COEFFICIENT ESTIMATES FOR FUNCTIONS ASSOCIATED WITH VERTICAL STRIP DOMAIN

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Abstract. In this paper, we consider a convex univalent function \( f_{\alpha,\beta} \) which maps the open unit disc \( \mathbb{U} \) onto the vertical strip domain
\[
\Omega_{\alpha,\beta} = \{ w \in \mathbb{C} : \alpha < \Re(w) < \beta \}
\]
and introduce new subclasses of both close-to-convex and bi-close-to-convex functions with respect to an odd starlike function associated with \( \Omega_{\alpha,\beta} \). Also, we investigate the Fekete-Szegö type coefficient bounds for functions belonging to these classes.

1. Introduction

Assume that \( \mathcal{H} \) is the class of analytic functions in the open unit disc
\[
\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \},
\]
and let the class \( \mathcal{P} \) be defined by
\[
\mathcal{P} = \{ p \in \mathcal{H} : p(0) = 1 \text{ and } \Re(p(z)) > 0 \ (z \in \mathbb{U}) \}.
\]
For two functions \( f, g \in \mathcal{H} \), we say that the function \( f \) is subordinate to \( g \) in \( \mathbb{U} \), and write
\[
f(z) \prec g(z) \quad (z \in \mathbb{U}),
\]
if there exists a Schwarz function
\[
\omega \in \Lambda := \{ \omega \in \mathcal{H} : \omega(0) = 0 \text{ and } |\omega(z)| < 1 \ (z \in \mathbb{U}) \},
\]
such that
\[
f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).
\]
Indeed, it is known that
\[
f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).
\]
Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence

$$f(z) < g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions $f$ normalized by

$$f(0) = f'(0) - 1 = 0.$$ 

Each function $f \in \mathcal{A}$ can be expressed as

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U).$$ 

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $U$.

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha \,(0 \leq \alpha < 1)$, if it satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U).$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are starlike of order $\alpha$ by $\mathcal{S}^*(\alpha)$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

Furthermore a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}(\beta) \,(\beta > 1)$ if it satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in U).$$

This class was introduced by Uralegaddi et al. [12].

Motivated by the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{M}(\beta)$, Kuroki and Owa [7] introduced the subclass $\mathcal{S}(\alpha,\beta)$ of analytic functions $f \in \mathcal{A}$ which is given by Definition 1 below.

**Definition 1** ([7]). Let $\mathcal{S}(\alpha,\beta)$ be a class of functions $f \in \mathcal{A}$ which satisfy the inequality

$$\alpha < \Re \left( \frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in U)$$

for some real number $\alpha \,(\alpha < 1)$ and some real number $\beta \,(\beta > 1)$.

The class $\mathcal{S}(\alpha,\beta)$ is non-empty. For example, the function $f \in \mathcal{A}$ given by

$$f(z) = z \exp \left\{ \frac{\beta - \alpha}{\pi} i \int_0^z \frac{1}{t} \log \left( \frac{1 - e^{2\pi i \frac{z-t}{\beta-\alpha}}}{1-t} \right) dt \right\}$$

is in the class $\mathcal{S}(\alpha,\beta)$.

Also for $f \in \mathcal{S}(\alpha,\beta)$, if $\alpha \geq 0$ then $f \in \mathcal{S}^*(\alpha)$ in $U$, which implies that $f \in \mathcal{S}$. 
Lemma 1.1 ([7]). Let \( f \in \mathcal{A} \) and \( \alpha < 1 < \beta \). Then \( f \in \mathcal{S}(\alpha, \beta) \) if and only if
\[
\frac{zf'(z)}{f(z)} < 1 + \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta}} z}{1 - z} \right) \quad (z \in \mathbb{U}).
\]

Lemma 1.1 means that the function \( f_{\alpha, \beta} : \mathbb{U} \to \mathbb{C} \) defined by
\[
f_{\alpha, \beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta}} z}{1 - z} \right)
\]
is analytic in \( \mathbb{U} \) with \( f_{\alpha, \beta}(0) = 1 \) and maps the open unit disk \( \mathbb{U} \) onto the vertical strip domain
\[
\Omega_{\alpha, \beta} = \{ w \in \mathbb{C} : \alpha < \Re(w) < \beta \}
\]
conformally.

