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SOME IRRATIONAL QUARTIC THREEFOLDS

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Dedicated to the late Professor Bumsig Kim

Abstract. We study the factoriality of a nodal quartic hypersurface V_4 in $\mathbb{P}^4$ when there is a hyperplane in $\mathbb{P}^4$ containing all the nodes of V_4. As an application, we obtain new examples of irrational quartic 3-folds.

All considered varieties are assumed to be projective, normal, and defined over the complex number field $\mathbb{C}$.

1. Introduction

A variety is called factorial if every Weil divisor on it is Cartier. This innocent definition is quite subtle when realized on a projective variety. It does depend both on the kind of singularities and on their position. Note that a smooth hypersurface in $\mathbb{P}^4$ is always factorial. A hypersurface is called nodal if all its singular points are only ordinary double points, i.e., nodes. The factoriality problem of a nodal hypersurface in $\mathbb{P}^4$ has been considered by several authors for a long time [3–6, 17, 22–24, 28].

We will restrict ourselves to the case where the degree of hypersurfaces in $\mathbb{P}^4$ is 4. Let $V_4 \subset \mathbb{P}^4$ be a nodal quartic hypersurface. Then the Grothendieck-Lefschetz theorem [16, Chapter IV, Corollary 3.3] says that Cartier divisors on V_4 are restrictions of Cartier divisors on $\mathbb{P}^4$, i.e., that Pic $V_4 \cong \mathbb{Z}[\mathcal{O}_{V_4}(1)]$. However, no such result holds for Cl V_4, where Cl V_4 denotes the class group of V_4, namely the group of linear equivalence classes of Weil divisors. More precisely, since V_4 is projectively normal and nonsingular in codimension 1, the restriction map

$$\mathrm{Cl}\ \mathbb{P}^4 \longrightarrow \mathrm{Cl}\ V_4$$

is an isomorphism precisely when Cl V_4 = Pic $V_4 = \mathbb{Z}$. In this case we say that V_4 is factorial. In general, we have

$$\mathrm{Cl}\ V_4 = \mathrm{Pic}\ V_4 \oplus \mathbb{Z}^{\delta},$$

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where the number δ is the defect of V_4. From the equivalent condition (3) in Section 2, the defect is a global topological invariant that measures how far V_4 is from being factorial, or, in other words, to what extent Poincaré duality fails on V_4. A lot of the birational geometry of singular varieties depends on the factoriality condition. For instance, Mella proved in [24] that every factorial nodal quartic 3-fold is irrational. In particular, the rationality of a nodal determinantal quartic 3-fold is due to the lack of factoriality and not to the presence of singularities [24].

Example 1.1 ([25])**.** Every general determinantal quartic 3-fold is nodal, non-factorial, rational, and it has 20 nodes.

Results of [24] generalize a classical result by Iskovskikh and Manin [18] that all smooth quartic 3-folds are irrational.

There exist non-factorial irrational nodal quartic 3-folds in $\mathbb{P}^4$ [4, Theorem 11].

Theorem 1.1. *If $V_4 \subset \mathbb{P}^4$ is a sufficiently general quartic 3-fold that contains a smooth del Pezzo surface $S_4 \subset \mathbb{P}^4$ of degree 4, then V_4 is nodal, non-factorial and irrational, and has $\#|\operatorname{Sing}(V_4)| = 16$.*

For a given variety, it is one of the most essential questions to decide whether it is rational or not. This question has been considered in depth for smooth 3-folds [1, 2, 7, 8, 10, 18, 19, 25–27, 29]. This is why it is important to study the factoriality of a nodal quartic hypersurface V_4 in $\mathbb{P}^4$.

Remark 1.1. Every quadric 3-fold in $\mathbb{P}^4$ is rational. Clemens and Griffiths showed that a smooth cubic 3-fold is irrational [10, Theorem 13.12]. Every nodal hypersurface in $\mathbb{P}^4$ of degree at least 5 is irrational.

The following theorem is the main result in the paper [4] by Cheltsov.

Theorem 1.2. *A nodal quartic V_4 is factorial if it has at most 9 nodes and contains no planes.*

Theorem 1.2 has been improved.

Theorem 1.3 ([28, Theorem 1.3])**.** *A nodal quartic V_4 is factorial if it has at most 11 nodes and contains no planes. If V_4 has 12 nodes, then V_4 is factorial with the exception of the case when V_4 contains a quadric surface.*

Theorem 1.4 ([17, Theorem 1.3])**.** *A nodal quartic V_4 is factorial if it has at most 13 nodes and contains neither planes nor quadric surfaces.*

Examples 2.1, 2.2, 2.3, 3.1 and Lemmas 3.3, 3.4 enable us to propose the conjecture below.

Conjecture 1.1. *A nodal quartic V_4 is factorial if it has at most 16 nodes, does not contain any of planes, and quadric surfaces, and (possibly singular) del Pezzo surfaces of degree 4.*

In this paper, we prove the following.

Theorem 1.5. *Assume that there is a hyperplane in* $\mathbb{P}^4$ *containing all the nodes of a nodal quartic* V_4.

