CO-UNIFORM AND HOLLOW S-ACTS OVER MONOIDS

ROGHAIIEH KHOSRAVI AND MOHAMMAD ROUEENTAN

Abstract. In this paper, we first introduce the notions of superfluous and coessential subacts. Then hollow and co-uniform S-acts are defined as the acts that all proper subacts are superfluous and coessential, respectively. Also it is indicated that the class of hollow S-acts is properly between two classes of indecomposable and locally cyclic S-acts. Moreover, using the notion of radical of an S-act as the intersection of all maximal subacts, the relations between hollow and local S-acts are investigated. Ultimately, the notion of a supplement of a subact is defined to characterize the union of hollow S-acts.

1. Introduction

A submodule $K$ of an $R$-module $M$ is called superfluous (small), if the equality $N + K = M$ implies that $N = M$. The notion of small submodule plays a fundamental role in the category of modules over rings. According to [2], a non-zero module $M$ is defined to be hollow if every submodule of $M$ is small (superfluous). The classical notion of hollow modules has been studied extensively for a long time in many papers (see for example [3, 10]). In the category of $S$-acts the notions of small (coessential) and superfluous subacts are distinct which we define both as follows. For $S$-acts, first we refer the reader to [7] and for preliminaries and basic results related monoids and $S$-acts. A subact $B_S$ of $A_S$ is called large in $A_S$ if any homomorphism $g : A_S \rightarrow C_S$ such that $g|_B$ is a monomorphism is itself a monomorphism. An extension $B$ of $A$ with the embedding $f : A_S \rightarrow B_S$ is called an essential extension of $A$ if $\text{Im}f$ is large in $B$.

The categorical dual of essential extension is called a coessential epimorphism which we recall as follows. Let $S$ be a monoid. An act $B_S$ is called a cover of an act $A_S$ if there exists an epimorphism $f : B_S \rightarrow A_S$ such that for any proper subact $C_S$ of $B_S$ the restriction $f|_{C_S}$ is not an epimorphism. An epimorphism with this property is called a coessential epimorphism. Indeed it is defined in order to investigate $\mathcal{X}$-perfect monoids as monoids over which every right $S$-act has an $\mathcal{X}$-cover, where $\mathcal{X}$ is an act property which is preserved...
under coproduct. More information about various kinds of cover of acts one can see [4–6,8].

As a dual of large subact, we call $B_S$ a coessential (small) subact of $A_S$ if $A_S$ is a cover of the Rees factor act $A_S/B_S$. According to the notion of superfluous submodule, a subact $B_S$ of an $S$-act $A_S$ shall be called superfluous if the union of $B_S$ with every proper subact of $A_S$ is also a proper subact of $A_S$. In Section 2, We consider the properties of coessential and superfluous subacts. In [9], the authors investigated uniform acts over a semigroup $S$, as $S$-acts that all their non-zero subacts are large. In module theory, the dual notion of a uniform module is that of a hollow module. In fact hollow and co-uniform modules are equal. For $S$-acts, as we mentioned earlier, the notion of coessential and superfluous are distinct, so we define co-uniform as a dual of uniform $S$-acts and hollow $S$-acts with respect to the definition of hollow in module theory. In Section 3, we characterize the classes of co-uniform and hollow acts with respect to the definition of hollow in module theory. In Section 4, we investigate radical of an $S$-acts and local $S$-acts, and consider the relationship between local and hollow $S$-acts. Finally, in Section 5, a supplement of a subact and supplemented $S$-acts are introduced and using these notions to characterize the union of hollow $S$-acts. The following lemma is clearly proved which is needed in the sequel.

**Lemma 1.1.** If $M$ is a maximal subact of a right $S$-act $A_S$, then $A/M$ is finitely generated.

### 2. Coessential or superfluous subacts

In this section we introduce the notions of coessential and superfluous subacts, and consider general properties of them.

**Definition.** A subact $B_S$ of an $S$-act $A_S$ is called

(i) coessential if the epimorphism $\pi : A_S \rightarrow A_S/B_S$ is a coessential epimorphism; in other words, $A_S$ is a cover of $A_S/B_S$. It is denoted by $B \ll A$.

(ii) superfluous if $B_S \cup C_S \neq A_S$ for each proper subact $C_S$ of $A_S$, and it is denoted by $B \leq_s A$.

In the following lemma we present an equivalent condition for being coessential.

**Lemma 2.1.** A subact $B_S$ of an $S$-act $A_S$ is coessential if and only if for each proper subact $C_S$ of $A_S$, $C \cap B \neq \emptyset$ implies that $C \cup B \neq A$.

