

ON THE CONFORMAL TRIHARMONIC MAPS

SEDDIK OUAKKAS AND YASMINA REGUIG

ABSTRACT. In this paper, we give the necessary and sufficient condition for the conformal mapping $\phi : (\mathbb{R}^n, g_0) \rightarrow (N^n, h)$ ($n \geq 3$) to be triharmonic where we prove that the gradient of its dilation is a solution of a fourth-order elliptic partial differential equation. We construct some examples of triharmonic maps which are not biharmonic and we calculate the trace of the stress-energy tensor associated with the triharmonic maps.

1. Introduction

Let (M^m, g) and (N^n, h) be smooth Riemannian manifolds with M compact. A smooth map $\phi : (M^m, g) \rightarrow (N^n, h)$ is called harmonic if it is a critical point of the energy functional:

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g,$$

where $|d\phi|$ is the Hilbert-Schmidt norm of ϕ . Equivalently, ϕ is harmonic if it satisfies the Euler-Lagrange equations:

$$\tau(\phi) = \text{Tr}_g \nabla d\phi = 0,$$

where $\tau(\phi)$ is the tension field of ϕ . A map $\phi : (M^m, g) \rightarrow (N^n, h)$ is called biharmonic if it is a critical point of the bi-energy functional:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g.$$

The corresponding Euler-Lagrange equations are the fourth-order elliptic system:

$$\tau_2(\phi) = -\text{Tr}_g (\nabla^\phi)^2 \tau(\phi) - \text{Tr}_g R^N(\tau(\phi), d\phi)d\phi = 0,$$

where R^N is the Riemannian curvature on N and ∇^ϕ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1 \leq i \leq m}$ is a local orthonormal frame field

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on M , then

$$Tr_g (\nabla^\phi)^2 \tau (\phi) = \left(\nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i}^M e_i}^\phi \right) \tau (\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2 (\phi)$, the bi-tension field of the map ϕ . A generalization of harmonic and biharmonic maps are polyharmonic maps. Such maps are critical points of the following functional

$$(1.1) \quad E_{2s} (\phi) = \int_M |\Delta^{s-1} \tau (\phi)|^2 dv_g,$$

and

$$(1.2) \quad E_{2s+1} (\phi) = \int_M |\nabla \Delta^{s-1} \tau (\phi)|^2 dv_g,$$

where $s = 1, 2, \dots$ is a natural number. The first variation of E_k is developed in [9]. The associated Euler-Lagrange equation is given by

(1) If $k = 2s, s = 1, 2, \dots$, the critical points of (1.1) are given by

$$(1.3) \quad \begin{aligned} \tau_{2s} (\phi) &= \Delta^{2s-1} \tau (\phi) - R^N (\Delta^{2s-2} \tau (\phi), d\phi (e_i)) d\phi (e_i) \\ &+ \sum_{l=1}^{s-1} R^N (\Delta^{s-l-1} \tau (\phi), \nabla_{e_i} \Delta^{s+l-2} \tau (\phi)) d\phi (e_i) \\ &- \sum_{l=1}^{s-1} R^N (\nabla_{e_i} \Delta^{s-l-1} \tau (\phi), \Delta^{s+l-2} \tau (\phi)) d\phi (e_i) = 0. \end{aligned}$$

(2) If $k = 2s + 1, s = 0, 1, 2, \dots$, the critical points of (1.2) are given by

$$(1.4) \quad \begin{aligned} \tau_{2s+1} (\phi) &= \Delta^{2s} \tau (\phi) - R^N (\Delta^{2s-1} \tau (\phi), d\phi (e_i)) d\phi (e_i) \\ &+ \sum_{l=1}^{s-1} R^N (\nabla_{e_i} \Delta^{s+l-1} \tau (\phi), \Delta^{s-l-1} \tau (\phi)) d\phi (e_i) \\ &- \sum_{l=1}^{s-1} R^N (\Delta^{s+l-1} \tau (\phi), \nabla_{e_i} \Delta^{s-l-1} \tau (\phi)) d\phi (e_i) \\ &- R^N (\nabla_{e_i} \Delta^{s-1} \tau (\phi), \Delta^{s-1} \tau (\phi)) d\phi (e_i) = 0. \end{aligned}$$

Where $(e_i)_{1 \leq i \leq m}$ is an orthonormal frame on M ,

$$\Delta \tau (\phi) = -Tr_g (\nabla^\phi)^2 \tau (\phi) = - \left(\nabla_{e_i}^\phi \nabla_{e_i}^\phi \tau (\phi) - \nabla_{\nabla_{e_i}^M e_i}^\phi \tau (\phi) \right)$$

and

$$\Delta^k \tau (\phi) = \Delta (\Delta^{k-1} \tau (\phi)), \quad \Delta^{-1} = 0.$$

There are several works concerning the construction of polyhamonic maps, see for example [4], [5], [8], [10] and [11]. Following Jiang [6], we define the stress-energy tensor associated with the functional E_k by taking variations of the domain metric. More precisely let $\phi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map and, for any $X, Y \in \Gamma (TM)$, we have

- (1) The stress-energy tensor for polyharmonic maps of even order is defined by

$$\begin{aligned}
 & S_{2s}(\phi)(X, Y) \\
 = & \left(\frac{1}{2} |\Delta^{s-1}\tau(\phi)|^2 - h(\Delta^{2s-2}\tau(\phi), \tau(\phi)) \right) g(X, Y) \\
 & - (Tr_g h(\nabla\Delta^{2s-2}\tau(\phi), d\phi)) g(X, Y) \\
 & + \left(\sum_{l=1}^{s-1} Tr_g h(\nabla\Delta^{s-l-1}\tau(\phi), \nabla\Delta^{s+l-2}\tau(\phi)) \right) g(X, Y) \\
 (1.5) \quad & - \left(\sum_{l=1}^{s-1} h(\Delta^{s-l}\tau(\phi), \Delta^{s+l-2}\tau(\phi)) \right) g(X, Y) \\
 & - \sum_{l=1}^{s-1} h(\nabla_X\Delta^{s-l-1}\tau(\phi), \nabla_Y\Delta^{s+l-2}\tau(\phi)) \\
 & - \sum_{l=1}^{s-1} h(\nabla_Y\Delta^{s-l-1}\tau(\phi), \nabla_X\Delta^{s+l-2}\tau(\phi)) \\
 & + h(\nabla_X\Delta^{2s-2}\tau(\phi), d\phi(Y)) + h(\nabla_Y\Delta^{2s-2}\tau(\phi), d\phi(X)).
 \end{aligned}$$

- (2) The stress-energy tensor for polyharmonic maps of odd order is defined by

$$\begin{aligned}
 & S_{2s+1}(\phi)(X, Y) \\
 = & \left(\frac{1}{2} |\nabla\Delta^{s-1}\tau(\phi)|^2 - h(\Delta^{2s-1}\tau(\phi), \tau(\phi)) \right) g(X, Y) \\
 & - (Tr_g h(\nabla\Delta^{2s-1}\tau(\phi), d\phi)) g(X, Y) \\
 & + \left(\sum_{l=1}^{s-1} Tr_g h(\nabla\Delta^{s-l-1}\tau(\phi), \nabla\Delta^{s+l-1}\tau(\phi)) \right) g(X, Y) \\
 (1.6) \quad & - \left(\sum_{l=1}^{s-1} h(\Delta^{s-l}\tau(\phi), \Delta^{s+l-1}\tau(\phi)) \right) g(X, Y) \\
 & - \sum_{l=1}^{s-1} h(\nabla_X\Delta^{s-l-1}\tau(\phi), \nabla_Y\Delta^{s+l-1}\tau(\phi)) \\
 & - \sum_{l=1}^{s-1} h(\nabla_Y\Delta^{s-l-1}\tau(\phi), \nabla_X\Delta^{s+l-1}\tau(\phi)) \\
 & + h(\nabla_X\Delta^{2s-1}\tau(\phi), d\phi(Y)) + h(\nabla_Y\Delta^{2s-1}\tau(\phi), d\phi(X)) \\
 & - h(\nabla_X\Delta^{s-1}\tau(\phi), \nabla_Y\Delta^{s-1}\tau(\phi)).
 \end{aligned}$$

Note that in both cases, the stress-energy tensor satisfies the following conservation law (see [3]):

$$\operatorname{div} S_k(\phi) = -h(\tau_k(\phi), d\phi), \quad k = 1, 2, 3, \dots$$

The harmonicity and the biharmonicity of the conformal map between equidimensional manifolds have been studied by several authors, see [1], [7] and [12] for more details. Recall that a mapping $\phi : (M^n, g) \rightarrow (N^n, h)$ is called conformal if there exists a smooth function $\lambda : M \rightarrow \mathbb{R}_+^*$ such that

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y)$$

for every $X, Y \in \Gamma(TM)$. The function λ is called the dilation for the conformal map ϕ . The tension field for a conformal map $\phi : (M^n, g) \rightarrow (N^n, h)$ of dilation λ is given by (see [2]):

$$\tau(\phi) = (2 - n) d\phi(\operatorname{grad} \ln \lambda).$$

Recall that in dimension 2 any conformal map is harmonic, whereas in other dimensions, it is harmonic if and only if its dilation is constant. Note that by a result in [2], any such map can have no critical points and so is a local conformal diffeomorphism. In [12], the authors calculated the bi-tension field and the stress bi-energy tensor for a conformal map and the following results are obtained:

Theorem 1.1 ([12]). *Let $\phi : (M^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . The bi-tension field and the stress bi-energy tensor of ϕ are given by*

$$\tau_2(\phi) = (n - 2) d\phi(H),$$

where

$$\begin{aligned} H = & \operatorname{grad} \Delta \ln \lambda - \frac{n-6}{2} \operatorname{grad} (|\operatorname{grad} \ln \lambda|^2) - 2(\Delta \ln \lambda) \operatorname{grad} \ln \lambda \\ & - (n-2) |\operatorname{grad} \ln \lambda|^2 \operatorname{grad} \ln \lambda + 2\operatorname{Ricci}^M(\operatorname{grad} \ln \lambda) \end{aligned}$$

and

$$S_2(\phi) = (2 - n) \lambda^2 \left\{ \left(\Delta \ln \lambda + \frac{n-2}{2} |\operatorname{grad} \ln \lambda|^2 \right) g - 2\nabla d \ln \lambda \right\}.$$

The biharmonicity of the conformal map is given by the following result (see [12]).