We note that the function \( f_{\alpha, \beta} \) defined by (2) is a convex univalent function in \( \mathbb{U} \) and has the form
\[
f_{\alpha, \beta}(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n,
\]
where
\[
\varphi_n = \frac{\beta - \alpha}{n\pi} i \left( 1 - e^{2n\pi i \frac{1-\alpha}{\beta}} \right) \quad (n \in \mathbb{N}).
\]

Kowalczyk and Leś-Bomba [6] introduced the subclass \( \mathcal{K}_s(\alpha) \) of close-to-convex analytic functions as follows:

Definition 2 ([6]). Let the function \( f \) be analytic in \( \mathbb{U} \) defined by (1). We say that \( f \in \mathcal{K}_s(\alpha) \) \( (0 \leq \alpha < 1) \), if there exists a function \( g \in \mathcal{S}^*(1/2) \) such that
\[
\Re \left( \frac{z^2 f'(z)}{g(z)g(-z)} \right) > \alpha \quad (z \in \mathbb{U}).
\]

In particular, we have the class \( \mathcal{K}_s(0) = \mathcal{K}_s \) introduced and studied by Gao and Zhou [2].

Lemma 1.2 ([2]). If \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2) \), then
\[
\psi(z) = -\frac{g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} - \mathcal{S}^* \subset \mathcal{S},
\]
where the coefficients of the odd-starlike function \( \psi \) satisfy the condition
\[
|B_{2n-1}| = \left| 2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + 2 (-1)^n b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2 \right| \leq 1 \quad \text{for } n \geq 2.
\]
The aforecited work of Kowalczyk and Leś-Bomba [6] was followed by such works as those by Goyal and Goswami [3], Goyal and Singh [4], Wang and Chen [13], Wang et al. [14] and Xu et al. [15].

Here, in our present sequel to the aforecited works of Kuroki and Owa [7] and Kowalczyk and Leś-Bomba [6], we first introduce the following subclasses of analytic functions.

Definition 3. Let \( \alpha \) and \( \beta \) be real such that \( 0 \leq \alpha < 1 < \beta \). We denote by \( K_s(\alpha, \beta) \) the class of functions \( f \in A \) satisfying
\[
\alpha < \Re \left( \frac{z^2 f'(z)}{-g(z)g(-z)} \right) < \beta \quad (z \in \mathbb{U}),
\]
where \( g \in S^*(1/2) \).

Remark 1.3. (i) If we let \( \beta \to \infty \) in Definition 3, then the class \( K_s(\alpha, \beta) \) reduces to the class \( K_s(\alpha) \).

(ii) If we let \( \alpha = 0 \) and \( \beta \to \infty \) in Definition 3, then the class \( K_s(\alpha, \beta) \) reduces to the class \( K_s \).

Using (3) and by the principle of subordination, we can immediately obtain Lemma 1.4.

Lemma 1.4. Let \( \alpha \) and \( \beta \) be real numbers such that \( 0 \leq \alpha < 1 < \beta \) and let the function \( f \in A \) be defined by (1). Then \( f \in K_s(\alpha, \beta) \) if and only if
\[
\frac{z^2 f'(z)}{-g(z)g(-z)} \prec f_{\alpha,\beta}(z),
\]
where \( f_{\alpha,\beta}(z) \) is defined by (2).

On the other hand, since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk \( \mathbb{U} \). In fact, the Koebe one-quarter theorem [1] ensures that the image of \( \mathbb{U} \) under every univalent function \( f \in S \) contains a disk of radius \( 1/4 \). Thus every function \( f \in A \) has an inverse \( f^{-1} \), which is defined by
\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{U})
\]
and
\[
f(f^{-1}(w)) = w \quad \left( |w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right).
\]
In fact, the inverse function \( F = f^{-1} \) is given by
\[
F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

If both the function \( f \) and its inverse function \( f^{-1} \) are univalent in \( \mathbb{U} \), then the function \( f \) is called bi-univalent. We will denote the class which consists of functions \( f \) that are bi-univalent by \( \Sigma \) [8].

