Commun. Korean Math. Soc. **37** (2022), No. 2, pp. 575–583
https://doi.org/10.4134/CKMS.c210178
pISSN: 1225-1763 / eISSN: 2234-3024

A NOTE ON GENERALIZATIONS OF BAILEY'S IDENTITY INVOLVING PRODUCTS OF GENERALIZED HYPERGEOMETRIC SERIES

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Abstract. In the theory of hypergeometric and generalized hypergeometric series, the well-known and very useful identity due to Bailey (which is a generalization of the Preece's identity) plays an important role. The aim of this research paper is to provide generalizations of Bailey's identity involving products of generalized hypergeometric series in the most general form. A few known, as well as new results, have also been obtained as special cases of our main findings.

1. Introduction and results required

We start with the definition of generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by [2, 8]

$$ {}_pF_q\left[\begin{array}{c} a_1,\ \ldots,\ a_p \\ b_1,\ \ldots,\ b_q \end{array};\ z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{1} $$

where $(a)_n$ is the well-known Pochhammer symbol defined for $a \in \mathbb{C}$ by

$$ \begin{aligned} (a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)},\ \left(a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right) \\ &= \begin{cases} a(a+1)\cdots(a+n-1), & (n \in \mathbb{N}) \\ 1, & (n = 0) \end{cases} \end{aligned} \tag{2} $$

where Γ is the well-known Gamma function. For further details about the generalized hypergeometric function ${}_pF_q$ and its special case, the well-known Gauss hypergeometric function ${}_2F_1$, its convergence conditions, various special cases and limiting cases, we refer the standard text [8].

In general there are four classical summation theorems for the series ${}_2F_1$ notably Gauss, Gauss second, Kummer and Bailey. Also for the series ${}_3F_2$, we are having four classical summation theorems namely Watson, Dixon and

Received May 13, 2021; Accepted September 13, 2021.
2020 *Mathematics Subject Classification.* Primary 33C20, 33C05.
Key words and phrases. Generalized hypergeometric series, Bailey identity, Kummer transformations.

Whipple and Saalschütz. Recently a good deal of progress has been done in the direction of generalizing and extending the above mentioned classical summation theorems. For this, we refer research papers by Kim et al. [4] and Rakha and Rathie [9].

In literature the following two transformation formulas [8] viz.

$$e^{-x}\ {}_1F_1\left[\begin{array}{c} a \\ b \end{array};x\right] = {}_1F_1\left[\begin{array}{c} b-a \\ b \end{array};-x\right] \tag{3}$$

and

$$e^{-\frac{x}{2}}\ {}_1F_1\left[\begin{array}{c} a \\ 2a \end{array};x\right] = {}_0F_1\left[\begin{array}{c} - \\ a+\frac{1}{2} \end{array};\frac{x^2}{16}\right] \tag{4}$$

are known as the Kummer's first and second theorems, respectively.

Bailey[1] obtained the result (3) with the help of following Gauss summation theorem [8] viz.

$${}_2F_1\left[\begin{array}{c} a,\ b \\ c \end{array};1\right] = \frac{\Gamma(c)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)} \tag{5}$$

provided $Re(c-a-b)>0$, and the result (4) with the help of the following Gauss's second summation theorem [2] viz.

$${}_2F_1\left[\begin{array}{c} a,\ b \\ \frac{1}{2}(a+b+1) \end{array};\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}. \tag{6}$$

In 1998, Rathie and Choi [11] derived the result (4) in the form

$${}_1F_1\left[\begin{array}{c} a \\ 2a \end{array};x\right] = e^{x}\ {}_0F_1\left[\begin{array}{c} - \\ a+\frac{1}{2} \end{array};\frac{x^2}{4}\right] \tag{7}$$

by utilizing classical Gauss's summation theorem (5).