(1) *If* $\#|\operatorname{Sing}(V_4)| > 20$, *then* V_4 *is not factorial*;
(2) V_4 *is factorial if* $\#|\operatorname{Sing}(V_4)| \leq 20$, *and* V_4 *contains neither planes nor quadric surfaces.*

Corollary 1.1. *A nodal quartic* V_4 *is irrational if it has at most* 20 *nodes, contains neither planes nor quadric surfaces, and there is a hyperplane in* $\mathbb{P}^4$ *containing all the nodes of* V_4.

Corollary 1.2. *Assume that there is a hyperplane in* $\mathbb{P}^4$ *containing all the nodes of* V_4. *Then Conjecture 1.1 is true.*

Proof. The statement immediately follows from Theorem 1.5. □

Remark 1.2. A nodal quartic V_4 cannot have more than 45 nodes [15, 30]. Moreover, there is a unique nodal quartic 3-fold with 45 nodes [12]. It is known as the Burkhardt quartic, which has too many nodes to be factorial. In fact, if V_4 is factorial, then it must have at most 35 nodes because $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 35$ (This immediately follows from the equivalent condition (4) in Section 2).

2. Preliminaries

Let V_d be a nodal hypersurface of degree d in $\mathbb{P}^4$ given by the equation

$$h(x, y, z, t, w) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where h is a homogeneous polynomial of degree d in $\mathbb{P}^4$. Then it is well-known that the following conditions are equivalent [9, 13, 16]:

(1) V_d is factorial;
(2) the quotient ring
$$\mathbb{C}[x, y, z, t, w]/\langle h(x, y, z, t, w)\rangle$$
is a unique factorization domain;
(3) $\dim H_4(V_d, \mathbb{Z}) = \dim H^2(V_d, \mathbb{Z})$;
(4) the nodes of V_d impose independent linear conditions on homogeneous forms of degree $2d-5$ in $\mathbb{P}^4$ (global sections of $H^0(\mathcal{O}_{\mathbb{P}^4}(2d-5))$);
(5) any surface in V_d is the complete intersection of V_d with a hypersurface of $\mathbb{P}^4$.

From the equivalent condition (2), we present some non-factorial hypersurfaces in $\mathbb{P}^4$.

Example 2.1. Let V_d be a nodal hypersurface of degree $d > 1$ in $\mathbb{P}^4 \cong \operatorname{Proj}\big(\mathbb{C}[x, y, z, t, w]\big)$ given by the equation

$$xf(x, y, z, t, w) + yg(x, y, z, t, w) = 0,$$

where f and g are general homogeneous polynomials of degree $d-1$ in $\mathbb{P}^4$. Then V_d has exactly $(d-1)^2$ nodes and contains the plane π defined by $\{x = y = 0\}$. Hence, by the condition (5), V_d is not factorial.

Example 2.2. Let V_d be a nodal hypersurface of degree $d > 2$ in $\mathbb{P}^4 \cong \operatorname{Proj}\big(\mathbb{C}[x, y, z, t, w]\big)$ given by the equation

$$xf(x, y, z, t, w) + (yz + tw)g(x, y, z, t, w) = 0,$$

where f and g are general homogeneous polynomials of degree $d-1$ and $d-2$ in $\mathbb{P}^4$, respectively. Then V_d has exactly $2(d-1)(d-2)$ nodes and contains the quadric surface U defined by $\{x = yz + tw = 0\}$. Hence, by the condition (5), V_d is not factorial.

Now, we present a factorial nodal quartic hypersurface in $\mathbb{P}^4$ which contains neither planes nor quadric surfaces. In particular, there is a hyperplane in $\mathbb{P}^4$ containing all the nodes of this nodal quartic 3-fold.

Example 2.3. Let S be a nodal quartic surface in $\mathbb{P}^3$. Then $\#|\operatorname{Sing}(S)| \leq 16$. Suppose that S is given by the equation

$$f(x_0, x_1, x_2, x_3) = 0$$

for some quartic homogeneous polynomial f. Here x_0, x_1, x_2, x_3 are coordinates on $\mathbb{P}^3$. Since we have $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) = 20$, one can find a cubic homogeneous polynomial $h(x_0, x_1, x_2, x_3)$ that vanishes at every nodes of the surface S. Consider the quartic hypersurface in $\mathbb{P}^4$ that is given by the equation $g(x_0, x_1, x_2, x_3, x_4) = 0$,

$$g(x_0, x_1, x_2, x_3, x_4) := x_4 h(x_0, x_1, x_2, x_3) + \alpha f(x_0, x_1, x_2, x_3),$$

where α is a general complex number, and x_0, x_1, x_2, x_3, x_4 are coordinates on $\mathbb{P}^4$. By Bertini theorem, this quartic 3-fold has exactly $s, s = \#|\operatorname{Sing}(S)|$, nodes, which we can identify with the nodes of the surface S contained in the hyperplane, $\{x_4 = 0\}$. Furthermore, one can show that this quartic 3-fold is nodal. If we take a general element of the pencil, this nodal quartic contains neither planes nor quadric surfaces in $\mathbb{P}^3$. Then, by Theorem 1.5(2), this nodal quartic is factorial, and hence, by Corollary 1.1, this nodal quartic is irrational.