Theorem 1.2. *Let $\phi : (M^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ , ϕ is biharmonic if and only if*

$$\begin{aligned} & \operatorname{grad} \Delta \ln \lambda - \frac{n-6}{2} \operatorname{grad} (|\operatorname{grad} \ln \lambda|^2) - 2(\Delta \ln \lambda) \operatorname{grad} \ln \lambda \\ & - (n-2) |\operatorname{grad} \ln \lambda|^2 \operatorname{grad} \ln \lambda + 2\operatorname{Ricci}^M(\operatorname{grad} \ln \lambda) = 0. \end{aligned}$$

Consequently, the biharmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) where the dilation λ is radial ($\ln \lambda = \alpha(r)$, $r = |x|$ and $\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$) is equivalent to an ordinary differential equation. We have (see [12]):

Corollary 1.3. *Let $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) to be a conformal map of dilation λ when we suppose that $\ln \lambda$ is radial ($\ln \lambda = \alpha(r), r = |x|$). Then ϕ is biharmonic if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation:*

$$(1.7) \quad \beta'' - (n - 4)\beta\beta' + \frac{n - 1}{r}\beta' - \frac{n - 1}{r^2}\beta - \frac{2(n - 1)}{r}\beta^2 - (n - 2)\beta^3 = 0.$$

Remark 1.4. If we consider particular solutions of the form $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$). By (1.7), a map $\phi : (\mathbb{R}^n \setminus \{0\}, g) \rightarrow (N^n, h)$ ($n \geq 3$) is biharmonic if and only if a is a solution of the algebraic equation

$$(n - 2)a^2 + (n + 2)a + 2(n - 2) = 0.$$

This equation has real solutions if and only if $n \in \{3, 4\}$.

- (1) If $n = 3$, we find $a = \frac{-5 + \sqrt{17}}{2}$ or $a = \frac{-5 - \sqrt{17}}{2}$. Then any conformal map $\phi : (\mathbb{R}^3 \setminus \{0\}, g) \rightarrow (N^3, h)$ of dilation $\lambda = \frac{C}{r^{\frac{5 - \sqrt{17}}{2}}}$ or $\lambda = \frac{C}{r^{\frac{5 + \sqrt{17}}{2}}}$ ($C > 0$) is biharmonic non-harmonic.
- (2) If $n = 4$, the solutions are $a = -2$ and $a = -1$. The conformal map $\phi : (\mathbb{R}^4 \setminus \{0\}, g) \rightarrow (N^4, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ ($C > 0$) is biharmonic non-harmonic.

In this paper, we are interested in triharmonic maps, that are the solutions of the following equation:

$$(1.8) \quad \tau_3(\phi) = \Delta^2\tau(\phi) - Tr_g R^N(\Delta\tau(\phi), d\phi)d\phi - Tr_g R^N(\nabla\tau(\phi), \tau(\phi))d\phi = 0.$$

In this case, the triharmonic stress-energy tensor is given by:

$$(1.9) \quad \begin{aligned} & S_3(\phi)(X, Y) \\ &= \left(\frac{1}{2}|\tau(\phi)|^2 - h(\Delta\tau(\phi), \tau(\phi)) \right) g(X, Y) \\ &\quad - (Tr_g h(\nabla\Delta\tau(\phi), d\phi))g(X, Y) + h(\nabla_X\Delta\tau(\phi), d\phi(Y)) \\ &\quad + h(\nabla_Y\Delta\tau(\phi), d\phi(X)) - h(\nabla_X\tau(\phi), \nabla_Y\tau(\phi)). \end{aligned}$$

As a first result, we calculate the expression of $\tau_3(\phi)$ where $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) is a conformal map of dilation λ . We show that a conformal mapping ϕ is triharmonic if and only if the gradient of its dilation, viewed as a section of the tangent bundle TM , satisfies a fourth-order elliptic partial differential equation and we give some basic examples of such mappings. We conclude our paper by calculating the trace of the stress-energy tensor for any conformal map $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) of dilation λ . As a consequence, we present some particular cases.

2. The main results

According to the properties of conformal maps, we have the following results:

Proposition 2.1 ([2]). *Let $\phi : (M^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . For any $X, Y \in \Gamma(TM)$, we have*

$$(2.1) \quad \begin{aligned} \nabla_Y d\phi(X) &= X(\ln \lambda) d\phi(Y) + Y(\ln \lambda) d\phi(X) \\ &\quad - g(X, Y) d\phi(\text{grad } \ln \lambda) + d\phi(\nabla_Y X). \end{aligned}$$

In particular, for any $f \in C^\infty(M)$, we obtain

$$(2.2) \quad \begin{aligned} \nabla_Y d\phi(\text{grad } f) &= d\ln \lambda(\text{grad } f) d\phi(Y) + Y(\ln \lambda) d\phi(\text{grad } f) \\ &\quad - Y(f) d\phi(\text{grad } \ln \lambda) + d\phi(\nabla_Y \text{grad } f), \end{aligned}$$

and if $f = \ln \lambda$, equation (2.2) becomes

$$(2.3) \quad \nabla_Y d\phi(\text{grad } \ln \lambda) = |\text{grad } \ln \lambda|^2 d\phi(Y) + d\phi(\nabla_Y \text{grad } \ln \lambda).$$

Equations (2.2) and (2.3) lead us to the following remark:

Remark 2.2. Let $\phi : (M^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . For any $X, Y \in \Gamma(TM)$ and for any $f \in C^\infty(M)$, we have

$$(2.4) \quad \begin{aligned} h(\nabla_X d\phi(\text{grad } f), d\phi(Y)) &= \lambda^2 X(\ln \lambda) Y(f) - \lambda^2 X(f) Y(\ln \lambda) \\ &\quad + \lambda^2 d\ln \lambda(\text{grad } f) g(X, Y) \\ &\quad + \lambda^2 g(\nabla_X \text{grad } f, Y) \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} h(\nabla_X d\phi(\text{grad } \ln \lambda), d\phi(Y)) &= h(\nabla_Y d\phi(\text{grad } \ln \lambda), d\phi(X)) \\ &= \lambda^2 |\text{grad } \ln \lambda|^2 g(X, Y) \\ &\quad + \lambda^2 g(\nabla_X \text{grad } \ln \lambda, Y). \end{aligned}$$

To study the triharmonicity of a conformal map, we need the following theorem:

Theorem 2.3. *Let $\phi : (M^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . Then for any $X \in \Gamma(TM)$ and for any function $f \in C^\infty(M)$, we have*

$$(2.6) \quad \begin{aligned} \text{Tr}_g(\nabla^\phi)^2 d\phi(X) &= d\phi(\text{Tr}_g \nabla^2 X) + (2-n) X(\ln \lambda) d\phi(\text{grad } \ln \lambda) \\ &\quad + (\Delta \ln \lambda) d\phi(X) + 2d\phi(\text{grad}(X(\ln \lambda))) \\ &\quad - 2(\text{div} X) d\phi(\text{grad } \ln \lambda) + 2d\phi(\nabla_{\text{grad } \ln \lambda} X) \\ &\quad - 2d\phi(\nabla_X \text{grad } \ln \lambda), \end{aligned}$$

$$\begin{aligned}
 (2.7) \quad Tr_g (\nabla^\phi)^2 f d\phi(X) &= f d\phi (Tr_g \nabla^2 X) + (2 - n) f X (\ln \lambda) d\phi (\text{grad } \ln \lambda) \\
 &\quad + f (\Delta \ln \lambda) d\phi (X) + 2f d\phi (\text{grad } (X (\ln \lambda))) \\
 &\quad + (\Delta f) d\phi (X) - 2f (\text{div} X) d\phi (\text{grad } \ln \lambda) \\
 &\quad + 2f d\phi (\nabla_{\text{grad } \ln \lambda} X) - 2f d\phi (\nabla_X \text{grad } \ln \lambda) \\
 &\quad + 2X (\ln \lambda) d\phi (\text{grad } f) + 2d \ln \lambda (\text{grad } f) d\phi (X) \\
 &\quad - 2X (f) d\phi (\text{grad } \ln \lambda) + 2d\phi (\nabla_{\text{grad } f} X)
 \end{aligned}$$

and

$$\begin{aligned}
 (2.8) \quad R^N (d\phi(X), d\phi(Y)) d\phi(Z) &= (n - 2) X (\ln \lambda) Y (\ln \lambda) d\phi(Z) \\
 &\quad - (n - 2) |\text{grad } \ln \lambda|^2 g(X, Y) d\phi(Z) \\
 &\quad - (n - 2) g(\nabla_X \text{grad } \ln \lambda, Y) d\phi(Z) \\
 &\quad - (\Delta \ln \lambda) g(X, Y) d\phi(Z) \\
 &\quad + d\phi(R^M(X, Y)Z).
 \end{aligned}$$

Proof. By definition, for any $X \in \Gamma(TM)$, we have

$$(2.9) \quad Tr_g (\nabla^\phi)^2 d\phi(X) = \nabla_{e_i} \nabla_{e_i} d\phi(X) - \nabla_{\nabla_{e_i} e_i} d\phi(X).$$

By (2.1), we obtain

$$\begin{aligned}
 \nabla_{e_i} d\phi(X) &= X (\ln \lambda) d\phi(e_i) + e_i (\ln \lambda) d\phi(X) \\
 &\quad - g(X, e_i) d\phi(\text{grad } \ln \lambda) + d\phi(\nabla_{e_i} X),
 \end{aligned}$$

then

$$\begin{aligned}
 \nabla_{e_i} \nabla_{e_i} d\phi(X) &= \nabla_{e_i} X (\ln \lambda) d\phi(e_i) + \nabla_{e_i} e_i (\ln \lambda) d\phi(X) \\
 &\quad - \nabla_{e_i} g(X, e_i) d\phi(\text{grad } \ln \lambda) + \nabla_{e_i} d\phi(\nabla_{e_i} X).
 \end{aligned}$$