In 1995, Rathie and Nagar [13] obtained the following two interesting results closely related to the Kummer's second theorem (4) viz.

$$\begin{aligned} &e^{-\frac{x}{2}}\ {}_1F_1\left[\begin{array}{c} a \\ 2a+1 \end{array};x\right] \\ &= {}_0F_1\left[\begin{array}{c} - \\ a+\frac{1}{2} \end{array};\frac{x^2}{16}\right] - \frac{x}{2(2a+1)}\ {}_0F_1\left[\begin{array}{c} - \\ a+\frac{3}{2} \end{array};\frac{x^2}{16}\right] \end{aligned} \tag{8}$$

and

$$\begin{aligned} &e^{-\frac{x}{2}}\ {}_1F_1\left[\begin{array}{c} a \\ 2a-1 \end{array};x\right] \\ &= {}_0F_1\left[\begin{array}{c} - \\ a-\frac{1}{2} \end{array};\frac{x^2}{16}\right] + \frac{x}{2(2a-1)}\ {}_0F_1\left[\begin{array}{c} - \\ a+\frac{1}{2} \end{array};\frac{x^2}{16}\right]. \end{aligned} \tag{9}$$

Also, we mention below the well-known and useful identity due to Preece [6] viz.

$${}_1F_1\left[\begin{array}{c} a \\ 2a \end{array};x\right] \times {}_1F_1\left[\begin{array}{c} a \\ 2a \end{array};x\right] = e^{x}\ {}_1F_2\left[\begin{array}{c} a \\ a+\frac{1}{2},\ 2a \end{array};\frac{x^2}{4}\right]. \tag{10}$$

In 1997, Rathie [10] gave a very short proof of the Preece's identity (10) and obtained the following two results closely related to it viz.

$$(11)\quad {}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix};x\right] \times {}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix};x\right] = e^x\left\{{}_1F_2\left[\begin{matrix} a \\ a+\frac{1}{2},\ 2a \end{matrix};\frac{x^2}{4}\right] - \frac{x}{2(2a+1)}\,{}_1F_2\left[\begin{matrix} a+1 \\ a+\frac{3}{2},\ 2a+1 \end{matrix};\frac{x^2}{4}\right]\right\}$$

and

$$(12)\quad {}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix};x\right] \times {}_1F_1\left[\begin{matrix} a \\ 2a-1 \end{matrix};x\right] = e^x\left\{{}_1F_2\left[\begin{matrix} a \\ a+\frac{1}{2},\ 2a-1 \end{matrix};\frac{x^2}{4}\right] + \frac{x}{2(2a-1)}\,{}_1F_2\left[\begin{matrix} a \\ a+\frac{1}{2},\ 2a \end{matrix};\frac{x^2}{4}\right]\right\}.$$

The Preece's identity (10) was generalized by Bailey [1] in the following form.

$$(13)\quad {}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix};x\right] \times {}_1F_1\left[\begin{matrix} b \\ 2b \end{matrix};x\right] = e^x\,{}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b),\ \frac{1}{2}(a+b+1) \\ a+\frac{1}{2},\ b+\frac{1}{2}, a+b \end{matrix};\frac{x^2}{4}\right].$$

For $b = a$, (13) reduces to (10).

Bailey [1] obtained the identity (13) by utilizing following classical Watson's summation theorem [2] viz.

$$(14)\quad {}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), 2c \end{matrix};1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\ \Gamma\left(c+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}$$

provided $Re(2c-a-b) > -1$.

In 1998, Rathie and Choi [12] re-derived Bailey's result by a short method and obtained five results closely related to (13). However we shall mention here two of them.

$$(15)\quad {}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix};x\right] \times {}_1F_1\left[\begin{matrix} b \\ 2b \end{matrix};x\right] = e^x\left\{{}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b), \frac{1}{2}(a+b+1) \\ a+\frac{1}{2},\ b+\frac{1}{2}, a+b \end{matrix};\frac{x^2}{4}\right] - \frac{x}{2(2a+1)}\,{}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+1),\ \frac{1}{2}(a+b+2) \\ a+\frac{3}{2},\ b+\frac{1}{2},\ a+b+1 \end{matrix};\frac{x^2}{4}\right]\right\}$$

and

$$
\begin{aligned}
&{}_1F_1\left[\begin{matrix} a \\ 2a-1 \end{matrix}\,;\,x\right] \times {}_1F_1\left[\begin{matrix} b \\ 2b \end{matrix}\,;\,x\right] \\
&= e^x\left\{{}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b-1),\ \frac{1}{2}(a+b) \\ a-\frac{1}{2},\ b+\frac{1}{2},\ a+b-1 \end{matrix}\,;\,\frac{x^2}{4}\right]\right. \\
&\left.+\frac{x}{2(2a-1)}+\ {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b),\ \frac{1}{2}(a+b+1) \\ a+\frac{1}{2},\ b+\frac{1}{2},\ a+b \end{matrix}\,;\,\frac{x^2}{4}\right]\right\}.
\end{aligned} \tag{16}
$$

