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ON VANISHING THEOREMS FOR LOCALLY CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we obtain some vanishing theorems for p-harmonic 1-forms on locally conformally flat Riemannian manifolds which admit an integral pinching condition on the curvature operators.

1. Introduction

It is well known that the theory of L^2-harmonic 1-forms has played an important role in the study of the structure of complete manifolds such as the topology at infinity of a complete Riemannian manifold or complete orientable δ-stable minimal hypersurface in $\mathbb{R}^{n+1}$ (see [1, 12] and others). By the appeal of this theory, many authors are interested in studying it and there have been a lot of remarkable results on this field. For instance, Lin [14] investigated the L^2 harmonic 1-form on locally conformally flat Riemannian manifolds and obtained some vanishing and finiteness theorems for L^2 harmonic 1-forms. Zhang [18] obtained vanishing results for p-harmonic 1-forms. Chang [2] obtained the compactness for any bounded set of p-harmonic 1-forms. Han-Pan [9] investigated L^p p-harmonic 1-forms on complete noncompact submanifolds in a Hadamard manifold, and obtained some vanishing and finiteness theorems for these forms.

For vanishing theorems, there are also many results of those for manifolds endowed with special analysis structure. By assuming that the Ricci curvature is bounded from below in terms of the dimension and the first eigenvalue, Li-Wang [13] proved a vanishing-type theorem of L^2 harmonic 1-forms. Later, Li-Wang's results were generalized by Lam [11] and were continued by many authors like Chen-Sung [3], Dung-Sung [5, 6] and Vieira [17]. By replacing the condition on bounded Ricci curvature by an integral pinching condition on the traceless Ricci tensor and the scalar curvature, Han-Zhang-Liang [10] obtained some vanishing results for L^p p-harmonic 1-forms on a locally conformally flat Riemannian manifold. To state their results, we recall some notations as follows.

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Let (M,g) be an m-dimensional Riemannian manifold. A function u on M is said to be p-harmonic on M if u satisfies the Euler-Lagrange equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0.$$

When $p=2$, the function u is a harmonic function. There is a general definition given as follows: an 1-form ω on M is said to be a p-harmonic 1-form if

$$\begin{cases} d\omega=0,\\ d^*(|\omega|^{p-2}\omega)=0,\end{cases}$$

in the distributional sense. Here d^* stands for the dual operator of the usual differential operator. The space of the L^p p-harmonic 1-form on M is defined by

$$H^{1,p}(M)=\left\{\omega:\int_M|\omega|^p<\infty,\ d\omega=0,\ d^*(|\omega|^{p-2}\omega)=0\right\}.$$

Denote by Ric, R and $T=\mathrm{Ric}-\frac{R}{m}g$ the Ricci curvature tensor, the scalar curvature and the traceless Ricci tensor respectively of (M,g).

When M is complete, Lin [14] gave a relation between these curvature operators as follows:

$$\mathrm{Ric}\ge -|T|g-\frac{|R|}{\sqrt{m}}g \tag{1.1}$$

in the sense of quadratic forms.

When M is simply connected, locally conformally flat, then it has a conformal immersion into S^m and according to [16], the Yamabe constant of M satisfies

$$Q(M)=Q(S^m)=\frac{m(m-2)\omega_m^{2/m}}{4},$$

where ω_m is the volume of the unit sphere in $\mathbb{R}^m$. Therefore, the following inequality

$$Q(S^m)\left(\int_M f^{2m/(m-2)}\right)^{(m-2)/m}\le\int_M|\nabla f|^2+\frac{m-2}{4(m-1)}\int_M Rf^2 \tag{1.2}$$

holds for all $f\in C_0^\infty(M)$.

Since (1.2), it is well-known that if $R\le 0$ or $\int_M|R|^{m/2}dv<\infty$, then we have the following Sobolev inequality (for example, see [14]).

$$\left(\int_M f^{2m/(m-2)}\right)^{(m-2)/m}\le S\int_M|\nabla f|^2 \tag{1.3}$$

holds for all $f\in C_0^\infty(M)$ with some constant $S>0$, which is equal to $Q(S^m)^{-1}$ in the case where $R\le 0$. In particular, M has infinite volume.