3. Useful tools

Let V_d be a nodal hypersurface of degree d in $\mathbb{P}^4$. From the equivalent condition (4) in Section 2, the factoriality of V_d is strongly related to the number and the position of its singularities. For instance, if V_d is factorial, then the number of nodes of V_d cannot exceed $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-5))$. Furthermore, we see that the nodes of V_d are located in $\mathbb{P}^4$ with the following nice properties.

Lemma 3.1. *Let V_d be a nodal hypersurface of degree d in $\mathbb{P}^4$.*

(1) *A curve of degree k contains at most $k(d-1)$ nodes of V_d.*

(2) *If a 2-plane contains* $\frac{d(d-1)}{2}+1$ *nodes of* V_d*, then the plane is contained in* V_d*.*

Proof. See [6, Lemma 2.9]. □

Lemma 3.2. *Let* V_d *be a nodal hypersurface of degree* d *in* $\mathbb{P}^4$*, let* $\Xi_{d,i} = \mathrm{Sing}(V_d) \cap \mathrm{Sing}(S_i)$*, where* S_i *is an irreducible surface of degree* i*, and let* $\#|\Xi_{d,i}|$ *be the cardinality of* $\Xi_{d,i}$*. If* S_i *contains* $\frac{id(d-1)}{2} - 2\#|\Xi_{d,i}| + 1$ *nodes of* V_d*, then* $S_i \subset V_d$*.*

Proof. Suppose that V_d is given by the equation

$$h(x_0, x_1, x_2, x_3, x_4) = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\left(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\right),$$

where h is a homogeneous polynomial of degree d in $\mathbb{P}^4$. Then the singular locus of V_d is contained in a general hypersurface $V_d' =: \{\Sigma\alpha_i \frac{\partial h}{\partial x_i} = 0\}$ of degree $d-1$ with $\alpha_i \in \mathbb{C}$. Because V_d has only isolated singularities, $S_i \cap V_d'$ is a curve of degree $i(d-1)$. Assume that $S_i \not\subset V_d$. Then $S_i \cap V_d' \not\subset V_d$. Note that the intersection number of the hypersurface V_d and the curve $S_i \cap V_d'$ is $id(d-1)$, and the curve $S_i \cap V_d'$ is singular at the points of $\mathrm{Sing}(V_d) \cap \mathrm{Sing}(S_i)$. Therefore, $S_i \cap V_d'$ cannot meet V_d at more than $\frac{id(d-1)}{2} - 2\#|\Xi_{d,i}|$ points of $\mathrm{Sing}(V_d)$. □

Also, the following theorem is an application of the modern Cayley-Bacharach theorem as stated in [14].

Theorem 3.1. *Let* Γ *be a subset of a zero-dimensional complete intersection of hypersurfaces* $X_{d_1}, X_{d_2}, \ldots, X_{d_N}$ *of degree* $d_i \geq 1$ *in* $\mathbb{P}^N$*, and let* $\#|\Gamma|$ *be the cardinality of* Γ*. Then the points of* Γ *impose dependent linear conditions on homogeneous forms of degree* $\Sigma_{i=1}^N d_i - N - 1$ *in* $\mathbb{P}^N$ *if and only if the equality* $\#|\Gamma| = \prod_{i=1}^N d_i$ *holds.*

Proof. See [23, Theorem 2.6]. □

Let V_d be a nodal hypersurface of degree d in $\mathbb{P}^4$. Recall that if the hypersurface V_d is factorial, then, for a surface $S_r \subset V_d$ of degree r, there is a hypersurface $F \subset \mathbb{P}^4$ such that S_r is a complete intersection of V_d and F, so that in particular the degree r of a surface S_r in V_d is a multiple of d. Thus, if a surface is contained in V_d and the surface is not a complete intersection of V_d with another hypersurface in $\mathbb{P}^4$, then V_d is not factorial. More precisely, for a nodal quartic hypersurface in $\mathbb{P}^4$, we have the following three results, i.e., Lemma 3.3, Example 3.1, and Lemma 3.4. The first result is that a non-factorial nodal quartic hypersurface in $\mathbb{P}^4$ contains a surface of degree r, $r \neq 4k$ with $k \in \mathbb{N}$, in a hyperplane in $\mathbb{P}^4$, and, in the other two cases, a non-factorial nodal quartic hypersurface in $\mathbb{P}^4$ contains a non-degenerate irreducible surface of degree r, $r = 3, 4$, in $\mathbb{P}^4$ which is not the complete intersection of V_4 with a hypersurface of $\mathbb{P}^4$.