Now, we introduce a new subclass of bi-univalent functions as follows:
Definition 4. Let $\alpha$ and $\beta$ be real such that $0 \leq \alpha < 1 < \beta$. A function $f \in \Sigma$ given by (1) is said to be in the class $K_{\Sigma}(\alpha, \beta)$ if there exist the functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^* (1/2), \quad G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in S^* (1/2)$$

and the following conditions are satisfied:

$$\alpha < \Re \left( \frac{z^2 f'(z)}{-g(z)g(-z)} \right) < \beta \quad (z \in \mathbb{U})$$

and

$$\alpha < \Re \left( \frac{w^2 F'(w)}{-G(w)G(-w)} \right) < \beta \quad (w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (8).

Remark 1.5. (i) Letting $\beta \to \infty$, we have the class $K_{\Sigma}^*(\alpha)$ of bi-close-to-convex functions of order $\alpha$ satisfying the conditions

$$\Re \left( \frac{z^2 f'(z)}{-g(z)g(-z)} \right) > \alpha \quad (z \in \mathbb{U})$$

and

$$\Re \left( \frac{w^2 F'(w)}{-G(w)G(-w)} \right) > \alpha \quad (w \in \mathbb{U}).$$

This class introduced and studied by Şeker and Sümer Eker [11].

(ii) Letting $\alpha = 0$ and $\beta \to \infty$, we have the class $K_{\Sigma}^*$ of bi-close-to-convex functions satisfying the conditions

$$\Re \left( \frac{z^2 f'(z)}{-g(z)g(-z)} \right) > 0 \quad (z \in \mathbb{U})$$

and

$$\Re \left( \frac{w^2 F'(w)}{-G(w)G(-w)} \right) > 0 \quad (w \in \mathbb{U}).$$

2. Preliminary lemmas

Lemma 2.1 ([10]). Let the function $g$ given by

$$g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. Also let the function $f$ given by

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \quad (z \in \mathbb{U})$$

be holomorphic in $\mathbb{U}$. If

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

and

$$\Re \left( \frac{f(z)}{g(z)} \right) > 0 \quad (z \in \mathbb{U}).$$
then

\[ |a_k| \leq |b_1| \quad (k \in \mathbb{N}). \]

**Lemma 2.2** ([9]). Let \( p \in \mathcal{P} \) with \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \). Then for any complex number \( \nu \),

\[ |c_2 - \nu c_1^2| \leq 2 \max \{1, |2\nu - 1|\}. \]

**Lemma 2.3** ([5]). For \( 0 \leq \beta < 1 \), let \( f \in \mathcal{A} \) given by (1) belong to the function class \( \mathcal{S}^*(\beta) \). Then for any real number \( \mu \),

\[ |a_3 - \mu a_2^2| \leq (1 - \beta) \max \{1, |3 - 2\beta - 4\mu (1 - \beta)|\}. \]

**Lemma 2.4** ([16]). Let \( k, l \in \mathbb{R} \) and \( z_1, z_2 \in \mathbb{C} \). If \( |z_1| < R \) and \( |z_2| < R \), then

\[
|k + l| z_1 + (k - l) z_2| \leq \begin{cases} 2R |k|, & |k| \geq |l| \\ 2R |l|, & |k| \leq |l| \end{cases} \quad (n \in \mathbb{N}).
\]

### 3. Coefficient estimates for functions in \( \mathcal{K}_s(\alpha, \beta) \)

In this section, we find the upper bound for general coefficient of functions belonging to the class \( \mathcal{K}_s(\alpha, \beta) \) and also solve Fekete-Szegő problem.