By basing on a precise estimate of the curvature operators which appear in the Bochner-Weitzenböck formula on p-harmonic 1-forms and together with the Sobolev inequality induced by the positivity of the Yamabe constant as well as Kato's inequality, Han-Zhang-Liang proved the following vanishing theorems.

Theorem A ([10]). *Let $(M^m, g)(m \geq 3)$ be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. If*

$$\left(\int_M |T|^{m/2}\right)^{2/m} + \frac{1}{\sqrt{m}}\left(\int_M |R|^{m/2}\right)^{2/m} < \frac{4[(m-1)(p-1)+1]}{Sp^2(m-1)},$$

then we have $H^{1,p}(M) = \{0\}$ for $p \geq 2$, where S is a positive constant appearing in Sobolev inequality (1.3).

Theorem B ([10]). *Let $(M^m, g)(m \geq p^4)$ be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold with $R \leq 0$. If*

$$\left(\int_M |T|^{m/2}\right)^{2/m} < \left[\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(m-1)}{\sqrt{m}(m-2)}\right] Q(S^m),$$

then we have $H^{1,p}(M) = \{0\}$ for $p \geq 2$, where $Q(S^m) = \frac{m(m-2)\omega_m^{2/m}}{4}$ is the Yamabe constant of S^m and ω_m is the volume of the unit sphere in $\mathbb{R}^m$.

By using the approach in [4] and the method in [10], we will give some extensions for the above vanishing theorems.

Theorem 1.1. *Let $(M^m, g)(m \geq 3)$ be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that there exists a positive constant*

$$\Lambda < \frac{4(p-1+k_p)}{Sp^2}$$

satisfying

$$\left(\int_M |T|^{m/2}\right)^{2/m} + \frac{1}{\sqrt{m}}\left(\int_M |R|^{m/2}\right)^{2/m} \leq \Lambda,$$

where $k_p = \min\{1, \frac{(p-1)^2}{m-1}\}$ with $p \geq 2$. Then every $L^{2\beta}$ p-harmonic 1-form vanishes for

$$\frac{p}{2} \leq \beta < \frac{1}{S\Lambda}\left(1 + \sqrt{1 - S\Lambda(1-k_p)}\right).$$

Remark 1.2. It is easy to see that

$$\frac{4(p-1+k_p)}{Sp^2} \geq \frac{4[(m-1)(p-1)+1]}{Sp^2(m-1)}.$$

Therefore, for $2\beta = p$, Theorem 1.1 can be considered as a refinement of Theorem 1. We also notice that assertion was obtained by Dung-Tien in [7]. In fact, in [7], the authors gave a vanishing theorem for p-harmonic 1-forms with L^Q finite energy, for any $Q \geq 2$ provided that

$$\Lambda \leq \frac{4(Q-1+k_p)}{SQ^2}.$$

The main difference between ours and theirs is that when $Q = p$, Dung and Tien obtained only vanishing property for forms with L^p finite energy while we can show vanishing property for forms with $L^{2\beta}$ finite energy.

Theorem 1.3. *Let $(M^m, g)(m \geq p^4)$ be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold with $R \leq 0$. Assume that there exists a positive constant*

$$\Lambda < \left[\frac{4(p-1+k_p)}{p^2} - \frac{4(m-1)}{(m-2)\sqrt{m}}\right] Q(S^m)$$

satisfying

$$\left(\int_M |T|^{m/2}\right)^{2/m} \leq \Lambda,$$

where $k_p = \min\{1, \frac{(p-1)^2}{m-1}\}$ with $p \geq 2$. Then every $L^{2\beta}$ p-harmonic 1-form vanishes for

$$\frac{p}{2} \leq \beta < \frac{1 + \sqrt{1 - \left(\frac{4(m-1)}{(m-2)\sqrt{m}} + \Lambda S\right)(1-k_p)}}{\frac{4(m-1)}{(m-2)\sqrt{m}} + \Lambda S}.$$

Remark 1.4. As $p \geq 2$, then $\frac{(p-1)^2}{m-1} \geq \frac{1}{m-1}$. It implies that $k_p \geq \frac{1}{m-1}$ and therefore

$$\frac{(m-1)(p-1)+1}{m-1} \leq p - 1 + k_p.$$

This means that the range of Λ in our paper is wider than that in Han-Zhang-Liang. Moreover, our results conclude the $L^{2\beta}$ p-harmonic 1-forms with $2\beta \geq p$.