**Proof.** Necessity. Let $C_S$ be a proper subact of $A_S$ and $C \cap B \neq \emptyset$. Since $\pi : A_S \rightarrow A_S/B_S$ is a coessential epimorphism, $\pi|_{C_S}$ is not an epimorphism, which implies the existence of $a \in A_S$ such that $[a] \notin \pi(C)$. Now we claim that $a \notin C \cup B$. Otherwise, either $a \in C$ which means $[a] \in \pi(C)$ or $a \in B$ which implies $[a] = [b] \in \pi(C)$ for some $b \in C \cap B$. Thus $C \cup B \neq A$. 
Sufficiency. Let $C_S$ be a proper subact of $A_S$. We show that for the epimorphism $\pi : A_S \to A_S/B_S$, $\pi|_{C_S}$ is not an epimorphism. If $C \cap B = \emptyset$, clearly for each $b \in B$ we have $[b] \notin \pi(C)$. Otherwise, if $C \cap B \neq \emptyset$, by assumption $C \cup B \neq A$. So we have $[a] \notin \pi(C)$ for each $a \in A \setminus (C \cup B)$. Therefore, $\pi|_{C_S}$ is not an epimorphism.

In view of the previous lemma, it is obvious that being a superfluous subact implies coessential. But the converse is not valid. For instance, let $B/D$ be an arbitrary monoid and $A_S = \Theta [\Theta = \{\theta_1, \theta_2\}$. Then $\{\theta_1\}$ is coessential but not superfluous.

**Lemma 2.2.** A coessential subact of each indecomposable right $S$-act is superfluous.

**Proof.** Suppose that $B$ is a coessential subact of an indecomposable right $S$-act $A_S$ and $B \cup C = A$ for a subact $C$ of $A$. If $B \cap C = \emptyset$, then $A = B \displaystyle\biguplus C$ which contradicts with being indecomposable. So $B \cap C \neq \emptyset$ and $B \cup C = A$ which imply that $C = A$. Therefore, $B$ is superfluous.

**Lemma 2.3.** Suppose that $A_S$, $B_S$, $C_S$, $D_S$ are $S$-acts such that $D_S \subseteq C_S \subseteq B_S \subseteq A_S$. The following hold.

(i) $B \leq_s A$ if and only if $C \leq_s A$ and $B/C \leq_s A/C$.

(ii) If $C \leq_s B$, then $C \leq_s A$.

(iii) $B \leq_s A$ if and only if for each $S$-act $X_S$ and $h : X \to A$, $\text{Im}(h) \cup B = A$ implies $\text{Im}(h) = A$.

(iv) $B/D \leq_s A/D$ if and only if $B/C \leq_s A/C$ and $C/D \leq_s A/D$.

**Proof.** (i) Necessity. The first part is obvious. Let $K$ be a subact of $A/C$ with $B/C \cup K = A/C$. So $D = \{t \in A \mid [t] \in B/C\}$ is a subact of $A_S$ and it is easily checked that $D \cup B = A$. By assumption, $D = A$, and thus $K = A/C$.

Sufficiency. Let $D$ be a subact of $A$ and $D \cup B = A$. So $B/C \cup (D \cup C)/C = A/C$ which implies $(D \cup C)/C = A/C$. Then $D \cup C = A$ implies that $D = A$, as desired.

Parts (ii) and (iii) are clear.

(iv) We only show the sufficiency. Suppose that $(B/D) \cup K = A/D$ for some subact $K$ of $A/D$. Get $X = \{t \in A \mid [t] \in K\}$ which is clearly a subact of $A_S$. Then $(B/C) \cup ((X \cup C)/C) = A/C$. Since $B/C \leq_s A/C$, we have $X \cup C = A$. So $(C/D) \cup K = A/D$ and since $C/D \leq_s A/D$, $K = A/D$. Therefore $B/D \leq_s A/D$.

Similar to the proof of the previous lemma, two following lemmas are easily checked.

**Lemma 2.4.** The following hold for a monoid $S$.

(i) If $C_S \subseteq B_S \subseteq A_S$ and $C \ll B$, then $C \ll A$.

(ii) If $C_S \subseteq B_S \subseteq A_S$ and $B \ll A$, then $C \ll A$ and $B/C \ll A/C$. 

(ii) If $B \preccurlyeq A$ ($B \leq_S A$) and $f : A \rightarrow C$ is a monomorphism, then $f(B) \preccurlyeq C$ ($f(B) \leq_S C$).

**Lemma 2.5.** Let $B, C$ be proper subacts of $A_S$. Then $B \cup C \leq_S A$ if and only if $B \leq_S A$ and $C \leq_S A$.

**Lemma 2.6.** Suppose that $B_i$ is a proper subact of $A_i$ for each $i \in I$. The following hold for a monoid $S$:

(i) $\prod_{i \in I} B_i \leq_S \prod_{i \in I} A_i$ if and only if $B_i \leq_S A_i$ for each $i \in I$.

(ii) If $\prod_{i \in I} B_i \ll \prod_{i \in I} A_i$, then $B_i \ll A_i$ for each $i \in I$.

