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CHARACTERIZATIONS OF JORDAN DERIVABLE MAPPINGS AT THE UNIT ELEMENT

Jiankui Li, Shan Li, and Kaijia Luo

Abstract. Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{M}$ a unital $\mathcal{A}$-bimodule, and δ a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. We prove that if δ satisfies $\delta(A)A^{-1}+A^{-1}\delta(A)+A\delta(A^{-1})+\delta(A^{-1})A=0$ for every invertible element A in $\mathcal{A}$, then δ is a Jordan derivation. Moreover, we show that δ is a Jordan derivable mapping at the unit element if and only if δ is a Jordan derivation. As an application, we answer the question posed in [4, Problem 2.6].

1. Introduction

Throughout this paper, let $\mathcal{A}$ be a unital Banach algebra over the real or complex field and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for every A, B in $\mathcal{A}$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be a *Jordan derivation* if $\delta(A^2) = \delta(A)A + A\delta(A)$ for every A in $\mathcal{A}$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is a *Jordan derivable mapping at G* if $A\delta(B)+\delta(B)A+\delta(A)B+B\delta(A) = \delta(G)$ for every $A, B \in \mathcal{A}$ with $AB + BA = G$.

Every C^*-algebra $\mathcal{A}$ is a JB^*-triple in the sense of [7] with respect to the triple product given by

$$\{A, B, C\} = \frac{1}{2}(AB^*C + CB^*A) \quad (A, B, C \in \mathcal{A}).$$

A linear mapping δ from $\mathcal{A}$ into $\mathcal{B}$ is called a *triple derivation* whenever it satisfies:

$$\delta\{A, B, C\} = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{A, B, \delta(C)\}$$

for every A, B, C in $\mathcal{A}$. Let Z be an element in $\mathcal{A}$. We say that δ is a *triple derivation at Z* if $Z = \{A, B, C\}$ in $\mathcal{A}$ implies that $\delta(Z) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{A, B, \delta(C)\}$.

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In [4], the authors considered linear mappings which are triple derivations at certain points. It is proved that a continuous linear mapping T on a unital C^*-algebra is a generalized derivation whenever it is a triple derivation at the unit element. Furthermore, if $T(1) = 0$, then T is a $*$-derivation and a triple derivation. It is further shown that a continuous linear mapping on a unital C^*-algebra which is a triple derivation at the unit element is a triple derivation. Similar conclusions are obtained for continuous linear mappings which are derivations or triple derivations at zero. Burgos et al. [2] showed that every continuous local triple derivation on a C^*-algebra is a triple derivation. For the basic notions and results in the theory of triple derivations we refer the reader to [3, 9, 10] where further references can be found.

In [4, Problem 2.6], Essaleh and Peralta posed the following problem on triple derivations on C^*-algebras:

Problem 2.6. Suppose $\mathcal{A}$ is a C^*-subalgebra of a unital C^*-algebra $\mathcal{B}$ containing the unit of $\mathcal{B}$. If $\delta : \mathcal{A} \to \mathcal{B}$ is a triple derivation at the unit element, then is δ continuous?

In [8], Li and Zhou showed that a linear mapping δ from a unital Banach algebra $\mathcal{A}$ into a unital $\mathcal{A}$-bimodule $\mathcal{M}$ satisfying $\delta(A)A^{-1} + A\delta(A^{-1}) = 0$ for every invertible element A in $\mathcal{A}$ is a Jordan derivation. The main purpose of this paper is to present a generalization of this result. More precisely, if δ satisfies $\delta(A)A^{-1} + A^{-1}\delta(A) + A\delta(A^{-1}) + \delta(A^{-1})A = 0$ for every invertible element A in $\mathcal{A}$, then δ is a Jordan derivation. Moreover, we show that δ is a Jordan derivable mapping at I if and only if δ is a Jordan derivation. As an application, we give a positive answer to the question posed in [4, Problem 2.6].