The explicit form of the last equation is reported below term by term. We have

$$\begin{aligned}
 \nabla_{e_i} X (\ln \lambda) d\phi(e_i) &= X (\ln \lambda) \nabla_{e_i} d\phi(e_i) + e_i (X (\ln \lambda)) d\phi(e_i) \\
 &= X (\ln \lambda) \nabla_{e_i} d\phi(e_i) + e_i (g(X, \text{grad } \ln \lambda)) d\phi(e_i) \\
 &= X (\ln \lambda) \nabla_{e_i} d\phi(e_i) + g(\nabla_{e_i} \text{grad } \ln \lambda, X) d\phi(e_i) \\
 &\quad + g(\text{grad } \ln \lambda, \nabla_{e_i} X) d\phi(e_i) \\
 &= X (\ln \lambda) \nabla_{e_i} d\phi(e_i) + d\phi(\nabla_X \text{grad } \ln \lambda) \\
 &\quad + g(\text{grad } \ln \lambda, \nabla_{e_i} X) d\phi(e_i) \\
 &= X (\ln \lambda) \nabla_{e_i} d\phi(e_i) + d\phi(\text{grad } (X (\ln \lambda))),
 \end{aligned}$$

$$\begin{aligned}
 \nabla_{e_i} e_i (\ln \lambda) d\phi(X) &= e_i (\ln \lambda) \nabla_{e_i} d\phi(X) + e_i (e_i (\ln \lambda)) d\phi(X) \\
 &= \nabla_{\text{grad } \ln \lambda} d\phi(X) + e_i (e_i (\ln \lambda)) d\phi(X) \\
 &= |\text{grad } \ln \lambda|^2 d\phi(X) + d\phi(\nabla_{\text{grad } \ln \lambda} X) \\
 &\quad + e_i (e_i (\ln \lambda)) d\phi(X),
 \end{aligned}$$

$$\begin{aligned}
\nabla_{e_i} g(X, e_i) d\phi(\text{grad } \ln \lambda) &= \nabla_X d\phi(\text{grad } \ln \lambda) + e_i(g(X, e_i)) d\phi(\text{grad } \ln \lambda) \\
&= |\text{grad } \ln \lambda|^2 d\phi(X) + d\phi(\nabla_X \text{grad } \ln \lambda) \\
&\quad + g(\nabla_{e_i} X, e_i) d\phi(\text{grad } \ln \lambda) \\
&\quad + g(X, \nabla_{e_i} e_i) d\phi(\text{grad } \ln \lambda)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{e_i} d\phi(\nabla_{e_i} X) &= g(\text{grad } \ln \lambda, \nabla_{e_i} X) d\phi(e_i) + d\phi(\nabla_{\text{grad } \ln \lambda} X) \\
&\quad - g(\nabla_{e_i} X, e_i) d\phi(\text{grad } \ln \lambda) + d\phi(\nabla_{e_i} \nabla_{e_i} X) \\
&= d\phi(\text{grad } (X(\ln \lambda))) - d\phi(\nabla_X \text{grad } \ln \lambda) + d\phi(\nabla_{\text{grad } \ln \lambda} X) \\
&\quad - g(\nabla_{e_i} X, e_i) d\phi(\text{grad } \ln \lambda) + d\phi(\nabla_{e_i} \nabla_{e_i} X),
\end{aligned}$$

which gives us

$$\begin{aligned}
&\nabla_{e_i} \nabla_{e_i} d\phi(X) \\
(2.10) \quad &= d\phi(\nabla_{e_i} \nabla_{e_i} X) + X(\ln \lambda) \nabla_{e_i}^\phi d\phi(e_i) \\
&\quad + 2d\phi(\text{grad } (X(\ln \lambda))) + 2d\phi(\nabla_{\text{grad } \ln \lambda} X) \\
&\quad - 2d\phi(\nabla_X \text{grad } \ln \lambda) + e_i(e_i(\ln \lambda)) d\phi(X) \\
&\quad - 2(\text{div } X) d\phi(\text{grad } \ln \lambda) - g(X, \nabla_{e_i} e_i) d\phi(\text{grad } \ln \lambda).
\end{aligned}$$

Finally, it is clear that

$$(2.11) \quad \nabla_{\nabla_{e_i} e_i} d\phi(X) = X(\ln \lambda) d\phi(\nabla_{e_i} e_i) + (\nabla_{e_i} e_i)(\ln \lambda) d\phi(X) \\
- g(X, \nabla_{e_i} e_i) d\phi(\text{grad } \ln \lambda) + d\phi(\nabla_{\nabla_{e_i} e_i} X).$$

By replacing (2.10) and (2.11) in (2.9), we find that

$$\begin{aligned}
Tr_g(\nabla^\phi)^2 d\phi(X) &= d\phi(Tr_g \nabla^2 X) + (2-n) X(\ln \lambda) d\phi(\text{grad } \ln \lambda) \\
&\quad + (\Delta \ln \lambda) d\phi(X) + 2d\phi(\text{grad } (X(\ln \lambda))) \\
&\quad - 2(\text{div } X) d\phi(\text{grad } \ln \lambda) + 2d\phi(\nabla_{\text{grad } \ln \lambda} X) \\
&\quad - 2d\phi(\nabla_X \text{grad } \ln \lambda).
\end{aligned}$$

For the term $Tr_g(\nabla^\phi)^2 f d\phi(X)$, a simple calculation gives us

$$Tr_g(\nabla^\phi)^2 f d\phi(X) = f Tr_g(\nabla^\phi)^2 d\phi(X) + 2\nabla_{\text{grad } f}^\phi d\phi(X) + (\Delta f) d\phi(X).$$

Using equation (2.6) and the fact that

$$\begin{aligned}
\nabla_{\text{grad } f}^\phi d\phi(X) &= X(\ln \lambda) d\phi(\text{grad } f) + d\ln \lambda(\text{grad } f) d\phi(X) \\
&\quad - X(f) d\phi(\text{grad } \ln \lambda) + d\phi(\nabla_{\text{grad } f} X),
\end{aligned}$$

we conclude that

$$\begin{aligned} Tr_g (\nabla^\phi)^2 fd\phi(X) &= fd\phi(Tr_g \nabla^2 X) + (2-n) fX(\ln \lambda) d\phi(\text{grad } \ln \lambda) \\ &\quad + f(\Delta \ln \lambda) d\phi(X) + 2fd\phi(\text{grad } (X(\ln \lambda))) \\ &\quad + (\Delta f) d\phi(X) - 2f(\text{div} X) d\phi(\text{grad } \ln \lambda) \\ &\quad + 2fd\phi(\nabla_{\text{grad } \ln \lambda} X) - 2fd\phi(\nabla_X \text{grad } \ln \lambda) \\ &\quad + 2X(\ln \lambda) d\phi(\text{grad } f) + 2d \ln \lambda (\text{grad } f) d\phi(X) \\ &\quad - 2X(f) d\phi(\text{grad } \ln \lambda) + 2d\phi(\nabla_{\text{grad } f} X). \end{aligned}$$

Finally, the proof of equation (2.8) is based on the following result (see [2])

$$\begin{aligned} & Ric^N(d\phi(X), d\phi(Y)) \\ &= (n-2) X(\ln \lambda) Y(\ln \lambda) - (n-2) g(\nabla_X \text{grad } \ln \lambda, Y) \\ &\quad - (\Delta \ln \lambda) g(X, Y) - (n-2) |\text{grad } \ln \lambda|^2 g(X, Y) + Ric^M(X, Y). \quad \square \end{aligned}$$

Theorem 2.4. *Let $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . Then the tri-tension field of ϕ is given by*

$$\tau_3(\phi) = -(n-2) d\phi(H(\lambda)),$$

where

$$\begin{aligned} & H(\lambda) \\ &= \text{grad } (\Delta^2 \ln \lambda) + 2\text{grad } \left(\Delta \left(|\text{grad } \ln \lambda|^2 \right) \right) - (\Delta \ln \lambda) \text{grad } \Delta \ln \lambda \\ &\quad - 2(n-1) |\text{grad } \ln \lambda|^2 \text{grad } \Delta \ln \lambda - 2(\Delta \ln \lambda) \text{grad } \left(|\text{grad } \ln \lambda|^2 \right) \\ &\quad - 6(n-2) |\text{grad } \ln \lambda|^2 \text{grad } \left(|\text{grad } \ln \lambda|^2 \right) - 3(\Delta^2 \ln \lambda) \text{grad } \ln \lambda \\ &\quad + 2(n-2) (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 \text{grad } \ln \lambda + (n-2)^2 |\text{grad } \ln \lambda|^4 \text{grad } \ln \lambda \\ &\quad + 2(\Delta \ln \lambda)^2 \text{grad } \ln \lambda - (n+2) \left(\Delta \left(|\text{grad } \ln \lambda|^2 \right) \right) \text{grad } \ln \lambda \\ &\quad - n \nabla_{\text{grad } \Delta \ln \lambda} \text{grad } \ln \lambda - \frac{(n-2)(n+6)}{2} \nabla_{\text{grad } (|\text{grad } \ln \lambda|^2)} \text{grad } \ln \lambda \\ &\quad + 4 \nabla_{\text{grad } \ln \lambda} \text{grad } \Delta \ln \lambda + 8 \nabla_{\text{grad } \ln \lambda} \text{grad } \left(|\text{grad } \ln \lambda|^2 \right). \end{aligned}$$

Proof. We prove this theorem in three steps. By definition, we have

$$(2.12) \quad \tau_3(\phi) = \Delta^2 \tau(\phi) - Tr_g R^N(\Delta \tau(\phi), d\phi) d\phi - Tr_g R^N(\nabla \tau(\phi), \tau(\phi)) d\phi.$$