(iii) If $B_i \leq_S A_i$ ($B_i \ll A_i$) for each $i \in \{1, \ldots, n\}$, then $\bigcup_{i=1}^n B_i \leq_S \bigcup_{i=1}^n A_i$ ($\bigcup_{i=1}^n B_i \ll \bigcup_{i=1}^n A_i$).

**Proof.** (i) Necessity. Suppose that $\prod_{i \in I} B_i \leq_S \prod_{i \in I} A_i$. Fix $i \in I$ and $D_i$ a subact of $A_i$ such that $B_i \cup D_i = A_i$. Then $D = (\prod_{i \notin J} A_i) \prod_{i \in I} D_i$ is a subact of $\prod_{i \in I} A_i$ and $\prod_{i \in I} B_i \cup D = \prod_{i \in I} A_i$. By assumption, $D = \prod_{i \in I} A_i$ which implies that $D_i = A_i$.

Sufficiency. Suppose that $B_i \leq_S A_i$ for each $i \in I$. Let $D$ be a subact of $\prod_{i \in I} A_i$ such that $\prod_{i \in I} B_i \cup D = \prod_{i \in I} A_i$. Since $B_i$ is a proper subact of $A_i$ for each $i \in I$, $D = \prod_{i \in I} D_i$ such that $D_i \neq \emptyset$ is a subact of $A_i$. Obviously, $B_i \cup D_i = A_i$ for every $i \in I$ and by assumption $D_i = A_i$ which gives that $D = \prod_{i \in I} A_i$.

By a similar argument one can prove part (ii). Part (iii) is a straightforward consequence of Lemmas 2.3 and 2.5.

\[
\square
\]

3. Co-uniform and hollow $S$-acts

In this section we study the classes of co-uniform and hollow $S$-acts.

**Definition.** An $S$-act $A_S$ is called co-uniform if all proper subacts of $A_S$ are coessential, and $A_S$ is said to be hollow if every its proper subact is superfluous.

Obviously, hollow implies co-uniform, but the converse is not valid. Let $S$ be an arbitrary monoid. It is easily checked that, $\Theta \prod \Theta$ is co-uniform but not hollow.

**Proposition 3.1.** Every factor act of a (co-uniform) hollow act is also (co-uniform) hollow.

**Proof.** Let $A$ be a hollow $S$-act and $f : A \rightarrow C$ an epimorphism. Let $D$ be a proper subact of $C$. We show that $D \leq_S C$. Clearly, $B = f^{-1}(D)$ is also a proper subact of $A$. So $B \leq_A A$. Now, suppose that $D \cup E = C$. It is easily checked that $B \cup f^{-1}(E) = A$. So by assumption, $f^{-1}(E) = A$, and thus $E = C$. By a similar argument one could prove for co-uniform acts.

\[
\square
\]

Recall that an $S$-act $A_S$ is called locally cyclic if for all $a, a' \in A_S$ there exists $a'' \in A$ such that $a, a' \in a''S$. Every locally cyclic $S$-act is indecomposable and every cyclic $S$-acts is locally cyclic.
Proposition 3.2. Every locally cyclic right $S$-act is hollow, and consequently, every cyclic right $S$-act is hollow.

Proof. Let $A_S$ be a locally cyclic $S$-act. If $A_S$ is simple, i.e., contains no proper subacts, the result follows. Otherwise, let $B$ be a proper subact of $A_S$. If $C \cup B = A$ for some proper subact $C$ of $A$, take $a \in A \setminus B$ and $a' \in A \setminus C$. So there exists $a'' \in A$ with $a, a' \in a''S$. Since $A = B \cup C$, we have $a'' \in B$ or $a'' \in C$ which implies that $a \in B$ or $a' \in C$, a contradiction. Thus $C = A$, and $B$ is a superfluous subact of $A_S$. □

Theorem 3.3. A right $S$-act $A_S$ is hollow if and only if $A_S$ is an indecomposable co-uniform right $S$-act.

Proof. Necessity. Suppose that $A_S$ is hollow, and $B,C$ are proper subacts of $A$ such that $A = B \bigsqcup C$. Thus $A = B \cup C$ which means that $B$ is not superfluous subact of $A$, a contradiction.

In view of Lemma 2.2, the following the sufficiency is deduced. □

In general being indecomposable does not imply being hollow. For instance, let $A_S$ be a cyclic $S$-act with a proper subact $B$, then $A \bigsqcup B$ is indecomposable but not hollow. In particular, for a proper right ideal $I$ of a monoid $S$, $S \bigsqcup I$ is indecomposable but not hollow. So we have the following strict implications,

cyclic $\implies$ locally cyclic $\implies$ hollow $\implies$ indecomposable.

In the following proposition we characterize co-uniform $S$-acts.

Proposition 3.4. Every co-uniform $S$-act $A$ is indecomposable or $A = A_1 \bigsqcup A_2$, where each $A_i$ is simple.