2. Main result

Theorem 2.1. *Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying $A^{-1}\delta(A) + \delta(A)A^{-1} + A\delta(A^{-1}) + \delta(A^{-1})A = 0$ for each invertible element A in $\mathcal{A}$, then δ is a Jordan derivation.*

Proof. By assumption, for each invertible element A in $\mathcal{A}$, we have that

$$A^{-1}\delta(A) + \delta(A)A^{-1} + A\delta(A^{-1}) + \delta(A^{-1})A = 0. \tag{2.1}$$

Therefore, for each invertible element A in $\mathcal{A}$,

$$\delta(A) = -A[\delta(A)A^{-1} + A\delta(A^{-1}) + \delta(A^{-1})A], \tag{2.2}$$

$$\delta(A) = -[A^{-1}\delta(A) + A\delta(A^{-1}) + \delta(A^{-1})A]A. \tag{2.3}$$

By $I^{-1}\delta(I) + \delta(I)I^{-1} + I\delta(I^{-1}) + \delta(I^{-1})I = 0$, we have $\delta(I) = 0$. Let $T \in \mathcal{A}$, $n \in \mathbb{N}$ with $n > \|T\| + 1$, and $A = nI + T$. Then A and $I - A$ are both invertible in $\mathcal{A}$.

By (2.2) and (2.3), we obtain the following relations:

$$\begin{aligned}
A^{-1}\delta(A) &= -A^{-1}\delta(I-A)\\
&= -A^{-1}[-(I-A)[\delta(I-A)(I-A)^{-1} + (I-A)\delta((I-A)^{-1})\\
&\quad + \delta(I-A)^{-1}(I-A)]]\\
&= -A^{-1}(I-A)\delta(A)(I-A)^{-1} + A^{-1}(I-A)^2\delta((I-A)^{-1})\\
&\quad + A^{-1}(I-A)\delta((I-A)^{-1})(I-A),\\
\delta(A)A^{-1} &= -\delta(I-A)A^{-1}\\
&= [(I-A)^{-1}\delta(I-A) + (I-A)\delta((I-A)^{-1})\\
&\quad + \delta((I-A)^{-1})(I-A)](I-A)A^{-1}\\
&= -(I-A)^{-1}\delta(A)(I-A)A^{-1} + (I-A)\delta((I-A)^{-1})(I-A)A^{-1}\\
&\quad + \delta((I-A)^{-1})(I-A)^2A^{-1},\\
A\delta(A^{-1}) &= A\delta(A^{-1}(I-A))\\
&= -A[A^{-1}(I-A)[\delta(A^{-1}(I-A))A(I-A)^{-1}\\
&\quad + A^{-1}(I-A)\delta(A(I-A)^{-1}) + \delta(A(I-A)^{-1})A^{-1}(I-A)]]\\
&= -(I-A)\delta(A^{-1}(I-A))A(I-A)^{-1} - A^{-1}(I-A)^2\delta(A(I-A)^{-1})\\
&\quad - (I-A)\delta(A(I-A)^{-1})A^{-1}(I-A)\\
&= -(I-A)\delta(A^{-1})A(I-A)^{-1} - A^{-1}(I-A)^2\delta((I-A)^{-1})\\
&\quad - (I-A)\delta((I-A)^{-1})A^{-1}(I-A),\\
\delta(A^{-1})A &= \delta(A^{-1}(I-A))A\\
&= -[A(I-A)^{-1}\delta(A^{-1}(I-A)) + (A^{-1}(I-A))\delta(A(I-A)^{-1})\\
&\quad + \delta(A(I-A)^{-1})(A^{-1}(I-A))](A^{-1}(I-A))A\\
&= -A(I-A)^{-1}\delta(A^{-1}(I-A))(I-A)\\
&\quad - A^{-1}(I-A)\delta(A(I-A)^{-1})(I-A) - \delta(A(I-A)^{-1})A^{-1}(I-A)^2\\
&= -A(I-A)^{-1}\delta(A^{-1})(I-A) - A^{-1}(I-A)\delta((I-A)^{-1})(I-A)\\
&\quad - \delta((I-A)^{-1})A^{-1}(I-A)^2.
\end{aligned}$$

Adding the above equations, we have that

$$\begin{aligned}
0 = &-A^{-1}(I-A)\delta(A)(I-A)^{-1} - (I-A)^{-1}\delta(A)(I-A)A^{-1}\\
&-(I-A)\delta(A^{-1})A(I-A)^{-1} - A(I-A)^{-1}\delta(A^{-1})(I-A).
\end{aligned} \tag{2.4}$$