**Theorem 3.1.** Let \( \alpha \) and \( \beta \) be real numbers such that \( 0 \leq \alpha < 1 < \beta \) and let the function \( f \in \mathcal{A} \) be defined by (1). If \( f \in \mathcal{K}_s(\alpha, \beta) \), then

\[ |a_{2n}| \leq \frac{(\beta - \alpha)}{\pi} \sin \frac{\pi (1 - \alpha)}{\beta - \alpha} \quad (n \in \mathbb{N}) \]

and

\[ |a_{2n+1}| \leq 1 + \frac{2(\beta - \alpha)n}{\pi} \sin \frac{\pi (1 - \alpha)}{\beta - \alpha} \quad (n \in \mathbb{N}). \]

**Proof.** Let the function \( f \in \mathcal{K}_s(\alpha, \beta) \) be of the form (1). Therefore, there exists a function \( g \in \mathcal{S}^*(1/2) \) so that

\[ \alpha < \Re \left( \frac{z^2 f'(z)}{-g(z)g(-z)} \right) < \beta. \]

Let us set

\[ (11) \quad z^2 f'(z) = z f'(z) \quad (z \in \mathbb{U}), \]

where the function \( \psi \) is defined by (6). Furthermore, by Lemma 1.2, we have following equations:

\[ (12) \quad \psi(z) = -\frac{g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \quad \text{and} \quad |B_{2n-1}| \leq 1. \]

Let us define the function \( p \) by

\[ (13) \quad p(z) = \frac{z f'(z)}{\psi(z)} \quad (z \in \mathbb{U}). \]
Then according to the assertion of Lemma 1.4, we get
\[ p(z) \prec f_{\alpha,\beta}(z) \quad (z \in U), \]
where \( f_{\alpha,\beta}(z) \) is defined by (2). Hence, using Lemma 2.1, we obtain
\[ \left| \frac{p^{(m)}(0)}{m!} \right| = |c_m| \leq |\varphi_1| \quad (m \in \mathbb{N}), \]
where
\[ p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in U) \]
and by (5)
\[ |\varphi_1| = \left| \frac{\beta - \alpha}{\pi} \left( 1 - e^{2\pi i \frac{z}{1 - \beta + \alpha}} \right) \right| = \frac{2 (\beta - \alpha)}{\pi} \sin \frac{\pi (1 - \alpha)}{\beta - \alpha}. \]
Also from (13), we find
\[ zf'(z) = p(z) \psi(z). \]
Since \( G \) is an odd starlike function with \( B_1 = 1 \), in view of (18), we obtain
\[ 2n a_{2n} = c_{2n-1} + c_{2n-3} B_3 + \cdots + c_1 B_{2n-1} \quad (n \in \mathbb{N}) \]
and
\[ (2n + 1) a_{2n+1} = c_{2n} + c_{2n-2} B_3 + \cdots + c_2 B_{2n-1} + B_{2n+1} \quad (n \in \mathbb{N}). \]
Using (15), we get from the equalities (19) and (20)
\[ 2n |a_{2n}| = n |\varphi_1| \quad (n \in \mathbb{N}) \]
and
\[ (2n + 1) |a_{2n+1}| = 1 + n |\varphi_1| \quad (n \in \mathbb{N}), \]
respectively. The desired result is obtain from the equalities (21) and (22) by considering (17).

Letting \( \beta \to \infty \) in Theorem 3.1, we have the coefficient bounds for functions belong to the class \( \mathcal{K}_s(\alpha) \).

**Corollary 3.2.** Let \( \alpha \) be a real number such that \( 0 \leq \alpha < 1 \) and let the function \( f \in \mathcal{A} \) be defined by (1). If \( f \in \mathcal{K}_s(\alpha) \), then
\[ |a_{2n}| \leq 1 - \alpha \quad (n \in \mathbb{N}) \]
and
\[ |a_{2n+1}| \leq \frac{1 + 2 (1 - \alpha) n}{2n + 1} \quad (n \in \mathbb{N}). \]

Letting \( \alpha = 0 \) and \( \beta \to \infty \) in Theorem 3.1, we have the coefficient bounds for functions belong to the class \( \mathcal{K}_s \).