Proof. Suppose that $A_S$ is a co-uniform decomposable $S$-act. Let $A = \bigsqcup_{i \in I} A_i$. If $|I| > 2$, fix $k \neq j \in I$ and put $B = A_k \bigsqcup A_j$. So $B \cup (\bigsqcup_{i \neq j} A_i) = A$ and $B \cap (\bigsqcup_{i \neq j} A_i) = A_k \neq \emptyset$. Then $B$ is not coessential which is a contradiction. Thus $|I| = 2$. Now, suppose that $A = A_1 \bigsqcup A_2$ such that $A_1$ is not simple. Let $B_1$ be a proper subact of $A_1$. Then $B = B_1 \bigsqcup A_2$ is a proper subact of $A$ such that $B \cap A_1 \neq \emptyset$ and $B \cup A_1 = A$ which means that $B$ is not coessential, a contradiction. Then $A = A_1 \bigsqcup A_2$ which $A_1, A_2$ are simple, as desired. □

Let $S$ be an arbitrary monoid and $A = \Theta \bigsqcup \Theta \bigsqcup \Theta$. Using Proposition 3.4, $A$ is not co-uniform. So for each arbitrary monoid $S$ there exists a finitely generated $S$-act which is not hollow or co-uniform.

An $S$-act $A$ is said to be a uniserial $S$-act if every two subacts of $A$ are comparable with respect to inclusion. In the next theorem we characterize an $S$-act all its subacts are hollow.

Theorem 3.5. For an $S$-act $A_S$ the following statements are equivalent.

(i) $A$ is a uniserial $S$-act.

(ii) Every subact of $A$ is hollow.

(iii) Every subact of $A$ generated by two elements is hollow.
Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are obvious.
(iii)⇒(i) Let $B$ and $C$ be subacts of $A$ and let $B \nsubseteq C$. Then there exists an element $x \in B \setminus C$. To show that $C \subseteq B$, suppose that $y \in C$. Put $N = xS \cup yS$. If $N = yS$, then $xS \subseteq N = yS \subseteq C$. So $x \in C$, a contradiction. Hence $yS$ is a proper subact of $N$, and since $N$ is hollow, then $N = xS$. Therefore, $yS \subseteq N = xS \subset B$ which implies that $y \in B$, and so $C \subseteq B$. □

Proposition 3.6. The following hold for a monoid $S$.

(i) Every hollow $S$-act with a minimal generating set is cyclic.
(iii) Every finitely generated hollow $S$-act is cyclic.

Proof. It suffices to prove part (i). Let $A_S$ be a right $S$-act with a minimal generating set \{ $a_i \mid i \in I$ \}. In contrary suppose that $|I| > 1$, and fix $i \in I$. Then $a_iS \cup (\cup_{j \neq i} a_jS) = A$, and since $A_S$ is hollow, $A_S = \cup_{j \neq i} a_jS$, a contradiction.

Recall that a monoid $S$ satisfies condition (A) if all right $S$-acts satisfy the ascending chain condition for cyclic subacts. In [5] it is shown that a monoid $S$ satisfies condition (A) if and only if every locally cyclic $S$-act is cyclic, equivalently, every right $S$-act contains a minimal generating set. Now, using this fact and the previous proposition we deduce the following result as a generalization of that result in [5].

Lemma 3.7. A monoid $S$ satisfies condition (A) if and only if every hollow $S$-act is cyclic.

We conclude this section considering the cover of hollow $S$-acts. In [5], it is shown that a cover of a locally cyclic right $S$-act is indecomposable. Now, we extend this to the following result.

Lemma 3.8. Each cover of a hollow $S$-act is indecomposable.

Proof. Let $A_S$ be a hollow $S$-act and $f : D_S \to A_S$ a coessential epimorphism. Suppose that $D = \coprod_{i \in I} D_i$ such that each $D_i$ is indecomposable. In contrary, suppose that $|I| > 1$ and choose $i \neq j \in I$. Since $f \mid_{D \setminus D_i}$ is not an epimorphism, $f(D \setminus D_i)$ is a proper subact of $A$ and $f(D \setminus D_i) \cup f(D \setminus D_j) = A$. Now since $A_S$ is hollow, $f(D \setminus D_i) = A$, and so $f \mid_{D \setminus D_j}$ is an epimorphism, a contradiction. Therefore $D$ is indecomposable. □

The following corollary is a straightforward result of the previous lemma.

Corollary 3.9. For a monoid $S$ the following hold.

(i) Every projective cover of a hollow $S$-act is cyclic.
(i) Every strongly flat (condition (P)) cover of a hollow $S$-act is locally cyclic.
4. The relation between hollow and radical of $S$-acts

In this section we consider local $S$-acts and the radical of an $S$-act. We also discuss the relationship between local and hollow $S$-acts.