Again, in [7], a vanishing property was obtained for p-harmonic 1-forms with L^Q finite energy ($Q \geq 2$). This is to say that the classes of energy in their paper are larger. Therefore the main contribution of Theorem 1.3 is that for $Q = p$, namely when

$$\Lambda < \left[\frac{4(p-1+k_p)}{p^2} - \frac{4(m-1)}{(m-2)\sqrt{m}}\right] Q(S^m)$$

we obtain vanishing results for any p-harmonic 1-forms with $L^{2\beta}$ finite energy while in [7], it is only to conclude vanishing results for any p-harmonic 1-forms with L^p finite energy.

2. Proof of theorems

To verify Theorem 1.1, we need to have the below Kato's inequality.

Lemma 2.1 ([4]). *Let ω be a p-harmonic 1-form on a Riemannian manifold M^m. Then we have the following inequality*

$$|\nabla(|\omega|^{p-2}\omega)|^2 \geq \left(1 + \frac{k_p}{(p-1)^2}\right) |\nabla|\omega|^{p-1}|^2, \tag{2.1}$$

where $p \geq 2$ and $k_p = \min\{1, \frac{(p-1)^2}{m-1}\}$.

Lemma 2.2 ([4, 9]). *Let $f : M^m \to \mathbb{R}$ be smooth function on Riemannian manifold M^m, and ω be a closed 1-form on M^m. Then we have $|d(f\omega)| \leq |df||\omega|$.*

Proof of Theorem 1.1. Let $M_+ = M \setminus \mathcal{S}$ where $\mathcal{S} := \{x \in M : \omega(x) = 0\}$. Let ω be any p-harmonic 1-form on M^m with finite $L^{2\beta}$ norm. Applying the Bochner-Weitzenböck formula for $|\omega|^{p-2}\omega$, we get

$$\frac{1}{2}\Delta(||\omega|^{p-2}\omega|^2) = |\nabla(|\omega|^{p-2}\omega)|^2 - \big\langle (d^*d + dd^*)|\omega|^{p-2}\omega, |\omega|^{p-2}\omega\big\rangle + \mathrm{Ric}(|\omega|^{p-2}\omega, |\omega|^{p-2}\omega).$$

By using (1.1), we have

$$\begin{aligned}&\frac{1}{2}\left(2|\omega|^{p-1}\Delta|\omega|^{p-1} + 2\big|\nabla|\omega|^{p-1}\big|^2\right)\\ \geq\ & |\nabla(|\omega|^{p-2}\omega)|^2 - \big\langle (d^*d + dd^*)|\omega|^{p-2}\omega, |\omega|^{p-2}\omega\big\rangle - (|T| + \frac{|R|}{\sqrt{m}})|\omega|^{2(p-1)}.\end{aligned}$$

Since $d^*\left(|\omega|^{p-2}\omega\right) = 0$, we obtain

$$\begin{aligned}|\omega|^{p-1}\Delta|\omega|^{p-1} \geq& \left|\nabla\left(|\omega|^{p-2}\omega\right)\right|^2 - \big|\nabla|\omega|^{p-1}\big|^2\\ &- \left\langle d^*d\left(|\omega|^{p-2}\omega\right), |\omega|^{p-2}\omega\right\rangle - (|T| + \frac{|R|}{\sqrt{m}})|\omega|^{2(p-1)}.\end{aligned}$$