Step 1. In the first step, we look at the first term of the equation (2.12). Since

$$\begin{aligned} \Delta \tau(\phi) &= -Tr_g (\nabla^\phi)^2 \tau(\phi) = (n-2) Tr_g (\nabla^\phi)^2 d\phi(\text{grad } \ln \lambda) \\ &= (n-2) d\phi(\text{grad } \Delta \ln \lambda) + 2(n-2) d\phi \left(\text{grad } \left(|\text{grad } \ln \lambda|^2 \right) \right) \\ &\quad - (n-2) (\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda) - (n-2)^2 |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda), \end{aligned}$$

then

$$\begin{aligned}
\Delta^2 \tau(\phi) &= \Delta(\Delta \tau(\phi)) \\
&= (n-2) \Delta(d\phi(\text{grad } \Delta \ln \lambda)) \\
&\quad + 2(n-2) \Delta\left(d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right)\right) \\
&\quad - (n-2) \Delta\left((\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda)\right) \\
&\quad - (n-2)^2 \Delta\left(|\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda)\right).
\end{aligned}$$

For the two terms $\Delta(d\phi(\text{grad } \Delta \ln \lambda))$ and $\Delta\left(d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right)\right)$ and using the equation (2.6), we obtain

$$\begin{aligned}
&\Delta(d\phi(\text{grad } \Delta \ln \lambda)) \\
&= -Tr_g(\nabla^\phi)^2 d\phi(\text{grad } \Delta \ln \lambda) \\
&= -d\phi(\text{grad } \Delta^2 \ln \lambda) - (\Delta \ln \lambda) d\phi(\text{grad } \Delta \ln \lambda) \\
&\quad + (n-2) d \ln \lambda(\text{grad } \Delta \ln \lambda) d\phi(\text{grad } \ln \lambda) \\
&\quad + 2(\Delta^2 \ln \lambda) d\phi(\text{grad } \ln \lambda) - 4d\phi(\nabla_{\text{grad } \ln \lambda} \text{grad } \Delta \ln \lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\Delta\left(d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right)\right) \\
&= -Tr_g(\nabla^\phi)^2 d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\
&= -d\phi\left(\text{grad } \Delta\left(|\text{grad } \ln \lambda|^2\right)\right) - (\Delta \ln \lambda) d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\
&\quad + 2\left(\Delta\left(|\text{grad } \ln \lambda|^2\right)\right) d\phi(\text{grad } \ln \lambda) \\
&\quad - 4d\phi\left(\nabla_{\text{grad } \ln \lambda} \text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\
&\quad + (n-2) d \ln \lambda\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) d\phi(\text{grad } \ln \lambda).
\end{aligned}$$

For the term $\Delta((\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda))$, we have

$$\begin{aligned}
&\Delta((\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda)) \\
&= -Tr_g(\nabla^\phi)^2 (\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda) \\
&= -(\Delta \ln \lambda) Tr_g(\nabla^\phi)^2 d\phi(\text{grad } \ln \lambda) - (\Delta^2 \ln \lambda) d\phi(\text{grad } \ln \lambda) \\
&\quad - 2\nabla_{\text{grad } \Delta \ln \lambda}^\phi d\phi(\text{grad } \ln \lambda) \\
&= -(\Delta \ln \lambda) d\phi(\text{grad } \Delta \ln \lambda) - 2|\text{grad } \ln \lambda|^2 d\phi(\text{grad } \Delta \ln \lambda) \\
&\quad - 2(\Delta \ln \lambda) d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) + (\Delta \ln \lambda)^2 d\phi(\text{grad } \ln \lambda) \\
&\quad - (\Delta^2 \ln \lambda) d\phi(\text{grad } \ln \lambda) - 2d\phi(\nabla_{\text{grad } \Delta \ln \lambda} \text{grad } \ln \lambda) \\
&\quad + (n-2) (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda).
\end{aligned}$$

Finally, for the term $\Delta \left(|\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda) \right)$, a similar calculation yields

$$\begin{aligned}
& \Delta \left(|\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda) \right) \\
&= -Tr_g (\nabla^\phi)^2 |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda) \\
&= -|\text{grad } \ln \lambda|^2 Tr_g (\nabla^\phi)^2 d\phi(\text{grad } \ln \lambda) \\
&\quad - \left(\Delta \left(|\text{grad } \ln \lambda|^2 \right) \right) d\phi(\text{grad } \ln \lambda) \\
&\quad - 2\nabla_{\text{grad } (|\text{grad } \ln \lambda|^2)}^\phi d\phi(\text{grad } \ln \lambda) \\
&= -|\text{grad } \ln \lambda|^2 d\phi(\text{grad } \Delta \ln \lambda) - 4|\text{grad } \ln \lambda|^2 d\phi \left(\text{grad } \left(|\text{grad } \ln \lambda|^2 \right) \right) \\
&\quad + (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda) + (n-2) |\text{grad } \ln \lambda|^4 d\phi(\text{grad } \ln \lambda) \\
&\quad - \left(\Delta \left(|\text{grad } \ln \lambda|^2 \right) \right) d\phi(\text{grad } \ln \lambda) - 2d\phi \left(\nabla_{\text{grad } (|\text{grad } \ln \lambda|^2)} \text{grad } \ln \lambda \right).
\end{aligned}$$

We deduce that the final result of this first step is given by the following equation

$$\begin{aligned}
(2.13) \quad \Delta^2 \tau(\phi) &= -(n-2) d\phi(\text{grad } \Delta^2 \ln \lambda) \\
&\quad + n(n-2) |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \Delta \ln \lambda) \\
&\quad - 2(n-2) d\phi \left(\text{grad } \Delta \left(|\text{grad } \ln \lambda|^2 \right) \right) \\
&\quad - (n-2)^3 |\text{grad } \ln \lambda|^4 d\phi(\text{grad } \ln \lambda) \\
&\quad + 4(n-2)^2 |\text{grad } \ln \lambda|^2 d\phi \left(\text{grad } \left(|\text{grad } \ln \lambda|^2 \right) \right) \\
&\quad + 3(n-2) (\Delta^2 \ln \lambda) d\phi(\text{grad } \ln \lambda) \\
&\quad - (n-2) (\Delta \ln \lambda)^2 d\phi(\text{grad } \ln \lambda) \\
&\quad + (n-2)(n+2) \left(\Delta \left(|\text{grad } \ln \lambda|^2 \right) \right) d\phi(\text{grad } \ln \lambda) \\
&\quad - 2(n-2)^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda) \\
&\quad + (n-2)^2 d \ln \lambda (\text{grad } \Delta \ln \lambda) d\phi(\text{grad } \ln \lambda) \\
&\quad + 2(n-2)^2 d \ln \lambda \left(\text{grad } \left(|\text{grad } \ln \lambda|^2 \right) \right) d\phi(\text{grad } \ln \lambda) \\
&\quad - 4(n-2) d\phi(\nabla_{\text{grad } \ln \lambda} \text{grad } \Delta \ln \lambda) \\
&\quad + 2(n-2) d\phi(\nabla_{\text{grad } \Delta \ln \lambda} \text{grad } \ln \lambda) \\
&\quad - 8(n-2) d\phi \left(\nabla_{\text{grad } \ln \lambda} \text{grad } \left(|\text{grad } \ln \lambda|^2 \right) \right) \\
&\quad + 2(n-2)^2 d\phi \left(\nabla_{\text{grad } (|\text{grad } \ln \lambda|^2)} \text{grad } \ln \lambda \right).
\end{aligned}$$

Step 2. In this step, simplify the term $Tr_g R^N(\Delta\tau(\phi), d\phi) d\phi$. Using the expression of $\Delta\tau(\phi)$, we have

$$\begin{aligned} & Tr_g R^N(\Delta\tau(\phi), d\phi) d\phi \\ &= (n-2) Tr_g R^N(d\phi(\text{grad } \Delta \ln \lambda), d\phi) d\phi \\ &\quad + 2(n-2) Tr_g R^N\left(d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right), d\phi\right) d\phi \\ &\quad - (n-2)(\Delta \ln \lambda) Tr_g R^N(d\phi(\text{grad } \ln \lambda), d\phi) d\phi \\ &\quad - (n-2)^2 |\text{grad } \ln \lambda|^2 Tr_g R^N(d\phi(\text{grad } \ln \lambda), d\phi) d\phi. \end{aligned}$$

From equation (2.8), we obtain

$$\begin{aligned} & Tr_g R^N(d\phi(\text{grad } \Delta \ln \lambda), d\phi) d\phi \\ &= -(\Delta \ln \lambda) d\phi(\text{grad } \Delta \ln \lambda) \\ &\quad - (n-2) |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \Delta \ln \lambda) \\ &\quad + (n-2) d \ln \lambda (\text{grad } \Delta \ln \lambda) d\phi(\text{grad } \ln \lambda) \\ &\quad - (n-2) d\phi(\nabla_{\text{grad } \Delta \ln \lambda} \text{grad } \ln \lambda), \\ & Tr_g R^N\left(d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right), d\phi\right) d\phi \\ &= -(\Delta \ln \lambda) d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\ &\quad - (n-2) |\text{grad } \ln \lambda|^2 d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\ &\quad + (n-2) d \ln \lambda \left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) d\phi(\text{grad } \ln \lambda) \\ &\quad - (n-2) d\phi\left(\nabla_{\text{grad}\left(|\text{grad } \ln \lambda|^2\right)} \text{grad } \ln \lambda\right) \end{aligned}$$

and

$$\begin{aligned} & Tr_g R^N(d\phi(\text{grad } \ln \lambda), d\phi) d\phi \\ &= -(\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda) - \frac{(n-2)}{2} d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right). \end{aligned}$$

It follows that

$$\begin{aligned} & Tr_g R^N(\Delta\tau(\phi), d\phi) d\phi \\ &= -(n-2)(\Delta \ln \lambda) d\phi(\text{grad } \Delta \ln \lambda) \\ &\quad - (n-2)^2 |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \Delta \ln \lambda) \\ &\quad + \frac{(n-2)(n-6)}{2} (\Delta \ln \lambda) d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\ &\quad + \frac{(n-2)^2(n-6)}{2} |\text{grad } \ln \lambda|^2 d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\ &\quad + (n-2)^2 d \ln \lambda (\text{grad } \Delta \ln \lambda) d\phi(\text{grad } \ln \lambda) \end{aligned}$$

$$\begin{aligned}
 &+ 2(n-2)^2 d \ln \lambda \left(\text{grad} \left(|\text{grad} \ln \lambda|^2 \right) \right) d\phi(\text{grad} \ln \lambda) \\
 &+ (n-2)^2 (\Delta \ln \lambda) |\text{grad} \ln \lambda|^2 d\phi(\text{grad} \ln \lambda) \\
 &+ (n-2) (\Delta \ln \lambda)^2 d\phi(\text{grad} \ln \lambda) \\
 &- (n-2)^2 d\phi(\nabla_{\text{grad} \Delta \ln \lambda} \text{grad} \ln \lambda) \\
 &- 2(n-2)^2 d\phi \left(\nabla_{\text{grad} (|\text{grad} \ln \lambda|^2)} \text{grad} \ln \lambda \right).
 \end{aligned}$$

