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A FUNDAMENTAL THEOREM OF CALCULUS FOR THE M_α-INTEGRAL

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Abstract. This paper presents a fundamental theorem of calculus, an integration by parts formula and a version of equiintegrability convergence theorem for the M_α-integral using the M_α-strong Lusin condition. In the convergence theorem, to be able to relax the condition of being point-wise convergent everywhere to point-wise convergent almost everywhere, the uniform M_α-strong Lusin condition was imposed.

1. Introduction

Recently, in [9], a new Henstock-type integral was introduced by Park, Ryu, and Lee, and they named it M_α-integral. This new integral uses McShane partition. Several properties of this new integral were proved in [8], [9] and [10]. It was shown further in [10] that it is equivalent to the C-integral. Most of the properties are parallel to the usual properties of an integral including the Saks-Henstock Lemma [9, Lemma 2.5]. Convergence theorems for this integral were discussed in [3] and [7]. Cauchy extension and absolute M_α-integrability were discussed in [4].

It is well known that in the real line f is Henstock-Kurzweil integrable on $[a,b]$ if and only if there exists a function F satisfying the strong Lusin (SL) condition with $F'(x) = f(x)$ almost everywhere. See for example the discussion in [6]. Since the M_α-integral is a Henstock-type integral it is natural to ask whether a similar type of characterization exists for the M_α-integral. An affirmative answer is given to this query and as a consequence an integration by parts and a convergence theorem are given.

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2. Preliminaries

Let $\alpha > 0$ be a constant, and $[a, b]$ a non-degenerate closed and bounded interval in $\mathbb{R}$. A subset S of $[a, b]$ is of measure zero when it is of Lebesgue measure zero. A McShane partial partition $D = \{(I, x)\}$ of $[a, b]$ is a finite collection of interval-point pairs such that $x \in [a, b]$, $I \subset [a, b]$ and $\{I : (I, x) \in D\}$ are non-overlapping. A positive function on $[a, b]$ is called a gauge on $[a, b]$. We say that a McShane partial partition D of $[a, b]$ is

(1) S-tagged, where $S \subset [a, b]$, if for all $(I, x) \in D$ we have $x \in S$,
(2) a McShane partition if $\bigcup_{(I,x)\in D} I = [a, b]$,
(3) δ-fine if for $(I, x) \in D$ we have $I \subset (x - \delta(x), x + \delta(x))$ for a gauge δ,
(4) a partial M_α-partition if

$$(D)\sum \operatorname{dist}(x, I) < \alpha,$$

where $\operatorname{dist}(x, I) = \inf\{|x - y| : y \in I\}$.

We say that a McShane partial partition $D = \{(I, x)\}$ is a Henstock partial partition, if for each $(I, x) \in D$, $x \in I$. Given a gauge δ on $[a, b]$, the existence of a δ-fine M_α-partition of $[a, b]$ is guaranteed by [5, Lemma 2.1]. The said lemma is known as the Cousin's Lemma.

We are now ready to present the definition of the M_α-integral.

Definition ([9, Definition 2.1])**.** A function $f : [a, b] \to \mathbb{R}$ is M_α-integrable if there exists a real number A such that for each $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that for any δ-fine M_α-partition $D = \{(I, x)\}$ of $[a, b]$

$$\left|(D)\sum f(x)|I| - A\right| < \epsilon.$$

The number A is called the M_α-integral of f on $[a, b]$ and we write $(M_\alpha)\int_a^b f = A$.

In the definition above, $(D)\sum f(x)|I|$ denotes the Riemann sum of f over the M_α-partition D.

It f is M_α-integrable on $[a, b]$, then f is M_α-integrable on any subinterval I of $[a, b]$ [9, Theorem 2.3(1)]. For an M_α-integrable function f, define its primitive function F by

$$F(x) = (M_\alpha)\int_a^x f, \text{ if } x \in (a, b]$$

and $F(a) = 0$. For any subinterval $I = [u, v]$ of $[a, b]$ and a function F on $[a, b]$, we put $F(I) = F(u, v) = F(v) - F(u)$. A primitive function is additive, in the sense that, for any subinterval I_1 and I_2 of $[a, b]$ whose union is also an interval, $F(I_1 \cup I_2) = F(I_1) + F(I_2)$.