**Corollary 3.3** ([2, Theorem 2]). Let the function \( f \in \mathcal{A} \) be defined by (1). If \( f \in \mathcal{K}_s \), then
\[ |a_n| \leq 1 \quad (n = 2, 3, \ldots). \]
Theorem 3.4. Let $\alpha$ and $\beta$ be real numbers such that $0 \leq \alpha < 1 < \beta$ and let the function $f \in A$ be defined by (1). If $f \in K_s(\alpha, \beta)$, then for any real number $\mu$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} + \frac{2(\beta - \alpha)}{3\pi} \sin \frac{\pi (1 - \alpha)}{\beta - \alpha} \times \max \left\{ 1, \left| \frac{\pi (1 - \alpha)}{\beta - \alpha} - \mu \frac{3(\beta - \alpha)}{2\pi} \sin \frac{\pi (1 - \alpha)}{\beta - \alpha} \right| \right\}. \quad (23)$$

Proof. Let $f \in K_s(\alpha, \beta)$ be given by (1) and by means of the function $f_{\alpha, \beta}$ given by (4), let us define the function $u(z)$ by

$$u(z) = \frac{1 + f_{\alpha, \beta}^{-1}(p(z))}{1 - f_{\alpha, \beta}^{-1}(p(z))} = 1 + u_1 z + u_2 z^2 + \cdots \in \mathcal{P} \quad (z \in \mathbb{U}),$$

where the function $p$ is given by (16) and it satisfies (13). So we have

$$p(z) = f_{\alpha, \beta} \left( \frac{u(z) - 1}{u(z) + 1} \right) \quad (z \in \mathbb{U}).$$

From (4) and (13) we obtain

$$f_{\alpha, \beta} \left( \frac{u(z) - 1}{u(z) + 1} \right) = 1 + \frac{1}{2} \varphi_1 u_1 z + \left[ \frac{1}{2} \varphi_1 \left( u_2 - \frac{1}{2} u_1^2 \right) + \frac{1}{4} \varphi_2 u_1^2 \right] z^2 + \cdots$$

and

$$p(z) = 1 + 2a_2 z + (3a_3 - B_3) z^2 + \cdots,$$

respectively, which implies that

$$2a_2 = \frac{1}{2} \varphi_1 u_1$$

and

$$3a_3 = B_3 + \frac{1}{2} \varphi_1 \left( u_2 - \frac{1}{2} u_1^2 \right) + \frac{1}{4} \varphi_2 u_1^2.$$

So we get

$$a_3 - \mu a_2^2 = \frac{B_3}{3} + \frac{1}{6} \varphi_1 \left( u_2 - \frac{1}{2} u_1^2 \right) + \frac{1}{12} \varphi_2 u_1^2 - \frac{\mu}{16} \varphi^2 u_1^2. \quad (24)$$

If we choose $n = 2$ in (7), then we find that

$$B_3 = 2b_3 - b_2^2. \quad (25)$$

Thus we have

$$a_3 - \mu a_2^2 = \frac{2}{3} \left( b_3 - \frac{1}{2} b_2^2 \right) + \frac{1}{6} \varphi_1 \left( u_2 - \nu u_1^2 \right),$$

where

$$\nu = \frac{1}{2} \left( 1 - \frac{\varphi_2}{\varphi_1} + \frac{3\mu}{4} \varphi_1 \right).$$

Hence, from Lemma 2.2 and Lemma 2.3, we obtain the inequality (23). $\square$
Letting $\beta \to \infty$ in Theorem 3.4, we have the coefficient bounds for functions belong to the class $K_s(\alpha)$.

**Corollary 3.5.** Let $\alpha$ be a real number such that $0 \leq \alpha < 1$ and let the function $f \in A$ be defined by (1). If $f \in K_s(\alpha)$, then for any real number $\mu$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} + \frac{2(1-\alpha)}{3} \max \left\{ 1, \left| 1 - \frac{3(1-\alpha)}{2} \mu \right| \right\}. $$

Letting $\alpha = 0$ and $\beta \to \infty$ in Theorem 3.4, we have the coefficient bounds for functions belong to the class $K_s$.

**Corollary 3.6.** Let the function $f \in A$ be defined by (1). If $f \in K_s$, then for any real number $\mu$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} + \frac{2}{3} \max \left\{ 1, \left| 1 - \frac{3}{2} \mu \right| \right\}. $$

4. Coefficient estimates for functions in $K_{\Sigma_s}(\alpha, \beta)$

In this section, we find the upper bounds for initial coefficients of functions belonging to the class $K_{\Sigma_s}(\alpha, \beta)$ and also solve Fekete-Szegö problem.