Multiplying the identity in (2.4) by $I-A$ from the left and right, it follows that

$$(I-A)^2[A^{-1}\delta(A) + \delta(A^{-1})A] + [\delta(A)A^{-1} + A\delta(A^{-1})](I-A)^2 = 0.$$

By (2.1),

$$(-2A+A^2)[\delta(A)A^{-1}+A\delta(A^{-1})]=[\delta(A)A^{-1}+A\delta(A^{-1})](-2A+A^2). \tag{2.5}$$

Expanding both sides of this equation, we obtain that

$$\begin{aligned}&(-2A+A^2)[\delta(A)A^{-1}+A\delta(A^{-1})]\\ =&-2A\delta(A)A^{-1}-2A^2\delta(A^{-1})+A^2\delta(A)A^{-1}+A^3\delta(A^{-1}),\end{aligned} \tag{2.6}$$

$$\begin{aligned}&[\delta(A)A^{-1}+A\delta(A^{-1})](-2A+A^2)\\ =&-2\delta(A)-2A\delta(A^{-1})A+\delta(A)A+A\delta(A^{-1})A^2.\end{aligned} \tag{2.7}$$

However, by (2.1) we have that

$$\begin{aligned}-2A\delta(A^{-1})A&=2A[A^{-1}\delta(A)+\delta(A)A^{-1}+A\delta(A^{-1})]\\ &=2\delta(A)+2A\delta(A)A^{-1}+2A^2\delta(A^{-1})\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}A\delta(A^{-1})A^2&=-A[A^{-1}\delta(A)+\delta(A)A^{-1}+A\delta(A^{-1})]A\\ &=-\delta(A)A-A\delta(A)-A^2\delta(A^{-1})A\\ &=-\delta(A)A-A\delta(A)+A^2[A^{-1}\delta(A)+\delta(A)A^{-1}+A\delta(A^{-1})]\\ &=-\delta(A)A+A^2\delta(A)A^{-1}+A^3\delta(A^{-1}).\end{aligned} \tag{2.9}$$

We deduce from (2.7), (2.8) and (2.9) that,

$$\begin{aligned}&[\delta(A)A^{-1}+A\delta(A^{-1})](-2A+A^2)\\ =&-2\delta(A)+2\delta(A)+2A\delta(A)A^{-1}+2A^2\delta(A^{-1})\\ &+\delta(A)A-\delta(A)A+A^2\delta(A)A^{-1}+A^3\delta(A^{-1})\\ =&\,2A\delta(A)A^{-1}+2A^2\delta(A^{-1})+A^2\delta(A)A^{-1}+A^3\delta(A^{-1}).\end{aligned} \tag{2.10}$$

Comparing (2.5), (2.6) and (2.10), we have that $-2A\delta(A)A^{-1}-2A^2\delta(A^{-1})=2A\delta(A)A^{-1}+2A^2\delta(A^{-1})$, i.e., $A\delta(A)A^{-1}+A^2\delta(A^{-1})=0$. Multiplying by A^{-1} from the left we get

$$\delta(A)A^{-1}+A\delta(A^{-1})=0.$$

By [8, Lemma 2.1], we obtain that $\delta(A^2)=\delta(A)A+A\delta(A)$. Since $\delta(I)=0$ and $A=nI+T$, we have

$$\delta(T^2)=\delta(T)T+T\delta(T)$$

for every $T\in\mathcal{A}$. Thus δ is a Jordan derivation. □

It is not difficult to deduce the following results from Theorem 2.1.

Corollary 2.2. *Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$, then δ is a Jordan derivable mapping at I if and only if δ is a Jordan derivation.*

Proof. The sufficiency is obvious. Therefore, we just need to prove the necessity. By assumption, we have that $\frac{1}{2}(I\delta(I) + \delta(I)I + \delta(I)I + I\delta(I)) = \delta(I)$. Thus $\delta(I) = 0$. For each invertible element A in $\mathcal{A}$, we have that

$$A^{-1}\delta(A) + \delta(A)A^{-1} + A\delta(A^{-1}) + \delta(A^{-1})A = 0.$$

By Theorem 2.1, δ is a Jordan derivation. □

Corollary 2.3. *Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying $\delta(A)A^{-1} + A\delta(A^{-1}) = 0$ for each invertible element A in $\mathcal{A}$, then δ is a Jordan derivation.*

Remark 2.4. It is worth noting that in Theorem 2.1 and Corollaries 2.2 and 2.3, $\mathcal{A}$ is a unital Banach algebra while $\mathcal{M}$ is merely a unital $\mathcal{A}$-bimodule, it is not assumed to be a Banach $\mathcal{A}$-bimodule.