Step 3. To complete the proof, we develop the term $Tr_g R^N(\nabla \tau(\phi), \tau(\phi)) d\phi$; we have

$$\begin{aligned}
 &Tr_g R^N(\nabla \tau(\phi), \tau(\phi)) d\phi \\
 &= R^N(\nabla_{e_i} \tau(\phi), \tau(\phi)) d\phi(e_i) \\
 &= (n-2)^2 R^N(\nabla_{e_i} d\phi(\text{grad} \ln \lambda), d\phi(\text{grad} \ln \lambda)) d\phi(e_i).
 \end{aligned}$$

Using the fact that

$$\nabla_{e_i} d\phi(\text{grad} \ln \lambda) = |\text{grad} \ln \lambda|^2 d\phi(e_i) + d\phi(\nabla_{e_i} \text{grad} \ln \lambda),$$

we obtain

$$\begin{aligned}
 &Tr_g R^N(\nabla \tau(\phi), \tau(\phi)) d\phi \\
 &= (n-2)^2 |\text{grad} \ln \lambda|^2 R^N(d\phi(e_i), d\phi(\text{grad} \ln \lambda)) d\phi(e_i) \\
 &\quad + (n-2)^2 R^N(d\phi(\nabla_{e_i} \text{grad} \ln \lambda), d\phi(\text{grad} \ln \lambda)) d\phi(e_i).
 \end{aligned}$$

Equation (2.8) leads us to

$$\begin{aligned}
 &R^N(d\phi(e_i), d\phi(\text{grad} \ln \lambda)) d\phi(e_i) \\
 &= -(\Delta \ln \lambda) d\phi(\text{grad} \ln \lambda) - \frac{(n-2)}{2} d\phi \left(\text{grad} \left(|\text{grad} \ln \lambda|^2 \right) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 &R^N(d\phi(\nabla_{e_i} \text{grad} \ln \lambda), d\phi(\text{grad} \ln \lambda)) d\phi(e_i) \\
 &= -\frac{(n-2)}{2} d\phi \left(\nabla_{\text{grad} (|\text{grad} \ln \lambda|^2)} \text{grad} \ln \lambda \right) \\
 &\quad - \frac{1}{2} (\Delta \ln \lambda) d\phi \left(\text{grad} \left(|\text{grad} \ln \lambda|^2 \right) \right),
 \end{aligned}$$

then

$$\begin{aligned}
 &Tr_g R^N(\nabla \tau(\phi), \tau(\phi)) d\phi \\
 &= -\frac{(n-2)^2}{2} (\Delta \ln \lambda) d\phi \left(\text{grad} \left(|\text{grad} \ln \lambda|^2 \right) \right) \\
 &\quad - \frac{(n-2)^3}{2} |\text{grad} \ln \lambda|^2 d\phi \left(\text{grad} \left(|\text{grad} \ln \lambda|^2 \right) \right) \\
 &\quad - (n-2)^2 (\Delta \ln \lambda) |\text{grad} \ln \lambda|^2 d\phi(\text{grad} \ln \lambda) \\
 &\quad - \frac{(n-2)^3}{2} d\phi \left(\nabla_{\text{grad} (|\text{grad} \ln \lambda|^2)} \text{grad} \ln \lambda \right).
 \end{aligned}$$

The results of the three steps and equation (2.12) allow us to conclude that

$$\tau_3(\phi) = -(n-2)d\phi(H(\lambda)),$$

where

$$\begin{aligned} H(\lambda) = & \operatorname{grad}(\Delta^2 \ln \lambda) + 2\operatorname{grad}\left(\Delta\left(|\operatorname{grad} \ln \lambda|^2\right)\right) \\ & - (\Delta \ln \lambda) \operatorname{grad} \Delta \ln \lambda - 2(n-1)|\operatorname{grad} \ln \lambda|^2 \operatorname{grad} \Delta \ln \lambda \\ & - 2(\Delta \ln \lambda) \operatorname{grad}\left(|\operatorname{grad} \ln \lambda|^2\right) \\ & - 6(n-2)|\operatorname{grad} \ln \lambda|^2 \operatorname{grad}\left(|\operatorname{grad} \ln \lambda|^2\right) - 3(\Delta^2 \ln \lambda) \operatorname{grad} \ln \lambda \\ & + 2(n-2)(\Delta \ln \lambda)|\operatorname{grad} \ln \lambda|^2 \operatorname{grad} \ln \lambda \\ & + (n-2)^2 |\operatorname{grad} \ln \lambda|^4 \operatorname{grad} \ln \lambda \\ & + 2(\Delta \ln \lambda)^2 \operatorname{grad} \ln \lambda - (n+2)\left(\Delta\left(|\operatorname{grad} \ln \lambda|^2\right)\right) \operatorname{grad} \ln \lambda \\ & - n\nabla_{\operatorname{grad} \Delta \ln \lambda} \operatorname{grad} \ln \lambda - \frac{(n-2)(n+6)}{2} \nabla_{\operatorname{grad}(|\operatorname{grad} \ln \lambda|^2)} \operatorname{grad} \ln \lambda \\ & + 4\nabla_{\operatorname{grad} \ln \lambda} \operatorname{grad} \Delta \ln \lambda + 8\nabla_{\operatorname{grad} \ln \lambda} \operatorname{grad}\left(|\operatorname{grad} \ln \lambda|^2\right). \quad \square \end{aligned}$$

A necessary and sufficient condition of the triharmonicity of a conformal map is given in the following theorem.

Theorem 2.5. *Let $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ be a conformal map of dilation λ between manifolds of the same dimension $n \geq 3$. Then ϕ is triharmonic if and only if the vector field $\operatorname{grad} \ln \lambda$ satisfies the fourth order elliptic partial differential equation:*

$$\begin{aligned} & \operatorname{grad}(\Delta^2 \ln \lambda) + 2\operatorname{grad}\left(\Delta\left(|\operatorname{grad} \ln \lambda|^2\right)\right) - (\Delta \ln \lambda) \operatorname{grad} \Delta \ln \lambda \\ & - 2(n-1)|\operatorname{grad} \ln \lambda|^2 \operatorname{grad} \Delta \ln \lambda - 2(\Delta \ln \lambda) \operatorname{grad}\left(|\operatorname{grad} \ln \lambda|^2\right) \\ & - 6(n-2)|\operatorname{grad} \ln \lambda|^2 \operatorname{grad}\left(|\operatorname{grad} \ln \lambda|^2\right) - 3(\Delta^2 \ln \lambda) \operatorname{grad} \ln \lambda \\ & + 2(n-2)(\Delta \ln \lambda)|\operatorname{grad} \ln \lambda|^2 \operatorname{grad} \ln \lambda + (n-2)^2 |\operatorname{grad} \ln \lambda|^4 \operatorname{grad} \ln \lambda \\ & + 2(\Delta \ln \lambda)^2 \operatorname{grad} \ln \lambda - (n+2)\left(\Delta\left(|\operatorname{grad} \ln \lambda|^2\right)\right) \operatorname{grad} \ln \lambda \\ & - n\nabla_{\operatorname{grad} \Delta \ln \lambda} \operatorname{grad} \ln \lambda - \frac{(n-2)(n+6)}{2} \nabla_{\operatorname{grad}(|\operatorname{grad} \ln \lambda|^2)} \operatorname{grad} \ln \lambda \\ & + 4\nabla_{\operatorname{grad} \ln \lambda} \operatorname{grad} \Delta \ln \lambda + 8\nabla_{\operatorname{grad} \ln \lambda} \operatorname{grad}\left(|\operatorname{grad} \ln \lambda|^2\right) = 0. \end{aligned}$$

A case of particular interest is when the dilation is radial, we prove that the triharmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) is equivalent to an ordinary differential equation of the fourth order. More precisely, we have:

Corollary 2.6. *Let $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ where we suppose that $\ln \lambda$ is radial ($\ln \lambda = \alpha(r), r = |x|$). Then ϕ is triharmonic if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation:*

$$\begin{aligned}
 & \beta^{(4)} + \frac{2(n-1)}{r}\beta^{(3)} + 5\beta\beta^{(3)} + \frac{(n-1)(n-5)}{r^2}\beta'' - (n-11)\beta'\beta'' \\
 & + \frac{(n-1)}{r}\beta\beta'' - (4n-14)\beta^2\beta'' - (n^2+6n-22)\beta(\beta')^2 \\
 (2.14) \quad & - \frac{(n-3)(n-1)}{r}(\beta')^2 - \frac{3(n-1)(n-3)}{r^3}\beta' - 10(n-2)\beta^3\beta' \\
 & - \frac{2(n-1)(2n+1)}{r}\beta^2\beta' - \frac{(3n+1)(n-1)}{r^2}\beta\beta' + \frac{3(n-1)(n-3)}{r^4}\beta \\
 & + \frac{(n-1)(4n-2)}{r^3}\beta^2 + \frac{4(n-1)^2}{r^2}\beta^3 + \frac{2(n-2)(n-1)}{r}\beta^4 \\
 & + (n-2)^2\beta^5 = 0.
 \end{aligned}$$