Lemma 3.3. *Let V_4 be a nodal quartic hypersurface in $\mathbb{P}^4$. If V_4 contains a surface S_r of degree r, $r = 1, 2$, in $\mathbb{P}^3$, then S_r contains at least $3r(4-r)$ points of* $\mathrm{Sing}(V_4)$*, and V_4 is not factorial.*

Proof. Suppose that, for $r = 1, 2$, V_4 is given by the equation

$$h_1 f_3 + u_r g_{4-r} = 0 \subset \mathbb{P}^4,$$

where h_1, f_3, u_r and g_{4-r} are homogeneous polynomials of degree $1, 3, r$ and $4-r$ in $\mathbb{P}^4$, respectively. Then V_4 contains the surface, $S_r := \{h_1 = 0\} \cap \{u_r = 0\}$, in $\mathbb{P}^3 \cong \{h_1 = 0\}$. Because V_4 has only ordinary double points as singularities, for any point $s \in \mathrm{Sing}(V_4)$, four hypersurfaces $\{h_1 = 0\}$, $\{f_3 = 0\}$, $\{u_r = 0\}$ and $\{g_{4-r} = 0\}$ meet transversally at the point s. Therefore, V_4 has at least $3r(4-r)$ nodes, and S_r contains at least $3r(4-r)$ nodes of V_4. Let $\Lambda =: \{h_1 = 0\} \cap \{f_3 = 0\} \cap \{u_r = 0\} \cap \{g_{4-r} = 0\}$. Then $\Lambda \subseteq \mathrm{Sing}\,(V_4)$. Because Λ is a zero-dimensional complete intersection of four hypersurfaces of degree $1, 3, r, 4-r$ in $\mathbb{P}^4$, the points of Λ impose dependent linear conditions on cubic forms on $\mathbb{P}^4$ by Theorem 3.1. This implies that the points of $\mathrm{Sing}(V_4)$ impose dependent linear conditions on cubic forms on $\mathbb{P}^4$. Thus, V_4 is not factorial by the equivalent condition (4) in Section 2. □

Example 3.1. The Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$ is isomorphic to the blow-up of $\mathbb{P}^2$ along a point. Consider the blow-up of $\mathbb{P}^2$ at one point p, giving exceptional divisor E. Then the intersection ring on $\mathbb{P}^2$ is given by $\mathbb{Z}[H, E]/H^2 = 1$, $HE = 0$, $E^2 = -1$. We can understand divisors and sections of divisors in terms of divisors on $\mathbb{P}^2$ with certain multiplicities in p. Let's consider the divisor class $2H - E$. This corresponds to conics in $\mathbb{P}^2$ through the point p, which gives a five-dimensional vector space. It separates points and tangent vectors. Therefore, we get an immersion of $\mathbb{F}_1$ into $\mathbb{P}^4$. Also, its degree is $(2H - E)(2H - E) = 3$, and hence we obtain a cubic surface in $\mathbb{P}^4$. More precisely, consider the smooth cubic surface S_3 given parametrically as the image of the map

$$\nu : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$$

which assigns to the homogeneous coordinate $[x : y : z]$ the value

$$\nu : [x : y : z] \mapsto [x^2 : y^2 : xy : xz : yz].$$

Equivalently, the cubic S_3 is a projective variety, defined as the zero locus of three irreducible quadratic hypersurfaces in $\mathbb{P}^4$. Given the homogeneous coordinates $[A : B : C : D : E]$ on $\mathbb{P}^4$, the cubic S_3 is the zero locus of the three homogeneous polynomials

$$AB - C^2 = 0, CE - BD = 0, AE - CD = 0.$$

Let V_4 be a nodal quartic hypersurface in $\mathbb{P}^4$ given by the equation

$$(AB - C^2)f_2 + (CE - BD)g_2 + (AE - CD)h_2 = 0,$$

where f_2, g_2, and h_2 are general homogeneous polynomials of degree 2 in $\mathbb{P}^4$. Then V_4 has exactly seventeen nodes and contains the smooth cubic surface

$S_3 \cong \mathbb{F}_1$, where $\mathbb{F}_1$ is a rational normal scroll. Because the cubic S_3 cannot be written as the complete intersection of V_4 with another hypersurface in $\mathbb{P}^4$, the quartic V_4 is not factorial.

Lemma 3.4. *Let V_4 be a nodal quartic hypersurface in $\mathbb{P}^4$. If V_4 contains a complete intersection surface S_4 of two quadratic hypersurfaces in $\mathbb{P}^4$, then S_4 contains at least 16 points of $\mathrm{Sing}(V_4)$, and V_4 is not factorial.*

Proof. Assume that V_4 is given by the equation

$$h_2 f_2 + u_2 g_2 = 0 \subset \mathbb{P}^4,$$

where h_2, f_2, u_2 and g_2 are quadratic homogeneous polynomials in $\mathbb{P}^4$. Then V_4 contains the surface, $S_4 := \{h_2 = 0\} \cap \{u_2 = 0\}$. Because V_4 has only ordinary double points as singularities, for any point $s \in \mathrm{Sing}(V_4)$, four hypersurfaces $\{h_2 = 0\}$, $\{f_2 = 0\}$, $\{u_2 = 0\}$ and $\{g_2 = 0\}$ meet transversally at the point s. Therefore, V_4 has at least 16 nodes, and S_4 contains at least 16 nodes of V_4. Let $\Sigma =: \{h_2 = 0\} \cap \{f_2 = 0\} \cap \{u_2 = 0\} \cap \{g_2 = 0\}$. Then $\Sigma \subseteq \mathrm{Sing}\,(V_4)$. Because Σ is a zero-dimensional complete intersection of four quadratic hypersurfaces in $\mathbb{P}^4$, the points of Σ impose dependent linear conditions on cubic forms on $\mathbb{P}^4$ by Theorem 3.1. This implies that the points of $\mathrm{Sing}(V_4)$ impose dependent linear conditions on cubic forms on $\mathbb{P}^4$. Thus, V_4 is not factorial by the equivalent condition (4) in Section 2. □