**Theorem 4.1.** Let $\alpha$ and $\beta$ be real such that $0 \leq \alpha < 1 < \beta$. If a function $f$ given by (1) is in $K_{\Sigma_s}(\alpha, \beta)$, then

$$|a_2| \leq \frac{(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha}$$

and

$$|a_3| \leq \frac{1 + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha}}{3}. $$

**Proof.** Let $f \in K_{\Sigma_s}(\alpha, \beta)$ be given by (1). Then by Definition 4, there exist the functions 

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*(1/2), \quad G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in S^*(1/2),$$

satisfying (9) and (10). Firstly, we will re-arrange the relations in (9) and (10) as follows:

$$p(z) = \frac{z^2 f'(z)}{-g(z)g(-z)} = \frac{zf'(z)}{-g(z)g(-z)} \psi(z) \prec f_{\alpha,\beta}(z) \quad (z \in U)$$

and

$$q(w) = \frac{w^2 F'(w)}{-G(w)G(-w)} = \frac{wF'(w)}{-G(w)G(-w)} \Omega(w) \prec f_{\alpha,\beta}(w) \quad (w \in U),$$

respectively, where

$$\psi(z) := -\frac{g(z)g(-z)}{z} \quad \text{and} \quad \Omega(w) := -\frac{G(w)G(-w)}{w}.$$
Let \( p \) and \( q \) be two functions with positive real part defined by
\[
p(z) := 1 + c_1 z + c_2 z^2 + \cdots
\]
and
\[
q(w) := 1 + q_1 w + q_2 w^2 + \cdots,
\]
respectively. The relations (28) and (29) imply by Lemma 2.1 that for all \( m \in \mathbb{N} \),
\[
|c_m| \leq |\varphi_1| \tag{30}
\]
and
\[
|q_m| \leq |\varphi_1| \tag{31}
\]
Furthermore, by Lemma 1.2, we have the following equations:
\[
\psi(z) = -g(z)g(-z) := z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in S^* \quad \text{and} \quad |B_{2n-1}| \leq 1, \tag{32}
\]
\[
\Omega(w) = -G(w)G(-w) := w + \sum_{n=2}^{\infty} D_{2n-1} w^{2n-1} \in S^* \quad \text{and} \quad |D_{2n-1}| \leq 1. \tag{33}
\]
Now, upon equating the coefficients in (28) and (29), we obtain
\[
2a_2 = c_1, \tag{34}
\]
\[
3a_3 - B_3 = c_2, \tag{35}
\]
\[
-2a_2 = q_1, \tag{36}
\]
\[
3(2a_2^2 - a_3) - D_3 = q_2. \tag{37}
\]
From (34) and (36), we get
\[
c_1 = -q_1
\]
and
\[
8a_2^2 = c_1^2 + q_1^2. \tag{38}
\]
We thus find (by (30), (31), (32) and (33)) that
\[
|a_2| \leq \frac{|\varphi_1|}{2}. \tag{39}
\]
Further, from the equalities (35) and (37), we find
\[
6a_2^2 - B_3 - D_3 = c_2 + q_2. \tag{40}
\]
Consequently (by (30), (31), (32) and (33)), we have
\[
|a_2| \leq \sqrt{\frac{1 + |\varphi_1|}{3}}. \tag{41}
\]
Hence we get the desired result on the coefficient \( a_2 \) as asserted in (26) from the inequalities (39) and (41).
Now, in order to obtain the bound on the coefficient $a_3$, we subtract (37) from (35). We thus get

$$6(a_3 - a^2_2) - B_3 + D_3 = c_2 - q_2$$

or

$$a_3 = a^2_2 + \frac{c_2 - q_2 + B_3 - D_3}{6}.$$  \hspace{1cm} (42)

Upon substituting the value of $a^2_2$ from (38) into (42), it follows that

$$a_3 = \frac{c^2_1 + q^2_1}{8} + \frac{c_2 - q_2 + B_3 - D_3}{6}.$$  \hspace{1cm} (43)

We thus find (by (30), (31), (32) and (33)) that

$$|a_3| \leq \left| \phi_1 \right|^2 \frac{1 + |\phi_1|}{3}.$$  \hspace{1cm} (44)