Proof. Let $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ where we suppose that $\ln \lambda$ is radial ($\ln \lambda = \alpha(r), r = |x|$). By Theorem 2.5, ϕ is triharmonic if and only if

$$\begin{aligned}
 & \text{grad} (\Delta^2 \ln \lambda) + 2\text{grad} \left(\Delta \left(|\text{grad} \ln \lambda|^2 \right) \right) - (\Delta \ln \lambda) \text{grad} \Delta \ln \lambda \\
 & - 2(n-1)|\text{grad} \ln \lambda|^2 \text{grad} \Delta \ln \lambda - 2(\Delta \ln \lambda) \text{grad} \left(|\text{grad} \ln \lambda|^2 \right) \\
 & - 6(n-2)|\text{grad} \ln \lambda|^2 \text{grad} \left(|\text{grad} \ln \lambda|^2 \right) - 3(\Delta^2 \ln \lambda) \text{grad} \ln \lambda \\
 & + 2(n-2)(\Delta \ln \lambda) |\text{grad} \ln \lambda|^2 \text{grad} \ln \lambda + (n-2)^2 |\text{grad} \ln \lambda|^4 \text{grad} \ln \lambda \\
 & + 2(\Delta \ln \lambda)^2 \text{grad} \ln \lambda - (n+2) \left(\Delta \left(|\text{grad} \ln \lambda|^2 \right) \right) \text{grad} \ln \lambda \\
 & + 4\nabla_{\text{grad} \ln \lambda} \text{grad} \Delta \ln \lambda + 8\nabla_{\text{grad} \ln \lambda} \text{grad} \left(|\text{grad} \ln \lambda|^2 \right) \\
 & - n\nabla_{\text{grad} \Delta \ln \lambda} \text{grad} \ln \lambda - \frac{(n-2)(n+6)}{2} \nabla_{\text{grad} (|\text{grad} \ln \lambda|^2)} \text{grad} \ln \lambda = 0.
 \end{aligned}$$

A rigorous calculation gives us

$$\begin{aligned}
 \text{grad} \alpha &= \alpha' \frac{\partial}{\partial r}, \quad |\text{grad} \alpha|^2 = (\alpha')^2, \quad \text{grad} \left(|\text{grad} \alpha|^2 \right) = 2\alpha'\alpha'' \frac{\partial}{\partial r}, \\
 \Delta \alpha &= \alpha'' + \frac{n-1}{r}\alpha', \quad \text{grad} \Delta \alpha = \left(\alpha^{(3)} + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha' \right) \frac{\partial}{\partial r}, \\
 \Delta^2 \alpha &= \alpha^{(4)} + \frac{2(n-1)}{r}\alpha^{(3)} + \frac{(n-1)(n-3)}{r^2}\alpha'' - \frac{(n-1)(n-3)}{r^3}\alpha', \\
 \text{grad} \Delta^2 \alpha &= \left(\alpha^{(5)} + \frac{2(n-1)}{r}\alpha^{(4)} + \frac{(n-1)(n-5)}{r^2}\alpha^{(3)} \right) \frac{\partial}{\partial r}
 \end{aligned}$$

$$- \left(\frac{3(n-1)(n-3)}{r^3} \alpha'' + \frac{3(n-1)(n-3)}{r^4} \alpha' \right) \frac{\partial}{\partial r},$$

$$\Delta \left(|\text{grad } \alpha|^2 \right) = 2\alpha' \alpha^{(3)} + 2(\alpha'')^2 + \frac{2(n-1)}{r} \alpha' \alpha''$$

and

$$\begin{aligned} \text{grad} \left(\Delta \left(|\text{grad } \alpha|^2 \right) \right) &= \left(2\alpha' \alpha^{(4)} + 6\alpha'' \alpha^{(3)} + \frac{2(n-1)}{r} \alpha' \alpha^{(3)} \right) \frac{\partial}{\partial r} \\ &+ \left(\frac{2(n-1)}{r} (\alpha'')^2 - \frac{2(n-1)}{r^2} \alpha' \alpha'' \right) \frac{\partial}{\partial r}. \end{aligned}$$

Hence, ϕ is triharmonic if and only if the function α satisfies the following differential equation

$$\begin{aligned} &\alpha^{(5)} + \frac{2(n-1)}{r} \alpha^{(4)} + 5\alpha' \alpha^{(4)} + \frac{(n-1)(n-5)}{r^2} \alpha^{(3)} - (n-11) \alpha'' \alpha^{(3)} \\ &+ \frac{(n-1)}{r} \alpha' \alpha^{(3)} - (4n-14) (\alpha')^2 \alpha^{(3)} - \frac{3(n-1)(n-3)}{r^3} \alpha'' \\ &- 10(n-2) (\alpha')^3 \alpha'' - \frac{2(n-1)(2n+1)}{r} (\alpha')^2 \alpha'' - \frac{(3n+1)(n-1)}{r^2} \alpha' \alpha'' \\ &- (n^2 + 6n - 22) \alpha' (\alpha'')^2 - \frac{(n-3)(n-1)}{r} (\alpha'')^2 + \frac{3(n-1)(n-3)}{r^4} \alpha' \\ &+ \frac{(n-1)(4n-2)}{r^3} (\alpha')^2 + \frac{4(n-1)^2}{r^2} (\alpha')^3 + \frac{2(n-2)(n-1)}{r} (\alpha')^4 \\ &+ (n-2)^2 (\alpha')^5 = 0. \end{aligned}$$

Let $\beta = \alpha'$. Then the last equation is equivalent to the fourth-order differential equation

$$\begin{aligned} &\beta^{(4)} + \frac{2(n-1)}{r} \beta^{(3)} + 5\beta \beta^{(3)} + \frac{(n-1)(n-5)}{r^2} \beta'' - (n-11) \beta' \beta'' \\ &+ \frac{(n-1)}{r} \beta \beta'' - (4n-14) \beta^2 \beta'' - (n^2 + 6n - 22) \beta (\beta')^2 \\ &- \frac{(n-3)(n-1)}{r} (\beta')^2 - \frac{3(n-1)(n-3)}{r^3} \beta' - 10(n-2) \beta^3 \beta' \\ &- \frac{2(n-1)(2n+1)}{r} \beta^2 \beta' - \frac{(3n+1)(n-1)}{r^2} \beta \beta' + \frac{3(n-1)(n-3)}{r^4} \beta \\ &+ \frac{(n-1)(4n-2)}{r^3} \beta^2 + \frac{4(n-1)^2}{r^2} \beta^3 + \frac{2(n-2)(n-1)}{r} \beta^4 \\ &+ (n-2)^2 \beta^5 = 0. \end{aligned} \quad \square$$

As a consequence of Corollary 2.6, we present some remarks that give us a particular solutions of equation (2.14) and that allows us to construct a triharmonic maps.

Remark 2.7. We are looking for particular solutions of the type $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$). By (2.14), we deduce that $\phi : (\mathbb{R}^n \setminus \{0\}, g) \rightarrow (N^n, h)$ ($n \geq 3$) is triharmonic if and only if a is solution of the algebraic equation

$$(n - 2)^2 a^4 + 2(n - 2)(n + 4)a^3 + (7n^2 - 24n + 52)a^2 + (6n^2 - 56)a + 8(n - 2)(n - 4) = 0.$$

This equation has real solutions if and only if $n \in \{3, 4, 5, 6\}$.

- (1) If $n = 3$, we find $a = \frac{-5+\sqrt{17}}{2}$, $a = \frac{-5-\sqrt{17}}{2}$, $a = \frac{-9+\sqrt{97}}{2}$ and $a = \frac{-9-\sqrt{97}}{2}$. Note that for $a = \frac{-5+\sqrt{17}}{2}$ or $a = \frac{-5-\sqrt{17}}{2}$ any conformal map $\phi : (\mathbb{R}^3 \setminus \{0\}, g) \rightarrow (N^3, h)$ of dilation $\lambda = \frac{C}{r \frac{5-\sqrt{17}}{2}}$ or $\lambda = \frac{C}{r \frac{5+\sqrt{17}}{2}}$ ($C > 0$) is biharmonic and triharmonic. For $\lambda = \frac{C}{r \frac{9-\sqrt{97}}{2}}$ or $\lambda = \frac{C}{r \frac{9+\sqrt{97}}{2}}$, the conformal map $\phi : (\mathbb{R}^3 \setminus \{0\}, g) \rightarrow (N^3, h)$ of dilation $\lambda = Cr \frac{-9+\sqrt{97}}{2}$ or $\lambda = \frac{C}{r \frac{9+\sqrt{97}}{2}}$ is triharmonic non-biharmonic.
- (2) If $n = 4$, the solutions are $a = -5$, $a = -2$ and $a = -1$. For $a = -2$ and $a = -1$, the conformal map $\phi : (\mathbb{R}^4 \setminus \{0\}, g) \rightarrow (N^4, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ ($C > 0$) is biharmonic and triharmonic. For $a = -5$, the conformal map $\phi : (\mathbb{R}^4 \setminus \{0\}, g) \rightarrow (N^4, h)$ of dilation $\lambda = \frac{C}{r^5}$ is triharmonic non-biharmonic.
- (3) If $n = 5$, we find $a = \frac{-11+\sqrt{73}}{6}$ and $a = \frac{-11-\sqrt{73}}{6}$ so $\lambda = \frac{C}{r \frac{11-\sqrt{73}}{6}}$ or $\lambda = \frac{C}{r \frac{11+\sqrt{73}}{6}}$ ($C > 0$), it follows that any conformal map $\phi : (\mathbb{R}^5 \setminus \{0\}, g) \rightarrow (N^5, h)$ of dilation $\lambda = \frac{C}{r \frac{11+\sqrt{73}}{6}}$ or $\lambda = \frac{C}{r \frac{11-\sqrt{73}}{6}}$ is triharmonic non-biharmonic.
- (4) The case where $n = 6$ gives us $a = -2$ and $a = -1$, so $\lambda = \frac{C}{r^2}$ and $\lambda = \frac{C}{r}$ ($C > 0$). Then, in this case any conformal map $\phi : (\mathbb{R}^6 \setminus \{0\}, g) \rightarrow (N^6, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ is triharmonic non-biharmonic.