Dividing both sides of the above inequality by $|\omega|^{p-2}$ and applying Lemma 2.1, we have

$$|\omega|\Delta|\omega|^{p-1} \geq k_p|\omega|^{p-2}|\nabla|\omega||^2 - \big\langle d^*d(|\omega|^{p-2}\omega), \omega\big\rangle - (|T| + \frac{|R|}{\sqrt{m}})|\omega|^p. \tag{2.2}$$

Choose any number $q \geq 0$ and a smooth nonnegative function φ with a compact support in M_+. Multiplying both sides of inequality (2.2) by $|\omega|^q\varphi^2$ and integrating by parts over M_+ gives

$$\begin{aligned}&\int_{M_+}\big\langle \nabla(|\omega|^{q+1}\varphi^2), \nabla|\omega|^{p-1}\big\rangle + \int_{M_+} k_p|\omega|^{p+q-2}\varphi^2|\nabla|\omega||^2\\ &- \int_{M_+}(|T| + \frac{|R|}{\sqrt{m}})|\omega|^{p+q}\varphi^2\\ \leq& \int_{M_+}\big\langle d(|\omega|^{p-2}\omega), d(|\omega|^q\varphi^2\omega)\big\rangle.\end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned}\int_{M_+}\big\langle \nabla(|\omega|^{q+1}\varphi^2), \nabla|\omega|^{p-1}\big\rangle =& (q+1)(p-1)\int_{M_+}|\omega|^{q+p-2}|\nabla|\omega||^2\varphi^2\\ &+ 2(p-1)\int_{M_+}\varphi|\omega|^{p+q-1}\left\langle \nabla\varphi, \nabla|\omega|\right\rangle.\end{aligned} \tag{2.4}$$

By Lemma 2.2, we have

$$|d(\varphi\omega)| = |d\varphi \wedge \omega| \leq |d\varphi||\omega|$$

for any smooth function $\varphi : M \to \mathbb{R}$ and any closed 1-form ω. So we get

$$\begin{aligned}\int_{M_+} \left\langle d(|\omega|^{p-2}\omega), d(|\omega|^q\varphi^2\omega)\right\rangle &\leq \int_{M_+} |\nabla(|\omega|^{p-2})| \cdot |\omega| \cdot |\nabla(|\omega|^q\varphi^2)| \cdot |\omega| \\ &\leq (p-2)q \int_{M_+} |\omega|^{p+q-2}|\nabla|\omega||^2\varphi^2 \\ &\quad + 2(p-2)\int_{M_+} |\omega|^{p+q-1}\varphi| \left\langle \nabla\varphi, \nabla|\omega|\right\rangle|. \end{aligned} \tag{2.5}$$

Combining inequalities (2.3), (2.4) and (2.5), we see that

$$\begin{aligned}&((q+1)(p-1) + k_p - (p-2)q)\int_{M_+} |\omega|^{p+q-2}|\nabla|\omega||^2\varphi^2 \\ &- \int_{M_+} (|T| + \frac{|R|}{\sqrt{m}})|\omega|^{p+q}\varphi^2 \\ \leq\ &(2(p-2) + 2(p-1))\int_{M_+} \varphi|\omega|^{p+q-1}| \left\langle \nabla\varphi, \nabla|\omega|\right\rangle|. \end{aligned} \tag{2.6}$$

Since

$$2\varphi|\omega|^{p+q-1}|\nabla\varphi||\nabla|\omega|| \leq \varepsilon|\omega|^{p+q-2}\varphi^2|\nabla|\omega||^2 + \frac{1}{\varepsilon}|\omega|^{p+q}|\nabla\varphi|^2$$

for $\varepsilon > 0$ and using (2.6), we get

$$\begin{aligned}&(p+q-1+k_p - \epsilon(2p-3))\int_{M_+} |\omega|^{p+q-2}|\nabla|\omega||^2\varphi^2 \\ &- \int_{M_+} (|T| + \frac{|R|}{\sqrt{m}})|\omega|^{p+q}\varphi^2 \\ \leq\ &\frac{2p-3}{\epsilon}\int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2. \end{aligned} \tag{2.7}$$