Peralta and Russo proved in [10, Corollary 17] that every Jordan derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is continuous. Johnson showed that every continuous Jordan derivation from a C^*-algebra into a Banach $\mathcal{A}$-bimodule is a derivation in [6, Theorem 6.3]. Then the following corollary is a direct consequence of Theorem 2.1.

Corollary 2.5. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a unital Banach $\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying $\delta(A)A^{-1} + A^{-1}\delta(A) + A\delta(A^{-1}) + \delta(A^{-1})A = 0$ for every invertible element A in $\mathcal{A}$, then δ is a continuous derivation.*

Next, we can answer the question posed in [4, Problem 2.6] through the following lemma.

Lemma 2.6. *Let $\mathcal{A}$ be a C^*-subalgebra of a unital C^*-algebra $\mathcal{B}$ containing the unit of $\mathcal{B}$. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{B}$ which is a triple derivation at the unit with $\delta(I) = 0$, then δ is a continuous derivation.*

Proof. By assumption, for every invertible element A in $\mathcal{A}$, we have that

$$\begin{aligned} 0 = \delta(I) = \delta\{A, I, A^{-1}\} &= \{\delta(A), I, A^{-1}\} + \{A, I, \delta(A^{-1})\} \\ &= \frac{1}{2}(\delta(A)A^{-1} + A^{-1}\delta(A) + A\delta(A^{-1}) + \delta(A^{-1})A). \end{aligned}$$

By Corollary 2.5, δ is a continuous derivation. □

Theorem 2.7. *Let $\mathcal{A}$ be a C^*-subalgebra of a unital C^*-algebra $\mathcal{B}$ containing the unit of $\mathcal{B}$. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{B}$ which is a triple derivation at the unit, then δ is a continuous triple derivation.*

Proof. We know from [4, Lemma 2.2] that $\delta(I)^* = -\delta(I)$. It is easy to show that the mapping $\tau(\delta(I), I) : \mathcal{A} \to \mathcal{B}$ defined by $\tau(\delta(I), I)(X) = \{\delta(I), I, X\} - \{I, \delta(I), X\}$ is a triple derivation (compare [4, proof of Corollary 2.5] or [5, proof of Lemma 1]). A deep result proved by Barton and Friedman in [1] shows that every triple derivation on a JB^*-triple is continuous. Then the mapping $\tau(\delta(I), I) : X \mapsto \delta(I)X + X\delta(I)$ is continuous although in this case we do not need the result by Barton and Friedman, since the continuity of this particular triple derivation follows from the continuity of the product of $\mathcal{B}$. Since a real linear combination of linear mappings which are triple derivations at I is a triple derivation at I, the mapping $\widetilde{\delta} = \delta - \frac{1}{2}\tau(\delta(I), I)$ is a triple derivation at I and $\widetilde{\delta}(I) = 0$. By Lemma 2.6, $\widetilde{\delta}$ is continuous. By [4, Proposition 2.4], $\widetilde{\delta}$ is a symmetric mapping (i.e., $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$) and a triple derivation. Thus $\delta = \widetilde{\delta} + \frac{1}{2}\tau(\delta(I), I)$ is a continuous triple derivation. □

Remark 2.8. Theorem 2.7 answers the question posed in [4, Problem 2.6]. Essaleh and Peralta showed in [4] that every continuous linear mapping on a unital C^*-algebra which is a triple derivation at the unit element is a triple derivation. We have proved in Theorem 2.7 that the hypothesis concerning the continuity can be relaxed.

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Jiankui Li
Department of Mathematics
East China University of Science and Technology
200237, Shanghai, P. R. China
Email address: jiankuili@yahoo.com

Shan Li
Department of Mathematics
East China University of Science and Technology
200237, Shanghai, P. R. China
Email address: lishan_math@163.com

Kaijia Luo
Department of Mathematics
East China University of Science and Technology
200237, Shanghai, P. R. China
Email address: kaijia_luo@163.com