Example 2.8. Using Remark 2.7, we give two examples:

- (1) For $a = -2$, we can cite the example of the inversion $\phi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ defined by $\phi(x) = \frac{x}{|x|^2}$; it is a conformal map of dilation $\lambda = \frac{1}{r^2}$. We deduce that the inversion is
 - harmonic if and only if $n = 2$.
 - triharmonic, biharmonic and non-harmonic if and only if $n = 4$.
 - triharmonic non- biharmonic if and only if $n = 6$.
- (2) The case $a = -1$ gives dilation $\lambda = \frac{1}{r}$. In this case we consider the conformal map $\phi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R} \times \mathbb{S}^{n-1}$, given in polar coordinates by $\phi(r\theta) = (\ln r, \theta)$ for $r > 0$, $\theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. This map is
 - harmonic if and only if $n = 2$.
 - triharmonic, biharmonic and non-harmonic if and only if $n = 4$.
 - triharmonic non-biharmonic if and only if $n = 6$.

We consider $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) a conformal map of dilation λ . To calculate the trace of the stress-energy tensor $S_3(\phi)$, we need some lemmas. In the first lemma, we consider a Riemannian manifold (M^n, g) and we give a relation between $\Delta |\text{grad } f|^2$ and $|\nabla \text{grad } f|^2$ for any function $f \in C^\infty(M)$.

Lemma 2.9. *Let (M^n, g) be a Riemannian manifold. Then for any function $f \in C^\infty(M)$, we have*

$$(2.15) \quad \Delta |\text{grad } f|^2 = 2df(\text{grad } \Delta f) + 2|\nabla \text{grad } f|^2 + 2df(\text{Ricci}(\text{grad } f)).$$

Proof. Let $(e_i)_{1 \leq i \leq n}$ be an orthonormal frame on M . By definition, we have

$$\Delta |\text{grad } f|^2 = e_i \left(e_i \left(|\text{grad } f|^2 \right) \right) - (\nabla_{e_i} e_i) \left(|\text{grad } f|^2 \right).$$

Then

$$\begin{aligned} \Delta |\text{grad } f|^2 &= e_i (e_i (g(\text{grad } f, \text{grad } f))) - (\nabla_{e_i} e_i) (g(\text{grad } f, \text{grad } f)) \\ &= 2e_i (g(\nabla_{e_i} \text{grad } f, \text{grad } f)) - 2(g(\nabla_{\nabla_{e_i} e_i} \text{grad } f, \text{grad } f)) \\ &= 2g(\nabla_{e_i} \nabla_{e_i} \text{grad } f, \text{grad } f) + 2g(\nabla_{e_i} \text{grad } f, \nabla_{e_i} \text{grad } f) \\ &\quad - 2(g(\nabla_{\nabla_{e_i} e_i} \text{grad } f, \text{grad } f)) \\ &= 2g(\text{Tr}_g \nabla^2 \text{grad } f, \text{grad } f) + 2|\nabla \text{grad } f|^2, \end{aligned}$$

which gives us

$$\Delta |\text{grad } f|^2 = 2df(\text{grad } \Delta f) + 2|\nabla \text{grad } f|^2 + 2df(\text{Ricci}(\text{grad } f)). \quad \square$$

Lemma 2.10. *Let $\phi : (M^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . Then*

$$(2.16) \quad \begin{aligned} &|\nabla \tau(\phi)|^2 \\ &= n(n-2)^2 \lambda^2 |\text{grad } \ln \lambda|^4 + 2(n-2)^2 \lambda^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 \\ &\quad + \frac{(n-2)^2}{2} \lambda^2 \Delta \left(|\text{grad } \ln \lambda|^2 \right) - (n-2)^2 \lambda^2 d \ln \lambda (\text{grad } \Delta \ln \lambda) \\ &\quad - (n-2)^2 \lambda^2 d \ln \lambda (\text{Ricci}(\text{grad } \ln \lambda)) \end{aligned}$$

and

$$(2.17) \quad \begin{aligned} \langle d\phi, \nabla \Delta \tau(\phi) \rangle &= (n-2)(n-1) \lambda^2 d \ln \lambda (\text{grad } \Delta \ln \lambda) \\ &\quad + (n-2)(n+2) \lambda^2 d \ln \lambda \left(\text{grad} \left(|\text{grad } \ln \lambda|^2 \right) \right) \\ &\quad - 2(n-2)(n-1) \lambda^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 \\ &\quad + 2(n-2) \lambda^2 \Delta \left(|\text{grad } \ln \lambda|^2 \right) + (n-2) \lambda^2 (\Delta^2 \ln \lambda) \\ &\quad - (n-2) \lambda^2 (\Delta \ln \lambda)^2 - n(n-2)^2 \lambda^2 |\text{grad } \ln \lambda|^4 \\ &\quad + n(n-2) \lambda^2 d \ln \lambda (\text{Ricci}(\text{grad } \ln \lambda)) \\ &\quad + (n-2) \lambda^2 \text{div}(\text{Ricci}(\text{grad } \ln \lambda)). \end{aligned}$$

Proof. For the term $|\nabla\tau(\phi)|^2$, we have

$$\begin{aligned} |\nabla\tau(\phi)|^2 &= h(\nabla_{e_i}\tau(\phi), \nabla_{e_i}\tau(\phi)) \\ &= (n-2)^2 h(\nabla_{e_i}d\phi(\text{grad } \ln \lambda), \nabla_{e_i}d\phi(\text{grad } \ln \lambda)). \end{aligned}$$

Using the fact that

$$\nabla_{e_i}d\phi(\text{grad } \ln \lambda) = |\text{grad } \ln \lambda|^2 d\phi(e_i) + d\phi(\nabla_{e_i}\text{grad } \ln \lambda),$$

we deduce that

$$\begin{aligned} |\nabla\tau(\phi)|^2 &= (n-2)^2 h(|\text{grad } \ln \lambda|^2 d\phi(e_i), |\text{grad } \ln \lambda|^2 d\phi(e_i)) \\ &\quad + 2(n-2)^2 h(|\text{grad } \ln \lambda|^2 d\phi(e_i), d\phi(\nabla_{e_i}\text{grad } \ln \lambda)) \\ &\quad + (n-2)^2 h(d\phi(\nabla_{e_i}\text{grad } \ln \lambda), d\phi(\nabla_{e_i}\text{grad } \ln \lambda)) \\ &= n(n-2)^2 \lambda^2 |\text{grad } \ln \lambda|^4 + (n-2)^2 \lambda^2 |\nabla\text{grad } \ln \lambda|^2 \\ &\quad + 2(n-2)^2 \lambda^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2. \end{aligned}$$

By Lemma 2.9, we have

$$\begin{aligned} |\nabla\text{grad } \ln \lambda|^2 &= \frac{1}{2}\Delta(|\text{grad } \ln \lambda|^2) - d\ln \lambda(\text{grad } \Delta \ln \lambda) \\ &\quad - d\ln \lambda(\text{Ricci}(\text{grad } \ln \lambda)). \end{aligned}$$

It follows that

$$\begin{aligned} |\nabla\tau(\phi)|^2 &= n(n-2)^2 \lambda^2 |\text{grad } \ln \lambda|^4 + 2(n-2)^2 \lambda^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 \\ &\quad + \frac{(n-2)^2}{2} \lambda^2 \Delta(|\text{grad } \ln \lambda|^2) - (n-2)^2 \lambda^2 d\ln \lambda(\text{grad } \Delta \ln \lambda) \\ &\quad - (n-2)^2 \lambda^2 d\ln \lambda(\text{Ricci}(\text{grad } \ln \lambda)). \end{aligned}$$

Now let's look at the term $\langle d\phi, \nabla\Delta\tau(\phi) \rangle$, we have

$$\langle d\phi, \nabla\Delta\tau(\phi) \rangle = h(d\phi(e_i), \nabla_{e_i}\Delta\tau(\phi)).$$

By Theorem 2.3, we obtain

$$\begin{aligned} \Delta\tau(\phi) &= (n-2)d\phi(\text{grad } \Delta \ln \lambda) + 2(n-2)d\phi(\text{grad } (|\text{grad } \ln \lambda|^2)) \\ &\quad - (n-2)(\Delta \ln \lambda)d\phi(\text{grad } \ln \lambda) \\ &\quad - (n-2)^2 |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda) \\ &\quad + (n-2)d\phi(\text{Ricci}(\text{grad } \ln \lambda)), \end{aligned}$$

then

$$\begin{aligned} \langle d\phi, \nabla\Delta\tau(\phi) \rangle &= (n-2)h(d\phi(e_i), \nabla_{e_i}d\phi(\text{grad } \Delta \ln \lambda)) \\ &\quad + 2(n-2)h(d\phi(e_i), \nabla_{e_i}d\phi(\text{grad } (|\text{grad } \ln \lambda|^2))) \\ &\quad - (n-2)h(d\phi(e_i), \nabla_{e_i}(\Delta \ln \lambda)d\phi(\text{grad } \ln \lambda)) \end{aligned}$$

$$\begin{aligned}
& - (n-2)^2 h\left(d\phi(e_i), \nabla_{e_i} |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda)\right) \\
& + (n-2) h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{Ricci}(\text{grad } \ln \lambda))\right).
\end{aligned}$$