Lemma 2.1 (Saks-Henstock, [9, Lemma 2.5])**.** *Let f be M_α-integrable on $[a, b]$ with primitive F. Then for every $\epsilon > 0$ there is a gauge δ on $[a, b]$ such that* *for any δ-fine M_α-partial partition D of $[a,b]$ we have*

$$(D)\sum |F(I) - f(x)|I|| < \epsilon.$$

Recall from [5] that a function $f : [a,b] \to \mathbb{R}$ is Henstock-Kurzweil integrable if there exists a real number A such that for each $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine Henstock partition $D = \{(I,x)\}$ of $[a,b]$

$$\left|(D)\sum f(x)|I| - A\right| < \epsilon.$$

Since every Henstock partition is an M_α-partition, every M_α-integrable function is also Henstock-Kurzweil integrable, [9, Theorem 2.10(b)] and [10, Theorem 2.10(b)].

3. Fundamental theorem of calculus

We say that an additive interval function F on $[a,b]$ satisfies the M_α-strong Lusin (M_α-SL) condition if given $\epsilon > 0$ and a set $S \subset [a,b]$ of measure zero there exists a gauge δ on $[a,b]$ such that for any δ-fine S-tagged partial M_α-partition $D = \{(I,x)\}$ of $[a,b]$ we have $(D)\sum |F(I)| < \epsilon$. Since every partial Henstock-partition is a partial M_α-partition, we have the following lemma.

Lemma 3.1. *If a function F is M_α-SL, then it is also SL.*

Theorem 3.2 (Main Result)**.** *Let f be a function on $[a,b]$ and F be an additive interval function on $[a,b]$. Then f is M_α-integrable on $[a,b]$ with primitive F if and only if $F'(x) = f(x)$ almost everywhere on $[a,b]$ and F satisfies the M_α-strong Lusin condition on $[a,b]$. In this case, we have $(M_\alpha)\int_a^b f = F(b) - F(a)$.*

Proof. Suppose f is M_α-integrable and F is its primitive. Let $\epsilon > 0$ and a subset S of $[a,b]$ with measure zero be given. By [9, Lemma 2.11] there exists a gauge δ on $[a,b]$ such that for any δ-fine S-tagged partial M_α-partition $D = \{(I,x)\}$ of $[a,b]$ we have

$$(D)\sum |f(x)||I| < \frac{\epsilon}{2}.$$

Since f is M_α-integrable on $[a,b]$ with primitive F, we may further choose δ appropriately so that for any δ-fine partial M_α-partition D of $[a,b]$ we have

$$(D)\sum |f(x)|I| - F(I)| < \frac{\epsilon}{2}.$$

Let D be any δ-fine S-tagged partial M_α-partition of $[a,b]$. Then

$$(D)\sum |F(I)| \leq (D)\sum |f(x)|I| - F(I)| + (D)\sum |f(x)||I| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore F is M_α-SL. By [9, Theorem 2.10(b)], M_α-integrability implies Henstock-Kurzweil integrability, it follows from [1, Theorem 5.9] that $F'(x) = f(x)$ almost everywhere on $[a,b]$.