Remark 3.1. Example 3.1 and Lemma 3.4 tell us that the statement of Lemma 3.2 is not sharp when a nodal quartic hypersurface in $\mathbb{P}^4$ contains a non-degenerate surface in $\mathbb{P}^4$.

To prove the factoriality of a nodal quartic hypersurface in $\mathbb{P}^4$ with at least 14 nodes, the following two lemmas are very helpful.

Lemma 3.5. *Let V_4 be a nodal quartic hypersurface in $\mathbb{P}^4$ with*

$$14 \leq \#|\,\mathrm{Sing}(V_4)| \leq 20.$$

Suppose that there is a hyperplane in $\mathbb{P}^4$ containing all the nodes of V_4, and the quartic V_4 contains a non-degenerate irreducible surface S_k of degree k such that $S_k \neq V_4 \cap F$, where F is a hypersurface in $\mathbb{P}^4$. Then one of the following holds;

(1) *V_4 contains a plane*;
(2) *V_4 contains a quadric surface*;
(3) *there is a cubic hypersurface in $\mathbb{P}^4$ containing the surface S_k.*

Proof. From the statements (1), and (2), we assume that the quartic V_4 does not contain planes and quadrics. Note that, by our assumption, the quartic V_4 is not factorial. Suppose that V_4 is given by the equation

$$f_4(x_0, x_1, x_2, x_3, x_4) = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\left(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\right),$$

where f_4 is a homogeneous polynomial of degree 4 in $\mathbb{P}^4$. Then, assume that there is a unique hyperplane, say H_1, in $\mathbb{P}^4$ containing all the nodes of V_4;

otherwise, if there is a hyperplane, say H_2, in $\mathbb{P}^4$ that is different from H_1 and contains all the nodes of V_4, then, by Lemma 3.1(2) and $\#|\operatorname{Sing}(V_4)| \geq 14$, the quartic V_4 must contain the plane, $H_1 \cap H_2$. Since $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 35$, and $\#|\operatorname{Sing}(V_4)| \leq 20$, there is an irreducible cubic hypersurface, say Z_3, in $\mathbb{P}^4$ containing all the nodes of V_4. Now, let $\widetilde{f_4} := h_1 z_3 + f_4$, where h_1, and z_3 are homogeneous polynomials of degree 1, and 3 in $\mathbb{P}^4$, respectively, such that H_1 is defined by the equation $h_1 = 0$, and Z_3 is defined by the equation $z_3 = 0$, and let $\widetilde{V_4}$ be defined by the equation $\widetilde{f_4}(x_0, x_1, x_2, x_3, x_4) = 0$. Here $\widetilde{f_4}$ is general in the pencil. Then, V_4, and $\widetilde{V_4}$ have the same nodes, i.e., $\operatorname{Sing}(V_4) = \operatorname{Sing}(\widetilde{V_4})$. This implies that V_4, and $\widetilde{V_4}$ are not factorial at the same time, and then we claim that the cubic Z_3 contains the surface S_k, i.e., V_4 ($\widetilde{V_4}$, respectively) contains the surface S_k of degree k such that S_k is not the complete intersection of V_4 ($\widetilde{V_4}$, respectively) with a hypersurface of $\mathbb{P}^4$; by the equation $h_1 z_3 + \alpha f_4 = 0$, $\alpha \in \mathbb{C}$, we have at least 1-dimensional family, say $\mathcal{F}_4$, of nodal quartic hypersurfaces in $\mathbb{P}^4$ such that an element of $\mathcal{F}_4$ has the same nodes as the quartic V_4, and it is defined by the equation of the form $h_1 z_3 + \alpha f_4 = 0$, $\alpha \in \mathbb{C}$. Let $F_4 \in \mathcal{F}_4$. Then we have $\operatorname{rank}\operatorname{Pic}(F_4) = \operatorname{rank}\operatorname{Pic}(V_4) = 1$. By the statements (1), and (2), we assume that F_4 and V_4 contain neither 2-planes nor quadric surfaces, and then, by [20, Theorem 1.1] and [21, Remark 11], we have $\operatorname{rank}\operatorname{Cl}(F_4) \leq 6$ ($\operatorname{rank}\operatorname{Cl}(V_4) \leq 6$, respectively), where $\operatorname{Cl}(F_4)$ ($\operatorname{Cl}(V_4)$, respectively) is the group of Weil divisors on F_4 (V_4, respectively). Moreover, by [21, Remark 9], the degree of generators of $\operatorname{Cl}(F_4)/\operatorname{Pic}(F_4)$ ($\operatorname{Cl}(V_4)/\operatorname{Pic}(V_4)$, respectively) is at most 10. Suppose that the quartic V_4 contains surfaces, $\{S_{d_1}, \ldots, S_{d_r}\}$, $r \leq 5$, of degree $d_i \leq 10$ such that each surface S_{d_i} is not hypersurface section, and S_{d_i} is a non-degenerate irreducible surface. Then, since $\operatorname{rank}\operatorname{Cl}(V_4) = \operatorname{rank}\operatorname{Cl}(F_4)$, and $\dim \mathcal{F}_4 \geq 1$, we assume that F_4 contains surfaces, $\{W_{d_1}, \ldots, W_{d_r}\}$, $r \leq 5$, of degree $d_i \leq 10$ such that each surface W_{d_i} is not hypersurface section, and W_{d_i} is a non-degenerate irreducible surface of degree d_i, i.e., $\deg(S_{d_i}) = \deg(W_{d_i})$. Now, suppose that $S_{d_i} \neq W_{d_i}$, i.e., $W_{d_i} \not\subset V_4$. Then, by the equation $h_1 z_3 + \alpha f_4 = 0, \alpha \in \mathbb{C}$, we have $H_1 \cap W_{d_i} \subset H_1 \cap F_4 = H_1 \cap V_4$, and hence the space curve $C_{d_i} := W_{d_i} \cap H_1$ is contained in the quartic V_4. However, by our assumption, W_{d_i} is not contained in the quartic surface, $H_1 \cap V_4$, and hence we have $C_{d_i} = (W_{d_i} \cap H_1) \cap V_4 \neq W_{d_i} \cap (H_1 \cap V_4) = E_{4d_i}$, where $C_{d_i}(E_{4d_i}$, respectively) is a curve of degree $d_i(4d_i$, respectively). This yields a contradiction. □