On the other hand, upon substituting the value of $a^2_2$ from (40) into (42), it follows that

$$a_3 = \frac{c_2 + q_2 + B_3 + D_3}{6} + \frac{c_2 - q_2 + B_3 - D_3}{6} = \frac{c_2 + B_3}{3}.\hspace{1cm}$$

Consequently (by (30), (31), (32) and (33)), we have

$$|a_3| \leq \frac{1 + |\phi_1|}{3}.$$  \hspace{1cm} (44)

Combining (43) and (44), we get the desired result on the coefficient $a_3$ as asserted in (27). \hspace{1cm} \Box

Letting $\beta \to \infty$ in Theorem 4.1, we have the coefficient bounds for functions belonging to the class $K_{\Sigma} (\alpha)$.

**Corollary 4.2.** Let $\alpha$ be real such that $0 \leq \alpha < 1$. If a function $f$ given by (1) is in $K_{\Sigma_1} (\alpha)$, then

$$|a_2| \leq 1 - \alpha$$

and

$$|a_3| \leq \frac{3 - 2\alpha}{3}.$$  \hspace{1cm} (44)

**Remark 4.3.** We note that Corollary 4.2 is an improvement of the estimates obtained by Şeker and Sümer Eker [11, Theorem 3.2].

Letting $\alpha = 0$ and $\beta \to \infty$ in Theorem 4.1, we have the coefficient bounds for functions belonging to the class $K_{\Sigma}$.

**Corollary 4.4.** If a function $f$ given by (1) is in $K_{\Sigma}$, then

$$|a_2| \leq 1 \quad \text{and} \quad |a_3| \leq 1.$$
Theorem 4.5. Let \( \alpha \) and \( \beta \) be real such that \( 0 \leq \alpha < 1 < \beta \). If a function \( f \) given by (1) is in \( K_{\Sigma}(\alpha, \beta) \), then for any real number \( \delta \),

\[
|a_3 - \delta a_2^3| \leq \frac{1}{3} \left( 1 + \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi (1 - \alpha)}{\beta - \alpha} \right) \begin{cases} 
|1 - \delta| \quad &, \delta \in (-\infty, 0] \cup [2, \infty), \\
1 \quad &, \delta \in [0, 2].
\end{cases}
\]

Proof. By using the equality (42) in the proof of Theorem 4.1, we obtain

\[
a_3 - \delta a_2^3 = (1 - \delta) a_2^3 + \frac{c_2 - q_2 + B_3 - D_3}{6}.
\]

Upon substituting the value of \( a_2^3 \) from (40) into the above equality, it follows that

\[
a_3 - \delta a_2^3 = (1 - \delta) \frac{c_2 + q_2 + B_3 + D_3}{6} + \frac{c_2 - q_2 + B_3 - D_3}{6}
\]

\[
= \frac{1}{6} [(2 - \delta) (c_2 + B_3) - \delta (q_2 + D_3)].
\]

Thus by Lemma 2.4, we get desired estimate.

Letting \( \beta \to \infty \) in Theorem 4.5, we have the coefficient bounds for functions belonging to the class \( K_{\Sigma}(\alpha) \).

Corollary 4.6. Let \( \alpha \) be real such that \( 0 \leq \alpha < 1 \). If a function \( f \) given by (1) is in \( K_{\Sigma}(\alpha) \), then for any real number \( \delta \),

\[
|a_3 - \delta a_2^3| \leq \frac{3 - 2\alpha}{3} \begin{cases} 
|1 - \delta| \quad &, \delta \in (-\infty, 0] \cup [2, \infty), \\
1 \quad &, \delta \in [0, 2].
\end{cases}
\]

Letting \( \alpha = 0 \) and \( \beta \to \infty \) in Theorem 4.5, we have the coefficient bounds for functions belong to the class \( K_{\Sigma} \).

Corollary 4.7. If a function \( f \) given by (1) is in \( K_{\Sigma} \), then for any real number \( \delta \),

\[
|a_3 - \delta a_2^3| \leq \begin{cases} 
|1 - \delta| \quad &, \delta \in (-\infty, 0] \cup [2, \infty), \\
1 \quad &, \delta \in [0, 2].
\end{cases}
\]

References


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