For the converse, suppose that F is M_α-SL and $F'(x) = f(x)$ almost everywhere. Then there exists $S \subset [a,b]$ with measure zero such that for all

$x \in [a,b]\backslash S$, $F'(x) = f(x)$. By [9, Lemma 2.11] given $\epsilon > 0$ there exists a gauge δ on S such that for any δ-fine X-tagged partial M_α-partition D of $[a,b]$ we have

$$(D)\sum |f(x)||I| < \epsilon_0, \tag{1}$$

where $\epsilon_0 = \frac{\epsilon}{3}$. Further, since F is M_α-SL we can choose δ so that

$$(D)\sum |F(I)| < \epsilon_0. \tag{2}$$

Now, since $F'(x) = f(x)$ for all $x \in [a,b]\backslash S$, we can further modify our gauge δ on $[a,b]\backslash S$ such that for any δ-fine M_α-pair (I,x) with $x \in [a,b] \setminus S$ we have

$$|F(I) - f(x)|I|| < \frac{\epsilon}{6(\alpha + (b-a))}|I|. \tag{3}$$

Let D be a δ-fine partial M_α-partition of $[a,b]$. Split D into D_1 and D_2, where D_1 contains those pairs with tags in S and D_2 otherwise. It follows from (1) and (2) that

$$(D_1)\sum |f(x)||I| < \epsilon_0 \quad \text{and} \quad (D_1)\sum |F(I)| < \epsilon_0$$

and from (3) that for D_2, we have

$$\begin{aligned}(D_2)\sum |f(x)|I| - F(I)| &\le \frac{\epsilon}{3(\alpha + (b-a))}(D_2)\sum(\operatorname{dist}(x,I) + |I|)\\ &< \frac{\epsilon}{3(\alpha + (b-a))}(\alpha + (b-a))\\ &= \frac{\epsilon}{3}.\end{aligned}$$

Hence, $(D)\sum |f(x)|I| - F(I)| < \epsilon$. Therefore f is integrable and F is its primitive. The proof is complete. ☐

Theorem 3.3. *A function F on E is a primitive of some M_α-integrable function if and only if F satisfies the M_α-SL condition.*

Proof. In view of Theorem 3.2, if F is a primitive of an M_α-integrable function, then F is M_α-SL.

For the converse, if F is M_α-SL on $[a,b]$, then, by Lemma 3.1, F is SL. It follows from [2] that F is differentiable almost everywhere on $[a,b]$. Define a function f on $[a,b]$ such that $f(x) = F'(x)$ whenever $F'(x)$ exists and $f(x) = 0$, otherwise. Then f is M_α-integrable and F is its primitive. ☐

Since every isolated point is of measure zero, we have the following result.

Lemma 3.4. *If a function F satisfies the M_α-SL condition on $[a,b]$, then F is continuous on $[a,b]$.*

Corollary 3.5. *If a function F satisfies the M_α-SL condition on $[a,b]$, then F is bounded on $[a,b]$.*

Theorem 3.6 (Integration by parts)**.** *If F and H satisfy the M_α-SL condition on $[a,b]$ and $F'(x) = f(x), H'(x) = h(x)$ almost everywhere on $[a,b]$, then*

$$(M_\alpha)\int_a^b (Fh + Hf) = F(b)H(b) - F(a)H(a).$$

Proof. We will first show that if F and H satisfy the M_α-SL condition on $[a,b]$, then the product FG also satisfies the M_α-SL condition on $[a,b]$. For $[u,v] \subset [a,b]$, we have

$$(FH)(u,v) = F(v)H(u,v) + H(u)F(u,v).$$

There exists $M > 0$ such that for any $x \in [a,b]$, $F(x), H(x) \leq M$. Given $\epsilon > 0$ and a subset S of $[a,b]$ of measure zero there exists a gauge δ_F on $[a,b]$ such that for any δ_F-fine S-tagged partial M_α-partition D of $[a,b]$ we have $(D)\sum|F(u,v)| < \frac{\epsilon}{2M}$. Also, there exists a gauge δ_H on $[a,b]$ such that for any δ_H-fine S-tagged partial M_α-partition D of $[a,b]$ we have $(D)\sum|H(u,v)| < \frac{\epsilon}{2M}$. Define $\delta(x) = \min\{\delta_F(x), \delta_H(x)\}$. For any δ-fine S-tagged partial M_α-partition $D = \{(x,[u,v])\}$ of $[a,b]$ we have

$$\begin{aligned}(D)\sum|(FH)(u,v)| &\leq (D)\sum|F(v)H(u,v)| + (D)\sum|H(u)F(u,v)|\\ &\leq M(D)\sum|H(u,v)| + M(D)\sum|F(u,v)|\\ &< M\cdot\frac{\epsilon}{2M} + M\cdot\frac{\epsilon}{2M}\\ &= \epsilon.\end{aligned}$$