**Definition.** A right $S$-act is called **local** if it contains exactly one maximal subact. A monoid $S$ is also called right (left) **local** if it contains exactly one maximal right (left) ideal.

The set of maximal subacts of a right $S$-act $A_S$ is denoted by $\text{Max}(A)$.

**Lemma 4.1.** Every cyclic right $S$-act is simple or local.

**Proof.** Suppose that $A = aS$ is cyclic, and $A_S$ is not simple. By using Zorn’s Lemma, $\text{Max}(A) \neq \emptyset$. Now, suppose that $M \neq N$ are maximal subacts of $A$. Then $M \cup N = A$ implies that $a \in M$ or $a \in N$, and so $N = A$ or $M = A$, a contradiction. Thus $A$ is local. □

Now, we deduce the following remark which was also discussed in [1].

**Remark 4.2.** Every monoid $S$ is a group or right local. Indeed the set
\[
\{ s \in S \mid s \text{ is not right invertible} \}
\]
is either empty or the unique maximal right ideal of $S$. Then the local monoid property is left-right symmetric. Thus we briefly call it a local monoid.

The following theorem establishes a relation to hollow $S$-acts with local and cyclic $S$-acts.

**Theorem 4.3.** Let $A_S$ be a right $S$-act. Then the following are equivalent:

(i) $A_S$ is a hollow right $S$-act and $\text{Max}(A) \neq \emptyset$;

(ii) $A_S$ is a cyclic and local right $S$-act;

(iii) $A_S$ is a finitely generated local right $S$-act;

(iv) Every proper subact of $A_S$ is contained in a maximal subact, and $A_S$ is a local right $S$-act;

(v) $A_S$ contains a maximal subact $N$ such that $N \leq_s A$;

(vi) $A_S$ contains the unique maximum subact $N$ such that $N \leq_s A$.

**Proof.** (i)⇒(ii) Let $N$ be a maximal subact of $A_S$ and let $L$ be an arbitrary subact of $A_S$ where $L \subset N$. Since $N \cup L = A$, and $A_S$ is a hollow right $S$-act, then $A = L$. Hence $A_S$ has just one maximal subact. If $a \in A \setminus N$ and $L = aS$, then $A = aS$.

The implications (ii)⇒(iii) and (iii)⇒(iv) are obvious.

(iv)⇒(v) Let $N$ be the unique maximal subact of $A$ and let $L$ be a proper subact of $A$. By assumption, $L \subset N$. Then $L \cup N = N \neq A$ and so $N \leq_s A$.

(v)⇒(vi) Let $N$ be a maximal subact of $A$ which $N \leq_s A$ and let $B$ be a proper subact of $A$. So $N \cup B \neq A$ and by maximality of $N$ we have $B \subset N$. So $N$ is maximum.
(vi)⇒(i) Let \(N\) be the maximum subact of \(A\) which \(N \leq_s A\). For each proper subact \(B\) of \(A\) we have \(B \subseteq N \leq_s A\), we deduce that \(B \leq_s A\). Therefore \(A_s\) is hollow. \(\square\)

In general, every hollow (indecomposable co-uniform) \(S\)-act is not cyclic or local. For instance, take \(S = (\mathbb{N}, \min) \cup \{\varepsilon\}\) where \(\varepsilon\) denotes the externally adjoined identity greater than each natural element. Then \(A = \{1, 2, 3, \ldots\}\) is not cyclic act and \(\text{Max}(A) = \emptyset\). But all its subacts are \(\{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\} \subseteq \cdots\), and so \(A\) is hollow.

Let \(S\) be a monoid and \(A\) a right \(S\)-act. The radical of the act \(A\) is the intersection of all maximal subacts of \(A\),

\[
\text{Rad}(A) = \cap \{N \mid N \text{ is a maximal subact of } A\}.
\]

If \(A\) contains no a maximal subact, we put \(\text{Rad}(A) = A\). If \(\text{Rad}(A) \neq \emptyset\), the \(\text{Rad}(A)\) is a subact of \(A\).

In module theory, the radical submodule is equal to the union of superfluous submodules. The next proposition demonstrates that it is also valid for \(S\)-acts.

To reach that we need the following lemma.

**Lemma 4.4.** If \(a \in A\) and \(C \leq A\) such that \(aS \cup C = A\), then \(C = A\) or there exists a maximal subact \(M\) of \(A\) such that \(C \subseteq M\) and \(a \notin M\).