In 2010, Rathie and Pogany [14] generalized Kummer's second theorem (7) in the following form:

$$
\begin{aligned}
&e^{-\frac{x}{2}}\ {}_2F_2\left[\begin{matrix} a,\ d+1 \\ 2a+1,\ d \end{matrix}\,;\,x\right] \\
&= {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\,;\,\frac{x^2}{16}\right] + \frac{\left(\frac{a}{d}-\frac{1}{2}\right)x}{(2a+1)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\,;\,\frac{x^2}{16}\right].
\end{aligned} \tag{17}
$$

For $d = 2a$, (17) reduces at once to (7).

In the same year 2010, Kim et al. [4] extended all the classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ out of which the extension of classical Watson's summation theorem (14) is recorded here.

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+1),\ 2c+1,\ d \end{matrix}\,;\,1\right] \\
&= 2^{a+b-2}\frac{\Gamma\left(c+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\ \Gamma(a)\ \Gamma(b)} \\
&\times\left\{\frac{\Gamma\left(\frac{1}{2}a\right)\ \Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} + \left(\frac{2c}{d}-1\right)\ \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+1\right)\ \Gamma\left(c-\frac{1}{2}b+1\right)}\right\}
\end{aligned} \tag{18}
$$

provided $Re(2c-a-b) > -1$.

Clearly for $d = 2c$, (18) reduces to (14).

Very recently Kim et al. [3] established the following very interesting generalization of the Bailey's result (13) viz.

$$
\begin{aligned}
&{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}\,;x\right] \times {}_2F_2\left[\begin{matrix} b,\ d+1 \\ 2b+1,\ d \end{matrix}\,;\,-x\right] \\
&= {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b),\ \frac{1}{2}(a+b+1) \\ a+\frac{1}{2},\ b+\frac{1}{2},\ a+b \end{matrix}\,;\,\frac{x^2}{4}\right] \\
&+\frac{\left(1-\frac{2b}{d}\right)x}{2(2b+1)}\ {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+1),\ \frac{1}{2}(a+b+2) \\ a+\frac{1}{2},\ b+\frac{3}{2},\ a+b+1 \end{matrix}\,;\frac{x^2}{4}\right]
\end{aligned} \tag{19}
$$

by employing extension of classical Watson's summation theorem (18).

Clearly, for $d = 2b$, the result (19) with the aid of the result (3) reduces to (13).

The aim of this paper is to provide a natural generalization of the well-known and useful Bailey's identity (19) involving product of generalized hypergeometric series. In order to obtain the generalization in the most general form, we have to construct two general formulas. For this, we shall require the following results in our present investigations.

Generalizations of Kummer's second theorem [7]

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a \\ 2a+i \end{matrix}\,;\,x\right]$$
$$= \Gamma\left(a-\frac{1}{2}\right)\left\{\sum_{k=0}^{i}\frac{(-i)_k(2a-1)_k}{(2a+i)_k\,\Gamma\left(a+k-\frac{1}{2}\right)\,k!}\left(\frac{x}{4}\right)^k \times\,{}_0F_1\left[\begin{matrix} - \\ a+k+\frac{1}{2} \end{matrix}\,;\,\frac{x^2}{16}\right]\right\} \tag{20}$$

for $i = 0, 1, 2\ldots$
and

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a \\ 2a-i \end{matrix}\,;\,x\right]$$
$$= \Gamma\left(a-i-\frac{1}{2}\right)\left\{\sum_{k=0}^{i}\frac{(-1)^k\,(-i)_k\,(2a-2i-1)_k}{(2a-i)_k\,\Gamma\left(a+k-i-\frac{1}{2}\right)\,k!}\left(\frac{x}{4}\right)^k \times\,{}_0F_1\left[\begin{matrix} - \\ a+k-i+\frac{1}{2} \end{matrix}\,;\,\frac{x^2}{16}\right]\right\} \tag{21}$$

for $i = 0, 1, 2\ldots$.

Bailey's result [1]

$$ {}_0F_1\left[\begin{matrix} - \\ a \end{matrix}\,;x\right] \times\,{}_0F_1\left[\begin{matrix} - \\ b \end{matrix}\,;\,x\right] = {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b-1),\ \frac{1}{2}(a+b) \\ a,\ b,\ a+b-1 \end{matrix}\,;\,4x\right]. \tag{22}$$

2. Main results

The natural generalizations of the Bailey's identity (13) to be established in this paper are given in the following theorem.