Now since $m \geq 3$, and by applying Hölder inequality, Sobolev inequality (1.3) and Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}&\int_{M^+} |T||\varphi|^2\omega|^{p+q} \\ \leq\ &\left(\int_{\mathrm{supp}(\varphi)} |T|^{m/2}\right)^{2/m} \left(\int_{M^+} \left(\varphi|\omega|^{(p+q)/2}\right)^{2m/(m-2)}\right)^{(m-2)/m} \\ \leq\ &S\left(\int_{\mathrm{supp}(\varphi)} |T|^{m/2}\right)^{2/m} \int_{M^+} |\nabla(\varphi|\omega|^{(p+q)/2})|^2 \\ \leq\ &\phi(\varphi)\left((1+\epsilon)\int_{M^+} \varphi^2|\nabla|\omega|^{(p+q)/2}|^2 + \left(1+\frac{1}{\epsilon}\right)\int_{M^+} |\omega|^{p+q}|\nabla\varphi|^2\right) \\ =\ &\phi(\varphi)(1+\epsilon)\left(\frac{p+q}{2}\right)^2\int_{M^+} \varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 \end{aligned}$$

$$+\phi(\varphi)\big(1+\frac{1}{\epsilon}\big)\int_{M^+}|\omega|^{p+q}|\nabla\varphi|^2 \tag{2.8}$$

for any $\epsilon > 0$, where $\phi(\varphi) = S\left(\int_{\mathrm{supp}(\varphi)}|T|^{m/2}\right)^{2/m}$. Similarly, we also get

$$\int_{M^+}|R|\varphi|^2\omega|^{p+q} \le \psi(\varphi)(1+\epsilon)\big(\frac{p+q}{2}\big)^2\int_{M^+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2$$
$$+\psi(\varphi)\big(1+\frac{1}{\epsilon}\big)\int_{M^+}|\omega|^{p+q}|\nabla\varphi|^2, \tag{2.9}$$

where $\psi(\varphi) = S\left(\int_{\mathrm{supp}(\varphi)}|R|^{m/2}\right)^{2/m}$.

Together (2.7) and (2.8) with (2.9), we get

$$C_\varepsilon\int_{M_+}|\omega|^{p+q-2}|\nabla|\omega||^2\varphi^2 \le D_\varepsilon\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2$$

for any $\varphi \in C_0^\infty(M_+)$, where

$$C_\varepsilon = p+q-1+k_p-\varepsilon(2p-3)-(1+\varepsilon)\left(\frac{p+q}{2}\right)^2\left(\phi(\varphi)+\frac{\psi(\varphi)}{\sqrt{m}}\right),$$

and

$$D_\varepsilon = \frac{2p-\epsilon}{\epsilon}+\left(1+\frac{1}{\varepsilon}\right)\left(\phi(\varphi)+\frac{\psi(\varphi)}{\sqrt{m}}\right).$$

Choose a sufficiently small $\varepsilon > 0$ in the above. Then there exists a positive constant $C = C(\varepsilon, n, p, q)$ such that for any $\varphi \in C_0^\infty(M_+)$

$$\int_{M_+}|\omega|^{p+q-2}|\nabla|\omega||^2\varphi^2 \le C\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2 \tag{2.10}$$

provided

$$p+q-1+k_p-\left(\frac{p+q}{2}\right)^2\left(\phi(\varphi)+\frac{\psi(\varphi)}{\sqrt{m}}\right)$$
$$\ge p+q-1+k_p-\left(\frac{p+q}{2}\right)^2 S\Lambda > 0. \tag{2.11}$$

Let $\beta = \frac{p+q}{2} \ge \frac{p}{2}$. Then inequality (2.11) is equivalent to the following condition:

$$S\Lambda\beta^2 - 2\beta + 1 - k_p < 0. \tag{2.12}$$

Moreover it is easy to see that inequality (2.12) is satisfied if and only if the assumption on Λ and β in Theorem 1.1 is satisfied, that is,

$$\frac{p}{2} \le \beta < \frac{1}{S\Lambda}\left(1+\sqrt{1-S\Lambda(1-k_p)}\right),$$
$$1-S\Lambda(1-k_p) > 0, \quad \text{and} \quad S\Lambda < (p-1+k_p)\,\frac{4}{p^2}.$$

Hence $\Lambda < \frac{1}{S}\min\left\{\frac{1}{1-k_p}, (p-1+k_p)\,\frac{4}{p^2}\right\} = (p-1+k_p)\,\frac{4}{Sp^2}$.

