generated and locally principal. Therefore, I is a multiplication ideal by [2, Theorem 3] and hence R is a multiplication ring, as wanted. □

None of the above implications is reversible in general as shown by the following example.

Example 2.7. (1) Let R be a non Noetherian von Neumann regular ring. Anderson in [2, p. 765] showed that $R[X]$ is a non multiplication almost multiplication ring.

(2) Let A be a von Neumann regular ring. Then by [18, Corollary 3.5], $R := A \propto A$ is an arithmetical ring. Assume that R is an almost multiplication ring. Let m be a maximal ideal of A. Then by [4, Theorem 3.2(1)], $M := m \propto A$ is a maximal ideal of R. Thus R_M is a multiplication ring. Therefore $A_m \propto A_m \cong R_M$ is a multiplication ring (cf. [4, Theorem 4.1(2)]). It is a contradiction by [16, Example 2.7], as desired.

Next, we study the transfer of multiplication property to the amalgamated duplication of a ring along an ideal. Let $A \bowtie I$ be the amalgamated duplication of A along I, it is easy to see that, if π_i $(i = 1, 2)$ are the projections of $A \times A$ on A, then $\pi_i(A \bowtie I) = A$ and hence if $O_i = ker(\pi_i \backslash A \bowtie I)$, then $A \bowtie I/O_i \cong A$. Moreover $O_1 = \{(0, i), i \in I\}$, $O_2 = \{(i, 0), i \in I\}$ and $O_1 \cap O_2 = (0)$.

Theorem 2.8. *Let A be a ring, let I be an ideal of A and let $R := A \bowtie I$ be the amalgamated duplication of A along I. Then*:

1. *If R is a multiplication ring, then A is so.*
2. *Assume that I is generated by an idempotent. Then, R is a multiplication ring if and only if A is a multiplication ring.*
3. *Assume that A is an integral domain. Then, R is a multiplication ring if and only if A is a multiplication ring and $I = (0)$.*

Proof. (1) Assume that R is a multiplication ring. By Proposition 2.1(1), R/O_i $(i = 1, 2)$ is a multiplication ring . Then, $A \cong R/O_i$ is a multiplication ring.

(2) By (1), we need only prove that if A is a multiplication ring (along with the hypothesis that I is generated by an idempotent), then R is a multiplication ring. By [20, Lemma 3] we have $A \bowtie I \cong A \times A/ann(I)$. Since A is a multiplication ring, by Proposition 2.1(1), $A/ann(I)$ is a multiplication ring and hence by Theorem 2.2, $A \times A/ann(I)$ is a multiplication ring. Thus, $A \bowtie I$ is so.

(3) Assume that R is a multiplication ring. Then by (1), A is a multiplication ring. On the other hand, since every multiplication ring is arithmetical, then R is an arithmetical ring. Since A is an integral domain, then by [15, Corollary 2.8], $I = (0)$. The converse is trivial since $A \bowtie 0 = A$. □

Next, we study the transfer of the almost multiplication property to the amalgamated duplication of a ring along an ideal.

Theorem 2.9. *Let A be a ring, let I be an ideal of A and let $R := A \bowtie I$ be the amalgamated duplication of A along I. Then*:

(1) *If R is an almost multiplication ring, then A is so.*
(2) *Assume that I is a nonzero pure ideal. Then, R is an almost multiplication ring if and only if A is an almost multiplication ring.*
(3) *Assume that A is an integral domain. Then, R is an almost multiplication ring if and only if A is an almost multiplication ring and $I = (0)$.*

Before proving Theorem 2.9, we need the following remark:

Remark 2.10. Let A be a commutative ring, let I be an ideal of A, and P a prime ideal of A. In [6, Propositions 5 and 7], D'Anna proved that if $I \nsubseteq P$, then $\breve{P} = \{(p+i, p) : p \in P,\ i \in I\}$ and $P \bowtie I = \{(p, p+i) : p \in P,\ i \in P\}$ are the only prime ideals of $A \bowtie I$ lying over P and we have $(A \bowtie I)_{P\bowtie I} \cong (A \bowtie I)_{\breve{P}} \cong A_P$. Notice that $P \bowtie I$ and $\breve{P}$ are incomparable. However, if $I \subseteq P$, then $P \bowtie I = \breve{P}$ is the unique prime ideal of $A \bowtie I$ lying over P and we have $(R \bowtie I)_{P\bowtie I} \cong R_P \bowtie I_P$.

Proof of Theorem 2.9. (1) Assume that R is an almost multiplication ring. Let m be a maximal ideal of A. Then by [8, Theorem 3.5 (1.d)], $M = m \bowtie I$ is a maximal ideal of R. Hence, R_M is a multiplication ring. Since $A_m \bowtie I_m \cong R_M$ if $I \subseteq m$ and $A_m \cong R_M$ if $I \nsubseteq m$. Then, by Theorem 2.8(1), A_m is a multiplication ring, as desired.

(2) Firstly, notice that if M be a maximal ideal of $R \bowtie I$ and if $m = M \cap A$, then necessarily $M \in \{M_1, M_2\}$, where $M_1 = \{(r, r+i) : r \in m, i \in I\}$ and $M_2 = \{(r+i, r) : r \in m, i \in I\}$ (by [8, Theorem 3.5]). On the other hand, $I_m \in \{0, A_m\}$ since I is pure and m is maximal in R (by [9, Theorem 1.2.15]). Then, testing all cases of Remark 2.10, we resume two cases;

(1) $(A \bowtie I)_M \cong A_m$ if $I_m = 0$ or $I \nsubseteq m$.
(2) $(A \bowtie I)_M \cong A_m \bowtie A_m$ if $I_m = A_m$ or $I \subseteq m$.