Theorem 2.1. *For $i = 0, 1, 2\ldots$, the following results hold true.*

$$ {}_1F_1\left[\begin{matrix} a+i \\ 2a+i \end{matrix}\,;x\right] \times\,{}_2F_2\left[\begin{matrix} b,\ d+1 \\ 2b+1,\ d \end{matrix}\,;\,-x\right] \tag{23}$$
$$= \Gamma\left(a-\frac{1}{2}\right)\sum_{k=0}^{i}\frac{(-i)_k(2a-1)_k}{(2a+i)_k\,\Gamma\left(a+k-\frac{1}{2}\right)\,k!}\left(-\frac{x}{4}\right)^k$$
$$\times\left\{\,{}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+k),\ \frac{1}{2}(a+b+k+1) \\ a+k+\frac{1}{2},\ b+\frac{1}{2},\ a+b+k \end{matrix}\,;\,\frac{x^2}{4}\right]\right.$$

$$- \frac{\left(\frac{b}{d}-\frac{1}{2}\right)x}{(2b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+k+1),\ \frac{1}{2}(a+b+k+2)\\ a+k+\frac{1}{2},\ b+\frac{3}{2},\ a+b+k+1\end{matrix};\ \frac{x^2}{4}\right]\Bigg\}$$

and

$$(24)\qquad {}_1F_1\left[\begin{matrix}a-i\\ 2a-i\end{matrix};x\right]\times\ {}_2F_2\left[\begin{matrix}b,\ d+1\\ 2b+1,\ d\end{matrix};\ -x\right]$$

$$= \Gamma\left(a-i-\frac{1}{2}\right)\sum_{k=0}^{i}\frac{(-i)_k\ (2a-2i-1)_k}{(2a-i)_k\ \Gamma\left(a+k-i-\frac{1}{2}\right)\ k!}\ \left(\frac{x}{4}\right)^k$$

$$\times\Bigg\{{}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+k-i),\ \frac{1}{2}(a+b+k-i+1)\\ a+k-i+\frac{1}{2},\ b+\frac{1}{2},\ a+b+k-i\end{matrix};\ \frac{x^2}{4}\right]$$

$$- \frac{\left(\frac{b}{d}-\frac{1}{2}\right)x}{(2b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+k-i+1),\ \frac{1}{2}(a+b+k-i+2)\\ a+k-i+\frac{1}{2},\ b+\frac{3}{2},\ a+b+k-i+1\end{matrix};\ \frac{x^2}{4}\right]\Bigg\}.$$

Proof. In order to establish the first result (23) of the theorem, it is sufficient to show

$$(25)\qquad {}_1F_1\left[\begin{matrix}a+i\\ 2a+i\end{matrix};-x\right]\times\ {}_2F_2\left[\begin{matrix}b,\ d+1\\ 2b+1,\ d\end{matrix};\ x\right]$$

$$= \Gamma\left(a-\frac{1}{2}\right)\sum_{k=0}^{i}\frac{(-i)_k\ (2a-1)_k}{(2a+i)_k\ \Gamma\left(a+k-\frac{1}{2}\right)\ k!}\ \left(-\frac{x}{4}\right)^k$$

$$\times\Bigg\{{}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+k),\ \frac{1}{2}(a+b+k+1)\\ a+k+\frac{1}{2},\ b+\frac{1}{2},\ a+b+k\end{matrix};\ \frac{x^2}{4}\right]$$

$$+ \frac{\left(\frac{b}{d}-\frac{1}{2}\right)x}{(2b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+k+1),\ \frac{1}{2}(a+b+k+2)\\ a+k+\frac{1}{2},\ b+\frac{3}{2},\ a+b+k+1\end{matrix};\frac{x^2}{4}\right]\Bigg\}.$$

Now, denoting the left hand side of (25) by S, we have

$$S = {}_1F_1\left[\begin{matrix}a+i\\ 2a+i\end{matrix};\ -x\right]\times\ {}_2F_2\left[\begin{matrix}b,\ d+1\\ 2b+1,\ d\end{matrix};\ x\right].$$