Remark 3.2. In the proof of Lemma 3.5, since the quartic V_4 is not factorial, by the equivalent condition (4) in Section 2, and $\#|\operatorname{Sing}(V_4)| \leq 20$, the dimension of the system, $|\mathcal{O}_{\mathbb{P}^4}(3) - \operatorname{Sing}(V_4)|$, is at least 16, and hence we see that there is at least 16-dimensional family of cubic hypersurfaces in $\mathbb{P}^4$ containing the surface S_k.

Lemma 3.6. *Let V_4 be a nodal quartic hypersurface in $\mathbb{P}^4$ with*

$$14 \leq \#|\operatorname{Sing}(V_4)| \leq 20.$$

Suppose that there is a hyperplane in $\mathbb{P}^4$ *containing all the nodes of* V_4*, and the quartic* V_4 *contains a non-degenerate irreducible surface* S_k *of degree* k *such that* $S_k \neq V_4 \cap F$*, where* F *is a hypersurface in* $\mathbb{P}^4$*. Then one of the following holds*;

(1) V_4 *contains a plane*;
(2) V_4 *contains a quadric surface*;
(3) $k = 3$*, and* S_3 *is a* 2*-fold rational normal scroll in* $\mathbb{P}^4$;
(4) $k = 4$*, and* S_4 *is a* (*possibly singular*) *del Pezzo surface of degree* 4;
(5) $k = 6$*, and* S_6 *is the complete intersection of an irreducible cubic hypersurface and an irreducible quadratic hypersurface in* $\mathbb{P}^4$*.*

Proof. By the statement (1), we assume that the quartic V_4 contains no planes. Then there is a unique hyperplane H_1 in $\mathbb{P}^4$ containing all the nodes of V_4. Also, by Lemma 3.5, we assume that there is a cubic hypersurface Z_3 in $\mathbb{P}^4$ containing the surface S_k. Note that S_k, $k \leq 10$, is a non-degenerate surface. Then, by Remark 3.2, we divide into two cases.

Suppose that $S_k = V_4 \cap Z_3 \cap A_n$, where A_n is a hypersurface of degree n, $n = 2$ or $n = 3$, in $\mathbb{P}^4$. Then the nodal quartic V_4 is defined by an equation of the form $h_1 z_3 + a_n b_{4-n} = 0$, where H_1 is defined by the equation $h_1 = 0$, and Z_3 is defined by the equation $z_3 = 0$, and A_n is defined by the equation $a_n = 0$, and b_{4-n} is a homogeneous polynomial of degree $4 - n$ in $\mathbb{P}^4$ such that the intersection points of $\{h_1 = 0\}$, $\{z_3 = 0\}$, $\{a_n = 0\}$, and $\{b_{4-n} = 0\}$ are singular points of V_4. In this case, the quartic V_4 contains the plane, $\{h_1 = 0\} \cap \{b_1 = 0\}$, or the quadric surface, $\{h_1 = 0\} \cap \{b_2 = 0\}$.