**Proof.** Let \(C \neq A\). Take \(B = \{D \mid D \subseteq A\text{ and }C \subseteq D\}\). Clearly \(C \subseteq B \neq \emptyset\) and \(B\) is a partially ordered set. Let \(\{D_i\}_{i \in I}\) be a chain in \(B\), so \(D_i \subseteq A\) and \(C \subseteq D_i\). Let \(D = \cup_{i \in I} D_i\). If \(D \subseteq A\), then \(D\) is an upper bound. Otherwise, if \(D = A\), \(a \in A\) implies \(a \in D\), and there exists \(i \in I\) such that \(a \in D_i\). Then \(aS \subseteq D_i\) which implies that \(aS \cup D_i = D_i = A\), a contradiction. Then by Zorn’s Lemma, \(B\) has a maximal element \(M\). So \(M\) is a maximal subact of \(A\) such that \(C \subseteq M\), \(a \notin M\). \(\square\)

As we know, \(A \leq_s A\) if and only if \(A\) is simple.

**Proposition 4.5.** Let \(A_s\) be a right \(S\)-act. Then

\[
\text{Rad}(A) = \cup \{B \mid B \leq_s A\}.
\]

**Proof.** Suppose that \(\Gamma = \cup \{B \mid B \leq_s A\}\). First we show that \(\Gamma \subseteq \text{Rad}(A)\).

If \(\text{Max}(A) = \emptyset\), clearly \(\Gamma \subseteq \text{Rad}(A) = A\). Otherwise, let \(B \leq_s A\) and \(N\) be an arbitrary maximal subact of \(A\). If \(B \not\subseteq N\), being maximal of \(N\) implies that \(B \cup N = A\). Since \(B \leq_s A\), \(N = A\), a contradiction. Thus \(B \subseteq N\), and so \(\Gamma \subseteq \text{Rad}(A)\). To show the converse, let \(a \in \text{Rad}(M)\). First we show that \(aS \leq_s A\). If \(aS = A\), then \(A = \text{Rad}(A)\) and by Lemma 4.1 \(A\) is simple. So \(aS = A \leq_s A\). Now, let \(aS\) be a proper subact of \(A\) and \(aS \cup C = A\). If \(C \neq A\) by previous lemma there exists a maximal subact \(M\) of \(A\) such that \(C \subseteq M\) and \(a \notin M\), but \(a \in \text{Rad}(M)\) implies \(a \in M\), a contradiction. Then \(C = A\) which means that \(aS \leq_s A\). We deduce \(aS \subseteq \cup \{B \mid B \leq_s A\}\), and therefore \(\text{Rad}(A) \subseteq \Gamma\). \(\square\)
Using the previous proposition, the following result is immediately deduced.

**Corollary 4.6.** For a monoid $S$ the following statements hold.

(i) Let $A_S$ be a right $S$-act. Then for each element $a \in \text{Rad}(A)$, $aS \leq_s A$.

(ii) Let $A$ and $B$ be right $S$-acts and let $f : A \to B$ be an $S$-monomorphism. Then $f(\text{Rad}(A)) \subseteq \text{Rad}(B)$.

(iii) $\text{Rad}(A) = A$ if and only if all finitely generated subact of $A$ are superfluous in $A$.

**Corollary 4.7.** Let $A_S$ be a right $S$-act. Then each non-cyclic hollow subact $B$ of $A$ is contained in $\text{Rad}(A)$.

**Proof.** Assume that $B$ is a hollow subact of $A$ and $b \in B$. So $bS$ is a proper subact of $B$ and $bS \leq_s B$, and by Lemma 2.3, $bS \leq_s A$. Using the previous proposition, $bS \subseteq \text{Rad}(A)$ which implies that $B \subseteq \text{Rad}(A)$. \hfill $\square$

Now, we give an equivalent condition for an $S$-act which its radical is superfluous.

**Theorem 4.8.** For a right $S$-act $A$ the following statements are equivalent.

(i) $\text{Rad}(A) \leq_s A$.

(ii) Every proper subact of $A$ is contained in a maximal subact.

**Proof.** (i)$\implies$(ii) Let $C$ be a proper subact of $A$. Since $\text{Rad}(A) \leq_s A$, $\text{Rad}(A) \cup C \neq A$. Suppose $\{M_i | i \in I\}$ is the family of all maximal subacts of $A$. So $\bigcap_{i \in I} M_i \cup C \neq A$, which implies that $\cap_{i \in I} (M_i \cup C) \neq A$. Then there exists $j \in I$ such that $M_j \cup C \neq A$. Now, maximality of $M_j$ implies that $C \subseteq M_j$, and the result follows.

(ii)$\implies$(i) Suppose that $C$ is an arbitrary proper subact of $A$. There exists a maximal subact $M$ of $A$ with $C \subseteq M$. Then we have $C \cup \text{Rad}(A) \subseteq M \cup \text{Rad}(A) = M \neq A$. Thus, $\text{Rad}(A) \leq_s A$. \hfill $\square$

**Proposition 4.9.** An $S$-act $A$ is finitely generated if and only if $A/\text{Rad}(A)$ is finitely generated and $\text{Rad}(A) \leq_s A$.