The development of each term of this last equation gives

$$\begin{aligned}
h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{grad } \Delta \ln \lambda)\right) &= n\lambda^2 d\ln \lambda(\text{grad } \Delta \ln \lambda) + \lambda^2 (\Delta^2 \ln \lambda), \\
h\left(d\phi(e_i), \nabla_{e_i} d\phi\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right)\right) &= n\lambda^2 d\ln \lambda\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\
& \quad + \lambda^2 \Delta\left(|\text{grad } \ln \lambda|^2\right),
\end{aligned}$$

$$\begin{aligned}
& h\left(d\phi(e_i), \nabla_{e_i} (\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda)\right) \\
&= (\Delta \ln \lambda) h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{grad } \ln \lambda)\right) \\
& \quad + h\left(d\phi(e_i), e_i (\Delta \ln \lambda) d\phi(\text{grad } \ln \lambda)\right) \\
&= n\lambda^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 + \lambda^2 (\Delta \ln \lambda)^2 \\
& \quad + \lambda^2 d\ln \lambda(\text{grad } \Delta \ln \lambda),
\end{aligned}$$

$$\begin{aligned}
& h\left(d\phi(e_i), \nabla_{e_i} |\text{grad } \ln \lambda|^2 d\phi(\text{grad } \ln \lambda)\right) \\
&= |\text{grad } \ln \lambda|^2 h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{grad } \ln \lambda)\right) \\
& \quad + h\left(d\phi(e_i), e_i \left(|\text{grad } \ln \lambda|^2\right) d\phi(\text{grad } \ln \lambda)\right) \\
&= n\lambda^2 |\text{grad } \ln \lambda|^4 + \lambda^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 \\
& \quad + \lambda^2 d\ln \lambda\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{Ricci}(\text{grad } \ln \lambda))\right) &= n\lambda^2 d\ln \lambda(\text{Ricci}(\text{grad } \ln \lambda)) \\
& \quad + \lambda^2 (\text{div}(\text{Ricci}(\text{grad } \ln \lambda))).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\langle d\phi, \nabla \Delta \tau(\phi) \rangle &= (n-2)(n-1)\lambda^2 d\ln \lambda(\text{grad } \Delta \ln \lambda) \\
& \quad + (n-2)(n+2)\lambda^2 d\ln \lambda\left(\text{grad}\left(|\text{grad } \ln \lambda|^2\right)\right) \\
& \quad - 2(n-2)(n-1)\lambda^2 (\Delta \ln \lambda) |\text{grad } \ln \lambda|^2 \\
& \quad + 2(n-2)\lambda^2 \Delta\left(|\text{grad } \ln \lambda|^2\right) + (n-2)\lambda^2 (\Delta^2 \ln \lambda) \\
& \quad - (n-2)\lambda^2 (\Delta \ln \lambda)^2 - n(n-2)^2 \lambda^2 |\text{grad } \ln \lambda|^4 \\
& \quad + n(n-2)\lambda^2 d\ln \lambda(\text{Ricci}(\text{grad } \ln \lambda)) \\
& \quad + (n-2)\lambda^2 \text{div}(\text{Ricci}(\text{grad } \ln \lambda)). \quad \square
\end{aligned}$$

Theorem 2.11. *Let $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . Then*

$$\begin{aligned} Tr_g S_3(\phi) = & -\frac{(n-2)^2(n-4)}{2} \lambda^2 d \ln \lambda (\operatorname{grad} \Delta \ln \lambda) - (n-2)^2 \lambda^2 (\Delta^2 \ln \lambda) \\ & + (n-2)^3 \lambda^2 d \ln \lambda \left(\operatorname{grad} \left(|\operatorname{grad} \ln \lambda|^2 \right) \right) + (n-2)^2 \lambda^2 (\Delta \ln \lambda)^2 \\ & + \frac{(n-2)^2(n-10)}{4} \lambda^2 \Delta \left(|\operatorname{grad} \ln \lambda|^2 \right) + \frac{n(n-2)^3}{2} \lambda^2 |\operatorname{grad} \ln \lambda|^4 \\ & + 2(n-2)^3 \lambda^2 (\Delta \ln \lambda) |\operatorname{grad} \ln \lambda|^2. \end{aligned}$$

Proof. By definition, for any vectors field X, Y , we have

$$\begin{aligned} S_3(\phi)(X, Y) = & \left(\frac{1}{2} |\nabla \tau(\phi)|^2 - h(\tau(\phi), \Delta \tau(\phi)) - \langle d\phi, \nabla \Delta \tau(\phi) \rangle \right) g(X, Y) \\ & + h(d\phi(X), \nabla_Y \Delta \tau(\phi)) + h(d\phi(Y), \nabla_X \Delta \tau(\phi)) \\ & - h(\nabla_X \tau(\phi), \nabla_Y \tau(\phi)), \end{aligned}$$

then

$$\begin{aligned} Tr_g S_3(\phi) = & n \left(\frac{1}{2} |\nabla \tau(\phi)|^2 - h(\tau(\phi), \Delta \tau(\phi)) - \langle d\phi, \nabla \Delta \tau(\phi) \rangle \right) \\ & + 2Tr_h h(d\phi, \nabla \Delta \tau(\phi)) - |\nabla \tau(\phi)|^2, \end{aligned}$$

which gives us

$$(2.18) \quad \begin{aligned} & Tr_g S_3(\phi) \\ = & \frac{n-2}{2} |\nabla \tau(\phi)|^2 - nh(\tau(\phi), \Delta \tau(\phi)) - (n-2) \langle d\phi, \nabla \Delta \tau(\phi) \rangle. \end{aligned}$$

The terms $|\nabla \tau(\phi)|^2$ and $\langle d\phi, \nabla \Delta \tau(\phi) \rangle$ are calculated in Lemma 2.10. For the term $h(\tau(\phi), \Delta \tau(\phi))$, we have

$$\tau(\phi) = (2-n) d\phi(\operatorname{grad} \ln \lambda)$$

and

$$\begin{aligned} \Delta \tau(\phi) = & (n-2) d\phi(\operatorname{grad} \Delta \ln \lambda) + 2(n-2) d\phi \left(\operatorname{grad} \left(|\operatorname{grad} \ln \lambda|^2 \right) \right) \\ & - (n-2) (\Delta \ln \lambda) d\phi(\operatorname{grad} \ln \lambda) \\ & - (n-2)^2 |\operatorname{grad} \ln \lambda|^2 d\phi(\operatorname{grad} \ln \lambda). \end{aligned}$$

It follows that

$$(2.19) \quad \begin{aligned} h(\tau(\phi), \Delta \tau(\phi)) = & -2(n-2)^2 \lambda^2 d \ln \lambda \left(\operatorname{grad} \left(|\operatorname{grad} \ln \lambda|^2 \right) \right) \\ & - (n-2)^2 \lambda^2 d \ln \lambda (\operatorname{grad} \Delta \ln \lambda) \\ & + (n-2)^2 \lambda^2 (\Delta \ln \lambda) |\operatorname{grad} \ln \lambda|^2 \\ & + (n-2)^3 \lambda^2 |\operatorname{grad} \ln \lambda|^4. \end{aligned}$$

By equations (2.16), (2.17) and (2.19), we deduce that

$$\begin{aligned} & Tr_g S_3(\phi) \\ = & -\frac{(n-2)^2(n-4)}{2}\lambda^2 d\ln\lambda(\operatorname{grad}\Delta\ln\lambda) - (n-2)^2\lambda^2(\Delta^2\ln\lambda) \\ & + (n-2)^3\lambda^2 d\ln\lambda\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right) + (n-2)^2\lambda^2(\Delta\ln\lambda)^2 \\ & + \frac{(n-2)^2(n-10)}{4}\lambda^2\Delta\left(|\operatorname{grad}\ln\lambda|^2\right) + \frac{n(n-2)^3}{2}\lambda^2|\operatorname{grad}\ln\lambda|^4 \\ & + 2(n-2)^3\lambda^2(\Delta\ln\lambda)|\operatorname{grad}\ln\lambda|^2. \quad \square \end{aligned}$$

Corollary 2.12. *Let $\phi : (\mathbb{R}^n, g) \rightarrow (N^n, h)$ ($n \geq 3$) be a conformal map of dilation λ . Then $Tr_g S_3(\phi) = 0$ if and only if*

$$\begin{aligned} & \Delta^2\ln\lambda - \frac{(n-10)}{4}\Delta\left(|\operatorname{grad}\ln\lambda|^2\right) - 2(n-2)(\Delta\ln\lambda)|\operatorname{grad}\ln\lambda|^2 \\ & - (\Delta\ln\lambda)^2 - \frac{n(n-2)}{2}|\operatorname{grad}\ln\lambda|^4 + \frac{(n-4)}{2}d\ln\lambda(\operatorname{grad}\Delta\ln\lambda) \\ & - (n-2)d\ln\lambda\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right) = 0. \end{aligned}$$

Moreover, if we suppose that $\ln\lambda$ is radial ($\ln\lambda = \alpha(r)$, $r = |x|$), we deduce that $Tr_g S_3(\phi) = 0$ if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation:

$$\begin{aligned} (2.20) \quad & \beta^{(3)} + \frac{2(n-1)}{r}\beta^{(2)} + 3\beta\beta^{(2)} + \frac{(n-1)(n-3)}{r^2}\beta' + \frac{(n-1)}{r}\beta\beta' \\ & - \frac{(n-8)}{2}(\beta')^2 - 4(n-2)\beta^2\beta' - \frac{(n-1)(n-3)}{r^3}\beta \\ & - \frac{3(n-2)(n-1)}{2r^2}\beta^2 - \frac{2(n-1)(n-2)}{r}\beta^3 - \frac{n(n-2)}{2}\beta^4 = 0. \end{aligned}$$

Remark 2.13. Let us look for particular solutions of equation (2.20), and are written in the form $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$), we find the following algebraic equation

$$a^3n^2 - 2a^3n + 4a^2n^2 - 20a^2n + 24a^2 + 3an^2 - 6an - 16a + 4n^2 - 24n + 32 = 0.$$

Example 2.14. Using Remark 2.13, we give two examples:

- (1) For $a = -2$, we can cite the example of the inversion

$$\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$$

defined by $\phi(x) = \frac{x}{|x|^2}$, it is a conformal map of dilation $\lambda = \frac{1}{r^2}$.

We conclude that $Tr_g S_3(\phi) = 0$ if and only if $n = 10$.

- (2) The case $a = -1$ gives dilation $\lambda = \frac{1}{r}$, in this case we consider the conformal map $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R} \times \mathbb{S}^{n-1}$, given in polar coordinates by $\phi(r\theta) = (\ln r, \theta)$, for $r > 0, \theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$.

We deduce that $Tr_g S_3(\phi) = 0$ if and only if $n = 3$ or $n = 6$.

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SEDDIK OUAKKAS

LABORATORY OF GEOMETRY, ANALYSIS, CONTROL AND APPLICATIONS

UNIVERSITY OF SAIDA, DR MOULAY TAHAR.

SAIDA 20000, ALGERIA

Email address: seddik.ouakkas@univ-saida.dz

YASMINA REGUIG

LABORATORY OF GEOMETRY, ANALYSIS, CONTROL AND APPLICATIONS

UNIVERSITY OF SAIDA, DR MOULAY TAHAR.

SAIDA 20000, ALGERIA

Email address: yasmina.reguig@univ-saida.dz