The results follow from Theorem 3.2, since

$$[(FG)(x)]' = F(x)h(x) + H(x)f(x)$$

almost everywhere on $[a,b]$. $\square$

We end this paper by presenting a convergence theorem. Let $\{f_n\}$ be a sequence of M_α-integrable functions on $[a,b]$ with primitives $\{F_n\}$. We say that f_n is *equi-integrable* if for every $\epsilon > 0$ there is a gauge δ on $[a,b]$ independent of n such that for any δ-fine M_α-partition D of $[a,b]$ we have $|(D)\sum f_n(x)|I| - F_n(a,b)| < \epsilon$.

The following result was presented in [7, Theorem 3.6] and [3, Corollary 2.2].

Theorem 3.7. *Let $\{f_n\}$ be a sequence of M_α-integrable functions on $[a,b]$. If $f_n \to f$ everywhere on $[a,b]$ and f_n is equi-integrable, then f is M_α-integrable and*

$$\lim_{n\to\infty}(M_\alpha)\int_E f_n = (M_\alpha)\int_E f.$$

In the theorem above we impose the condition that $f_n \to f$ everywhere. In order to relax the condition everywhere to almost everywhere we will use the concept of M_α-SL. A collection $\mathcal{F}$ of M_α-SL functions on $[a,b]$ is said to be M_α-USL if given $\epsilon > 0$ and a subset S of $[a,b]$ with measure zero, then there exists a gauge δ on $[a,b]$ such that for any δ-fine S-tagged partial M_α-partition $D = \{(I,x)\}$, and any $F \in \mathcal{F}$, we have $(D)\sum F(I) < \epsilon$.

Theorem 3.8. *Let $\{f_n\}$ be a sequence of functions on $[a,b]$ with corresponding primitives $\{F_n\}$. If $f_n \to f$ almost everywhere on $[a,b]$, f_n is equi-integrable and F_n satisfies the M_α-USL condition, then f is M_α-integrable and*

$$\lim_{n\to\infty}(M_\alpha)\int_{[a,b]} f_n = (M_\alpha)\int_{[a,b]} f.$$

Proof. Let $X = \{x \in [a,b] : f_n(x) \to f(x)\}$. Define

$$f_n^* = f_n\chi_X \quad \text{and} \quad f^* = f\chi_X,$$

where χ_X is the characteristic function of X. One can notice that $f_n^* \to f^*$ everywhere on $[a,b]$. It remains to show that f_n^* is equi-integrable.

Let $\epsilon > 0$. Since f_n is equi-integrable and F_n satisfies M_α-USL there is a gauge δ on $[a,b]$ independent of n such that (i) for any δ-fine M_α-partition $D = \{(I,x)\}$ of $[a,b]$ we have $(D)\sum |f_n(x)|I| - F_n(I)| < \epsilon$ and (ii) for any δ-fine $\{[a,b]\backslash X\}$-tagged partial M_α-partition $D = \{(I,x)\}$ of $[a,b]$, we have $(D)\sum |F_n(I)| < \epsilon$.

Then for any δ-fine M_α-partition $D = \{(I,x)\}$ of $[a,b]$, we have

$$\begin{aligned}(D)\sum |f_n^*(x)|I| - F_n(I)| &\le (D)\sum_{x\in X} |f_n(x)|I| - F_n(I)| + (D)\sum_{x\notin X} |F_n(I)| \\ &< 2\epsilon.\end{aligned}$$

It follows from Theorem 3.7 that f^* is M_α-integrable and therefore, since $f = f^*$ almost everywhere and considering [9, Lemma 2.11], f is M_α-integrable. $\square$

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