**Proof.** Let $A$ be finitely generated, clearly $A/\text{Rad}(A)$ is finitely generated. Let $C \leq A$, $\text{Rad}(A) \cup C = A$, by Proposition 4.5, $\text{Rad}(A) = \cup \{B | B \leq_s A, \cup B = C\} = A$. Since $A$ is finitely generated, there exist $B_1, \ldots, B_m \leq_s A$ such that $B_1 \cup B_2 \cup \cdots \cup B_m \cup C = A$. Since $B_1 \leq_s A$ and $B_1 \cup (B_2 \cup \cdots \cup B_m \cup C) = A$, we imply that $B_2 \cup \cdots \cup B_m \cup C = A$. Since $B_2, \ldots, B_m \leq_s A$, we continue this manner to imply $C = A$. Thus $\text{Rad}(A) \leq_s A$.

Sufficiency. Suppose that $A/\text{Rad}(A) = \cup_{i=1}^n [a_i]S$. So $\text{Rad}(A) \cup \cup_{i=1}^n a_i S = A$. Now, since $\text{Rad}(A) \leq_s A$, $\cup_{i=1}^n a_i S = A$. Thus $A$ is finitely generated. \hfill $\square$

5. **Supplemented acts**

In this section we introduce the notions of a supplement of a subact and supplemented $S$-acts, and general properties of them are discussed. Our aim is...
to use the notion of a supplement of a subact to investigate the union of hollow $S$-acts.

**Definition.** Let $B, C$ be proper subacts of a right $S$-act $A$. We call $C$ is a *supplement* of $B$ in $A$, or $B$ has a supplement $C$ in $A$ if the following two conditions are satisfied.

(i) $B \cup C = A$.
(ii) If $D \subseteq C$ and $B \cup D = A$, then $D = C$.

If every proper subact of $A$ has a supplement in $A$, then $A$ is called a *supplemented* $S$-act.

Clearly, if an $S$-act $A = B \coprod C$, then $C$ is a supplement of $B$ in $A$ if and only if $C \cap B = \emptyset$ or $C \cap B \leq_s A$.

**Lemma 5.1.** Let $A = B \cup C$. If $B \cap C \neq \emptyset$, then $C$ is a supplement of $B$ in $A$ if and only if $C \cap B = \emptyset$ or $C \cap B \leq_s C$.

**Proof.** Let $E$ be a subact of $C$. Then $(C \cap B) \cup E = C$ is equivalent to $A = B \cup E$ and so the result is easily checked.

The following result presents that co-uniform implies supplemented.

**Proposition 5.2.** Every co-uniform $S$-act is supplemented.

**Proof.** Let $A$ be a right $S$-act and $B$ be a proper subact of $A$. First suppose that $A$ is indecomposable. By Theorem 3.3, $A$ is hollow. Then $B \cup A = A$ and $(B \cap A) = B \leq_s A$ imply that $A$ is a supplemented $S$-act. In the case that $A$ is not indecomposable, by Proposition 3.4, $A = B \coprod C$ where $B, C$ are simple acts. Thus $C$ is a supplement of $B$.

The converse of Proposition 5.2 is not valid. For instance, let $S$ be an arbitrary monoid and $A = \Theta \coprod \Theta \coprod \Theta$. Using Proposition 3.4, $A$ is not co-uniform. But, as all subsets of $A$ are also subacts, for each subact $B$ of $A$ we have $A \setminus B$ is a supplement of $B$.

Let $C$ be a proper subact of an $S$-act $A$. By Lemma 2.3, each superfluous subact of $C$ is also superfluous in $A$. So clearly $\text{Rad}(C) \subseteq C \cap \text{Rad}(A)$.

**Proposition 5.3.** Suppose that $C$ is a proper subact of an $S$-act $A$ such that $C$ is a supplement of a proper subact $B$ of $A$. Then the following hold.

(i) If $D \cup C = A$ for some $D \subset B$, then $C$ is a supplement of $D$.
(ii) If $A$ is finitely generated, then $C$ is also finitely generated.
(iii) If $E$ is a subact of $C$ such that $E \leq_s A$, then $E \leq_s C$.
(iv) If $N \leq_s A$, then $N \cap C \leq_s C$.
(v) If $N \leq_s A$, then $C$ is a supplement of $N \cup B$.
(vi) $\text{Rad}(C) = C \cap \text{Rad}(A)$.

**Proof.** (i) It is easily proved by using Lemmas 5.1 and 2.3.
(ii) Let $A$ be finitely generated. Since $B \cup C = A$, there is a finitely generated subact $X \subseteq C$ such that $B \cup X = A$. By the minimality of $C$, we imply that $C = X$.

(iii) Let $X$ be a subact of $C$ with $E \cup X = C$. Since $B \cup C = A$, we have $B \cup E \cup X = A$. Now, since $E \subseteq A$, $B \cup X = A$ and so $X = C$.

(iv) Using part (iii) and Lemma 2.3, it is clearly checked.