Now, suppose that $S_k = V_4 \cap Z_3 \cap A_n \cap C_m$, where C_m is a hypersurface of degree m, $m = 2$ or $m = 3$, in $\mathbb{P}^4$; otherwise, if the surface S_k is the intersection of V_4 with four or more hypersurfaces of $\mathbb{P}^4$, then one can prove in the same way. Then the nodal quartic V_4 is defined by an equation of the form $h_1 z_3 + a_n b_{4-n} + c_m d_{4-m} + e_4 = 0$, where C_m is defined by the equation $c_m = 0$, and d_{4-m} is a homogeneous polynomial of degree $4-m$ in $\mathbb{P}^4$ such that $S_k \subset \{h_1 z_3 = 0\} \cap \{a_n b_{4-n} = 0\} \cap \{c_m d_{4-m} = 0\}$, and $\mathrm{Sing}(V_4) \subset \{h_1 z_3 = 0\} \cap \{a_n b_{4-n} = 0\} \cap \{c_m d_{4-m} = 0\}$, and e_4 is a quartic homogeneous polynomial in $\mathbb{P}^4$ such that $S_k \subset \{e_4 = 0\}$, and $\mathrm{Sing}(V_4) = \mathrm{Sing}(\{e_4 = 0\})$. The existence of the equation, $e_4 = 0$, follows from the proof of Lemma 3.5. Then, for the value n, we divide into two subcases.

If $n = 3$, then, assume that V_4 contains no planes; otherwise, the quartic V_4 contains the plane, $H_1 \cap B_1$, where the hyperplane B_1 in $\mathbb{P}^4$ is defined by the equation $b_1 = 0$. Then, by Lemma 3.1(2) and $\#|\mathrm{Sing}(V_4)| \geq 14$, we have $\#|(\mathrm{Sing}(V_4) \cap \mathrm{Sing}(A_3)) \setminus B_1| \geq 8$. Since $H_1 \cap \mathrm{Sing}(A_3) \subset \mathrm{Sing}(H_1 \cap A_3)$, we have $\#|\mathrm{Sing}(V_4) \cap \mathrm{Sing}(H_1 \cap A_3)| \geq 8$. Also, since a nodal cubic surface has at most 4 nodes, the cubic surface, $H_1 \cap A_3$, must be reducible. In this case, we divide into two subcases. Suppose that $H_1 \cap A_3 = \pi \cup S_2$, where π is a plane, and S_2 is an irreducible quadric surface. Then, since, by Lemma 3.1(1), a conic curve passes through at most 6 nodes of V_4, and S_2 has at most one node, we have $\#|\operatorname{Sing}(V_4) \cap \operatorname{Sing}(H_1 \cap A_3)| \leq 7$, and hence this yields a contradiction. Now, suppose that $H_1 \cap A_3 = \pi \cup \hat{\pi} \cup \bar{\pi}$, where π, $\hat{\pi}$, and $\bar{\pi}$ are planes. Note that $\operatorname{Sing}(V_4) \subset H_1 \cap A_3$. Then, by $\#|\operatorname{Sing}(V_4)| \geq 14$, we assume that $\pi \neq \hat{\pi} \neq \bar{\pi}$; otherwise, the quartic V_4 contains a plane in $H_1 \cap A_3$. Then, since $\#|\operatorname{Sing}(V_4) \cap \operatorname{Sing}(H_1 \cap A_3)| \geq 8$, and $\operatorname{Sing}(V_4) \subset H_1 \cap A_3$, by Lemma 3.1, the quartic V_4 must contain a plane in $H_1 \cap A_3$.

If $n = 2$, then, we assume that Z_3 is an irreducible cubic; otherwise, we obtain the statements (3), and (4). Moreover, we assume that $Z_3 \cap A_2$ is irreducible; otherwise, we obtain the statements (1), (2), and (3). Then, we get the statement (5), or the quartic V_4 contains the quadric surface, $H_1 \cap B_2$, in $\mathbb{P}^3 \cong H_1$, where the quadric B_2 is defined by the equation $b_2 = 0$. As before, if $m = 3$, then the quartic V_4 must contain a plane. Thus, we assume that $m = 2$. Then $n = m = 2$, and hence the nodal quartic V_4 is defined by an equation of the form

$$h_1 z_3 + a_2 b_2 + c_2 d_2 + e_4 = 0, \tag{3.1}$$

where $h_1, z_3, a_2, b_2, c_2, d_2$ and e_4 are homogeneous polynomials of degree 1, 3, 2, 2, 2, 2 and 4 in $\mathbb{P}^4$, respectively, such that $S_k \subset \{h_1 z_3 = 0\} \cap \{a_2 b_2 = 0\} \cap \{c_2 d_2 = 0\}$. Since S_k is a non-degenerate surface, and it is contained in the intersection of two quadratic hypersurfaces in $\mathbb{P}^4$. Therefore, the surface S_k is a 2-fold rational normal scroll in $\mathbb{P}^4$ (in this case, $Z_3 = H \cup Q$, where H is a hyperplane in $\mathbb{P}^4$, and Q is an irreducible quadratic hypersurface in $\mathbb{P}^4$), or the surface S_k is a del Pezzo surface of degree 4. □

4. Proof of Theorem 1.5

By our assumption, there is a hyperplane in $\mathbb{P}^4$ containing all the nodes of a nodal quartic hypersurface V_4 in $\mathbb{P}^4$. Therefore, if V_4 is factorial, then, by the equivalent condition (4) in Section 2, it must have at most 20 nodes because $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) = 20$.