By a variation of the Duzaar-Fuchs cut-off method (see in [4, 5, 8, 15]), we will show that inequality (2.10) holds for every $\psi \in C_0^\infty(M)$. Indeed, define

$$\eta_{\tilde{\epsilon}} = \min\left\{\frac{|\omega|}{\tilde{\epsilon}}, 1\right\}$$

for $\tilde{\epsilon} > 0$. Let $\varphi_{\tilde{\epsilon}} = \psi^2\eta_{\tilde{\epsilon}}$. Hence, by the assumption, $\varphi_{\tilde{\epsilon}}$ is a compactly supported continuous function and $\varphi_{\tilde{\epsilon}} = 0$ on $M \setminus M_+$. Moreover, $\varphi_{\tilde{\epsilon}} \in W_0^{1,2}(M_+)$. As $\tilde{\epsilon} \to 0, \eta_{\tilde{\epsilon}} \to 1$ pointwisely in M_+, by the similar argument as in [8, 15], noting that ω is differentiable outside $\mathcal{S}$ almost everywhere and $C_0^\infty(M)$ is dense in $C_0^1(M)$, we can replace φ by $\varphi_{\tilde{\epsilon}}$ in (2.10) to obtain

$$\begin{aligned} &\int_{M_+} \psi^4(\eta_{\tilde{\epsilon}})^2|\omega|^{p+q-2}|\nabla|\omega||^2 \\ &\le 6C\int_{M_+}|\omega|^{p+q}|\nabla\psi|^2\psi^2(\eta_{\tilde{\epsilon}})^2 + 3C\int_{M_+}|\omega|^{p+q}|\nabla\eta_{\tilde{\epsilon}}|^2\psi^4. \end{aligned} \tag{2.13}$$

Noting that

$$\int_{M_+}|\omega|^{p+q}|\nabla\eta_{\tilde{\epsilon}}|^2\psi^4 \le \tilde{\epsilon}^{p+q-2}\int_{M_+}|\nabla|\omega||^2\psi^4\chi_{\{|\omega|\le\tilde{\epsilon}\}} \tag{2.14}$$

and the right hand side of (2.14) vanishes by dominated convergence as $\tilde{\epsilon} \to 0$, $|\nabla|\omega|| \in L^2_{loc}(M)$. Letting $\tilde{\epsilon} \to 0$ and applying Fatou lemma to the integral on the left hand side and dominated convergence to the first integral in the right hand side of (2.13), we arrive at

$$\int_{M_+}\psi^4|\omega|^{p+q-2}|\nabla|\omega||^2 \le 6C\int_{M_+}|\omega|^{p+q}|\nabla\psi|^2\psi^2, \tag{2.15}$$

where $\psi \in C_0^\infty(M)$.

Choose a nonnegative smooth function ψ such that

$$\psi = \begin{cases} 1 & \text{on } B(R), \\ 0 & \text{on } M \setminus B(2R), \end{cases}$$

and $|\nabla\psi| \le \frac{2}{R}$. Note that $\beta = \frac{p+q}{2}$, then inequality (2.15) implies

$$\int_{M_+\cap B(R)}|\omega|^{2\beta-2}|\nabla|\omega||^2 \le \frac{24C}{R^2}\int_{M_+}|\omega|^{2\beta}.$$