(v) Let $N \leq_s A$. We have $(N \cup B) \cup C = A$. Let $X \subseteq C$ with $(N \cup B) \cup X = A$. Then $N \leq_s A$ implies that $B \cup X = A$, and hence $X = A$.

(vi) We have $\text{Rad}(C) \subseteq C \cap \text{Rad}(A)$. To show the converse, if $N \leq_s A$, by part (iv), $E = N \cap C \leq_s C$, and $E \subseteq \text{Rad}(C)$. Therefore, $C \cap \text{Rad}(A) = C \cap (\cup \{N \mid N \leq_s A\}) = \cup \{N \cap C \mid N \leq_s A\} \subseteq \text{Rad}(C)$. □

Now, we turn our attention to the concept of supplement in a projective $S$-act.

**Proposition 5.4.** Let $P$ be a projective $S$-act, and $C$ be a supplement of $B$ in $P$. Then $C$ is projective or there exists an epimorphism $f : P \to C$ such that $f(B) \leq_s C$.

**Proof.** Let $C$ be a supplement of $B$ in $P$. So $P = B \cup C$. If $B \cap C = \emptyset$, then $P = B \coprod C$, and $C$ is projective. Now, suppose that $B \cap C \neq \emptyset$. Let $\pi_1 : C \to C/(B \cap C)$ be the canonical epimorphism, and define $\pi_2 : P \to C/(B \cap C)$ by $\pi_2(p) = \begin{cases} [p], & p \in C \\ \emptyset, & p \in B \end{cases}$. So since $P$ is projective, there exists a homomorphism $f : P \to C$ with $\pi_1 f = \pi_2$. It is easily checked that $\text{Im} f \cup B = P$, and by assumption, $\text{Im} f = C$. Moreover, since $f(B) \subseteq B \cap C \leq_s C$, by Lemma 2.3, $f(B) \leq_s C$. □

Finally, we conclude this paper by considering the union of hollow acts.

**Theorem 5.5.** Let $A$ be a right $S$-act such that $\text{Rad}(A) \leq_s A$. The following statements are equivalent.

(i) $A$ is a union of hollow acts.

(ii) Each proper subact $B$ of $A$ whose $A/B$ is finitely generated has a supplement.

(iii) Every maximal subact of $A$ has a supplement.

**Proof.** (i)$\Rightarrow$(ii) Suppose $A = \cup_{i \in I} L_i$ such that each $L_i$ is hollow $S$-act. Let $B$ be a proper subact of $A$ such that $A/B$ is finitely generated. Then $A/B = \cup_{i \in I} (L_i \cup B)/B$. Since $A/B$ is finitely generated, $A = B \cup L_1 \cup L_2 \cup \cdots \cup L_n$ for some hollow $S$-acts $L_1, L_2, \ldots, L_n$ with $B \cap L_i \neq L_i$ for each $1 \leq j \leq n$. Take $L = L_1 \cup L_2 \cup \cdots \cup L_n$. To show that $L$ is a supplement of $B$, let $X$ be a proper subact. There exists $1 \leq j \leq n$ such that $X \cap L_j$ is a proper subact of $L_j$. Now, since $L_j$ is hollow, $(B \cap L_j) \cup (X \cap L_j) \neq L_j$. Thus $B \cup X \neq A$, and the result follows.

(ii)$\Rightarrow$(iii) follows by Lemma 1.1. (iii)$\Rightarrow$(i) Let $B$ be the union of all hollow subacts of $A$. In contrary, suppose that $B$ is a proper subact of $A$. So there
exists a maximal subact $N$ of $A$ with $B \subseteq N$. Let $L$ be a supplement of $N$ in $A$. If $L$ is simple, then $L \subseteq B$. Otherwise, let $X$ be a proper subact of $L$. So $N \cup X \neq A$, and maximality of $N$ implies that $X$ is contained in $N$. So by Lemma 5.1, $N \cap L \leq s \lesssim L$, and using Lemma 2.3, $X \subseteq N \cap L \subseteq L$ implies $X \leq s \lesssim L$. Then $L$ is a hollow act. Therefore $L$ is contained in $B$, and so $A = L \cup N \subseteq B \cup N = N$, a contradiction. Therefore, $B = A$. Now suppose that $C$ is an arbitrary proper subact of $A$. There exists a maximal subact $M$ of $A$ with $C \subseteq M$. Then we have $C \cup \text{Rad}(A) \subseteq M \cup \text{Rad}(A) = M \neq A$. Thus, $\text{Rad}(A) \leq s \lesssim A$. □

References


Roghaieh Khosravi
Department of Mathematics
Faculty of Sciences
Fasa University
Fasa, Iran.

Email address: khosravi@fasau.ac.ir

Mohammad Roueentan
College of Engineering
Lamerd Higher Education Center
Lamerd, Iran

Email address: rooeintan@lamerdhec.ac.ir