Recall that a nodal quartic hypersurface V_4 in $\mathbb{P}^4$ is factorial if any surface in V_4 is the complete intersection of V_4 with a hypersurface of $\mathbb{P}^4$. By Theorem 1.3 in [17], we assume that $14 \leq \#|\operatorname{Sing}(V_4)| \leq 20$. Also, by our assumption and Lemma 3.6, the quartic V_4 is factorial if it does not contain any of cubic surfaces, and non-degenerate quartic surfaces, and non-degenerate irreducible sextic surfaces. From now on, we will prove that the quartic V_4 does not contain any of cubic surfaces in $\mathbb{P}^4$, and non-degenerate quartic surfaces in $\mathbb{P}^4$, and non-degenerate irreducible sextic surfaces in $\mathbb{P}^4$.

At first, suppose that V_4 contains no a surface of degree r, $r \leq 2$, and a cubic surface S_3 is contained in V_4. Then we have $S_3 \not\subset \mathbb{P}^3$; otherwise, one can find a hyperplane H_1 in $\mathbb{P}^4$ containing S_3, and hence, the quartic V_4 must contain the plane, $\overline{(V_4 \cap H_1) \setminus S_3}$, and this contradicts our assumption. Also, we assume that S_3 is irreducible; otherwise, the quartic V_4 contains a plane in S_3. Then the cubic S_3 is a variety of minimal degree. Since $S_3 \subset \mathbb{P}^4$, by [11, Theorem 1], the cubic S_3 is a 2-fold rational normal scroll and hence, the cubic S_3 can be written as the intersection of three quadratic hypersurfaces in $\mathbb{P}^4$, i.e., $S_3 = Q_{2_1} \cap Q_{2_2} \cap Q_{2_3}$, where Q_{2_1}, Q_{2_2}, and Q_{2_3} are linearly independent irreducible quadratic hypersurfaces in $\mathbb{P}^4$. Then we have $Q_{2_1} \cap V_4 = S_3 \cup T_5$, where T_5 is a quintic surface. By Lemma 3.6, the quintic T_5 must be reducible. Then, the quartic V_4 contains a plane in T_5, or an irreducible quadric surface in T_5, and hence this contradicts our assumption.

From now, suppose that V_4 contains no a surface of degree r, $r \leq 3$, and a non-degenerate quartic surface S_4 is contained in V_4. Then, we assume that S_4 is irreducible; otherwise, V_4 contains a plane in S_4, or an irreducible quadric surface in S_4, and hence this contradicts our assumption. Then, S_4 is a non-degenerate irreducible surface of degree 4 in $\mathbb{P}^4$. Furthermore, by Lemma 3.6 (4), the quartic S_4 can be written as the intersection of two quadratic hypersurfaces in $\mathbb{P}^4$, i.e., $S_4 = Q_{2_1} \cap Q_{2_2}$, where Q_{2_1}, and Q_{2_2} are linearly independent irreducible quadratic hypersurfaces in $\mathbb{P}^4$. Since V_4 contains no surface of degree r, $r \leq 3$, we have $Q_{2_1} \cap V_4 = S_4 \cup S'_4$. Here S'_4 is a non-degenerate irreducible surface of degree 4 in $\mathbb{P}^4$; if a hyperplane Y in $\mathbb{P}^4$ contains the quartic S'_4, then $S'_4 \subset Y \cap Q_{2_1}$, and hence this yields a contradiction. Then, by the equation (3.1), the quartic V_4 should be defined by an equation of the form

$$h_1 z_3 + q_{2_1} b_2 + q_{2_2} d_2 + e_4 = 0,$$

where h_1, z_3, and e_4 are homogeneous polynomials of degree $1, 3$, and 4 in $\mathbb{P}^4$, respectively, such that $S_4 \cup S'_4 \subset \{z_3 = 0\}$, and q_{2_j}, $j = 1, 2$, b_2, and d_2 are quadratic homogeneous polynomials in $\mathbb{P}^4$ such that Q_{2_j} is given by the equation $q_{2_j} = 0$, and $S'_4 = Q_{2_1} \cap \{d_2 = 0\}$. Note that $Q_{2_1} \cap V_4 \subseteq Q_{2_1} \cap \{h_1 z_3 = 0\}$. However, since S_4, and S'_4 are irreducible, we have $S_4 \cup S'_4 \not\subset Q_{2_1} \cap \{h_1 z_3 = 0\}$, and hence this yields a contradiction.

Finally, suppose that V_4 contains no a surface of degree r, $r \leq 4$, and a non-degenerate irreducible sextic surface S_6 is contained in V_4. Then, by Lemma 3.6, the sextic S_6 lives in some quadratic hypersurface Q_2 in $\mathbb{P}^4$. Then we have $Q_2 \cap V_4 = S_6 \cup T_2$, where T_2 is a quadric surface, and hence the quartic V_4 contains a plane in T_2, or the irreducible quadric surface T_2. This contradicts our assumption.

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