By (1), we need only prove that if A is an almost multiplication ring (along with the hypothesis that I is pure), then R is an almost multiplication ring. Since A is an almost multiplication, then A_m is a multiplication ring and hence $A_m \bowtie A_m$ is so (by Theorem 2.2). Therefore $(A \bowtie I)_M$ is a multiplication ring. Thus $R := A \bowtie I$ is an almost multiplication.

(3) Assume that A is an integral domain. Since every almost multiplication ring is arithmetical, then by (1) and [15, Corollary 2.8], A is an almost multiplication ring and $I = (0)$. The converse is trivial since $A \bowtie 0 = A$. □

The following example illustrates Theorem 2.8(2) and 2.9(2).

Example 2.11. The following statements hold:

(1) $\mathbb{Z}_6 \bowtie (\overline{4})$ is a multiplication ring by Theorem 2.8(2) since $\mathbb{Z}_6$ is principal and hence a multiplication ring and $\overline{4}$ is idempotent in $\mathbb{Z}_6$.
(2) $\mathbb{Z}_{15} \bowtie (\overline{5})$ and $\mathbb{Z}_{12} \bowtie (\overline{3})$ are both almost multiplication rings by Theorem 2.9(2) since $\mathbb{Z}_{15}$ and $\mathbb{Z}_{12}$ are almost multiplication rings and easily we can verifies that $(\overline{5})$ and $(\overline{3})$ are pure ideals of $\mathbb{Z}_{15}$ and $\mathbb{Z}_{12}$ respectively.

We said that a commutative ring R has few zero-divisors if $Z(R)$ the set of zero-divisors of R is a finite union of prime ideals. The following theorem studies the transfer of the few zero-divisors property in the amalgamated duplication along an ideal.

Theorem 2.12. *Let A be a ring and let I be a nonzero proper ideal of A. Assume that $Z(A)^2 = 0$ and $I \subseteq Z(A)$. If A has few zero-divisors, then $A \bowtie I$ has few zero-divisors. The converse is true if $I \subseteq N(A)$ (where $N(A)$ denotes the nilradical ideal of A).*

Proof. Assume that $Z(A)^2 = 0$ and $I \subseteq Z(A)$. Then by [14, Corollary 2.2], $Z(A \bowtie I) = Z(A) \bowtie I$. Now assume that A has few zero-divisors. Then $Z(A) = \bigcup_{j=1}^{n} P_j$, where P_j is a prime ideal of A for each $j = 1, \ldots, n$. Then, $Z(A \bowtie I) = Z(A) \bowtie I = (\bigcup_{j=1}^{n} P_j) \bowtie I = (\bigcup_{j=1}^{n} P_j \bowtie I)$, where $P_j \bowtie I$ is a prime ideal of $A \bowtie I$ for each $j = 1, \ldots, n$. Then $A \bowtie I$ has few zero-divisors. Conversely, assume that $I \subseteq N(A)$. Then $I \subseteq P$ for all prime ideal P of A and therefore by [5, Lemma 2.2(2.b)], every prime ideal Q of $A \bowtie I$ has the form $P \bowtie I$ with P a prime ideal of A. Assume that $A \bowtie I$ has few zero-divisors. Then $Z(A \bowtie I) = \bigcup_{j=1}^{n} P_j \bowtie I$, where P_j is a prime ideal of A for each $j = 1, \ldots, n$. Thus $Z(A) \bowtie I = Z(A \bowtie I) = (\bigcup_{j=1}^{n} P_j \bowtie I) = (\bigcup_{j=1}^{n} P_j) \bowtie I$ and so $Z(A) = \bigcup_{j=1}^{n} P_j$. Thus A has few zero-divisors. □

Example 2.13. Let $A := \mathbb{Z}_4$. Then since A is principal, A is a multiplication ring and hence an almost multiplication ring. It is clear that the minimal prime ideals of A are finitely generated. Therefore A has few zero-divisors by [2, Theorem 5]. Let $I := 2\mathbb{Z}_4 = \{0, 2\}$ be a nonzero proper ideal of A. It is easy to show that $Z(A) = \{0, 2\} = I$ and $Z(A)^2 = \{0\}$. Then, by Theorem 2.12, $A \bowtie I$ has few zero-divisors.

Proposition 2.14. *Let A be an almost multiplication ring, let I be an ideal of A generated by an idempotent and let $R := A \bowtie I$ be the amalgamated duplication of A along I. If R has few zero-divisors, then A has few zero-divisors. The converse is true if $I \subseteq N(A)$ (where $N(A)$ denotes the nilradical of A).*

Proof. According to [20, Lemma 3] and Theorem 2.3, we show that R is an almost multiplication ring. Suppose that R has few zero-divisors. Then by [2, Theorem 5], the minimal prime ideals of R are finitely generated. Let P be a minimal prime ideal of A. Then following [5, Lemma 2.2], we can easily show that $P \bowtie I$ is a minimal prime ideal of R. Then $P \bowtie I$ is finitely generated. Therefore P is finitely generated and hence A has few zero divisors by [2, Theorem 5]. Conversely, assume that $I \subseteq N(A)$. Then $I \subseteq P$ for all prime ideal P of A. Therefore, every prime ideal of R has the form $P \bowtie I$, where P is a prime ideal of A by [5, Lemma 2.2]. Suppose that A has few zero-divisors. Then A has only finitely many minimal prime ideals by [2, Theorem 5]. Let $P_1, \ldots, P_n$ be the only finitely many minimal prime ideals of A. Then $P_1 \bowtie I, \ldots, P_n \bowtie I$ are the only finitely many minimal prime ideals of R and hence R has few zero-divisors by [2, Theorem 5]. □

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