Using (3) in the first ${}_1F_1$, we get

$$S = e^{-x}\ {}_1F_1\left[\begin{matrix}a\\ 2a+i\end{matrix};x\right]\times\ {}_2F_2\left[\begin{matrix}b,\ d+1\\ 2b+1,\ d\end{matrix};x\right]$$

which can be written as

$$S = e^{-\frac{x}{2}}\ {}_1F_1\left[\begin{matrix}a\\ 2a+i\end{matrix};\ x\right]\times\ e^{-\frac{x}{2}}\ {}_2F_2\left[\begin{matrix}b,\ d+1\\ 2b+1,\ d\end{matrix};\ x\right].$$

Now, using the known results (19) and (20) and after some simplification, we have

$$S=\Gamma\left(a-\frac{1}{2}\right)\sum_{k=0}^{i}\frac{(-i)_k\,(2a-1)_k}{(2a+i)_k\,\Gamma\left(a+k-\frac{1}{2}\right)\,k!}\left(\frac{x}{4}\right)^k$$
$$\times\left\{{}_0F_1\left[\begin{matrix}-\\ a+k+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]\right.$$
$$\left.+\frac{\left(\frac{b}{d}-\frac{1}{2}\right)x}{(2b+1)}\,{}_0F_1\left[\begin{matrix}-\\ a+k+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\right\}.$$

Finally, applying the result (22) we easily arrive at the right hand side of (23). This completes the proof of the first result (23) of theorem. □

In exactly the same manner, the second result (24) of theorem can be established.

3. Special cases

(i) In (23) or (24), if we take $i=0$, we get a result (19) recently obtained by Kim et al. [3] by following a different method.

(ii) In (23), if we take $i=1$, we get the following result:

$$(26)\qquad {}_1F_1\left[\begin{matrix}a+1\\ 2a+1\end{matrix};x\right]\times{}_2F_2\left[\begin{matrix}b,\ d+1\\ 2b+1,\ d\end{matrix};-x\right]$$
$$={}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b),\ \frac{1}{2}(a+b+1)\\ a+\frac{1}{2},\ b+\frac{1}{2},\ a+b\end{matrix};\frac{x^2}{4}\right]$$
$$+\frac{x}{2(2a+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+1),\ \frac{1}{2}(a+b+2)\\ a+\frac{3}{2},b+\frac{1}{2},\ a+b+1\end{matrix};\frac{x^2}{4}\right]$$
$$-\frac{\left(\frac{b}{d}-\frac{1}{2}\right)x}{(2b+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+1),\ \frac{1}{2}(a+b+2)\\ a+\frac{1}{2},\ b+\frac{3}{2},\ a+b+1\end{matrix};\frac{x^2}{4}\right]$$
$$-\frac{\left(\frac{b}{d}-\frac{1}{2}\right)x^2}{2(2a+1)(2b+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+2),\ \frac{1}{2}(a+b+3)\\ a+\frac{3}{2},\ b+\frac{3}{2},\ a+b+2\end{matrix};\frac{x^2}{4}\right].$$

(iii) In (24), if we take $i=1$, we get the following result:

$$(27)\qquad {}_1F_1\left[\begin{matrix}a-1\\ 2a-1\end{matrix};x\right]\times{}_2F_2\left[\begin{matrix}b,\ d+1\\ 2b+1,\ d\end{matrix};-x\right]$$
$$={}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b-1),\frac{1}{2}(a+b)\\ a-\frac{1}{2},\ b+\frac{1}{2},\ a+b\end{matrix};\frac{x^2}{4}\right]$$
$$-\frac{x}{2(2a-1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b),\ \frac{1}{2}(a+b+1)\\ a+\frac{1}{2},\ b+\frac{1}{2},\ a+b\end{matrix};\frac{x^2}{4}\right]$$
$$-\frac{\left(\frac{b}{d}-\frac{1}{2}\right)x}{(2b+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b),\ \frac{1}{2}(a+b+1)\\ a-\frac{1}{2},b+\frac{3}{2},\ a+b\end{matrix};\frac{x^2}{4}\right]$$

$$+\frac{\left(\frac{b}{d}-\frac{1}{2}\right)\ x^2}{2(2a-1)(2b+1)}\ {}_2F_3\left[\begin{array}{c}\frac{1}{2}(a+b+1),\ \frac{1}{2}(a+b+2)\\ a+\frac{1}{2},\ b+\frac{3}{2},\ a+b+1\end{array};\ \frac{x^2}{4}\right].$$

The results (26) and (27) have recently been obtained by Kim and Rohira [5].

Similarly, other results can be obtained.

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