Letting $R \to \infty$, then for all $|\omega| \in L^{2\beta}(M)$ is constant on each connected component of M_+. Observe that $\omega \in C^0(M)$ and $\omega = 0$ on ∂M_+. Thus $\omega = 0$ on each connected component of M_+ provided $\partial M_+ \neq \emptyset$, which is a contradiction. It follows that $M_+ = M$ and hence $|\omega|$ is a nonzero constant on M. Since M has infinite volume and $|\omega| \in L^{2\beta}(M)$, we have $\omega = 0$. This completes the proof. □

Proof of Theorem 1.3. By the assumption $R \leq 0$, inequality (1.2) implies that

$$\frac{m-2}{4(m-1)}\int_M |R|f^2 \leq \int_M |\nabla f|^2 \tag{2.16}$$

for all $f \in C_0^\infty(M)$. Replacing f by $\varphi|\omega|^{(p+q)/2}$ in (2.16), after that using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\frac{m-2}{4(m-1)}\int_{M_+} |R|\varphi^2|\omega|^{p+q} \\
&\leq \int_{M_+} |\nabla(\varphi|\omega|^{(p+q)/2})|^2 \\
&\leq (1+\epsilon)\int_{M_+} \varphi^2|\nabla|\omega^{(p+q)/2}||^2 + \big(1+\frac{1}{\epsilon}\big)\int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2 \\
&= (1+\epsilon)\big(\frac{p+q}{2}\big)^2\int_{M_+} \varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 \\
&\quad + \big(1+\frac{1}{\epsilon}\big)\int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2
\end{aligned} \tag{2.17}$$

with any $\epsilon > 0$. Together (2.7) and (2.8) with (2.17), we get

$$C_\varepsilon\int_{M_+} |\omega|^{p+q-2}|\nabla|\omega||^2\varphi^2 \leq D_\varepsilon\int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2$$

for any $\varphi \in C_0^\infty(M_+)$, where

$$C_\varepsilon = p+q-1+k_p-\varepsilon(2p-3)-(1+\varepsilon)\left(\frac{p+q}{2}\right)^2\left(\frac{4(m-1)}{(m-2)\sqrt{m}}+\phi(\varphi)\right),$$

and

$$D_\varepsilon = \frac{2p-3}{\epsilon}+\left(1+\frac{1}{\varepsilon}\right)\left(\phi(\varphi)+\frac{4(m-1)}{(m-2)\sqrt{m}}\right).$$

Since $m \geq p^4$, we have

$$(p-1+k_p)\,\frac{4}{p^2}-\frac{4(m-1)}{(m-2)\sqrt{m}} \geq \left(p-1+\frac{1}{n-1}\right)\frac{4}{p^2}-\frac{4(m-1)}{(m-2)\sqrt{m}} > 0.$$

Therefore, there exists a number Λ satisfying

$$0 < \Lambda \leq (p-1+k_p)\,\frac{4}{Sp^2}-\frac{4(m-1)}{S(m-2)\sqrt{m}}.$$

Then, by similar calculations to those in Theorem 1.1, we easily show that the following inequality

$$\begin{aligned}
&p+q-1+k_p-\left(\frac{p+q}{2}\right)^2\left(\frac{4(m-1)}{(m-2)\sqrt{m}}+\phi(\varphi)\right) \\
&\geq p+q-1+k_p-\left(\frac{p+q}{2}\right)^2\left(\frac{4(m-1)}{(m-2)\sqrt{m}}+S\Lambda\right)
\end{aligned}$$

$$\geq p+q-1+k_p-\left(\frac{p+q}{2}\right)^2\frac{4(p-1+k_p)}{p^2}>0$$

holds if and only if the assumption on $\beta=\frac{p+q}{2}$ in Theorem 1.3 is satisfied.

Now, we can take a sufficiently small $\varepsilon>0$ such that $C_\varepsilon>0$. Then there exists a positive constant $C=C(\varepsilon,n,p,q)$ such that for any $\varphi\in C_0^\infty(M_+)$

$$\int_{M_+}|\omega|^{p+q-2}|\nabla|\omega||^2\varphi^2\leq C\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2.$$

This inequality and the proof of Theorem 1.1 help us complete this proof. □

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