Commun. Korean Math. Soc. **37** (2022), No. 2, pp. 503–520
https://doi.org/10.4134/CKMS.c210098
pISSN: 1225-1763 / eISSN: 2234-3024

A STUDY OF THE RIGHT LOCAL GENERAL TRUNCATED M-FRACTIONAL DERIVATIVE

Rajendrakumar B. Chauhan and Meera H. Chudasama

Abstract. We introduce a new type of fractional derivative, which we call as the right local general truncated M-fractional derivative for α-differentiable functions that generalizes the fractional derivative type introduced by Anastassiou. This newly defined operator generalizes the standard properties and results of the integer order calculus viz. the Rolle's theorem, the mean value theorem and its extension, inverse property, the fundamental theorem of calculus and the theorem of integration by parts. Then we represent a relation of the newly defined fractional derivative with known fractional derivative and in context with this derivative a physical problem, Kirchoff's voltage law, is generalized. Also, the importance of this newly defined operator with respect to the flexibility in the parametric values is described via the comparison of the solutions in the graphs using MATLAB software.

1. Introduction

The study of non-integer order calculus was discovered in 1695 by L'Hospital and Leibniz [7]. Due to its vast applications in the fields like engineering, sciences etc., it has become more popular and interesting among the researchers. The various types of fractional derivatives and integrals have been defined and investigated through the unification of the classical integration and differentiation.

Many varieties of fractional derivatives have been introduced, amongst which the Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, Grünwald-Letnikov, Riesz and other types [9, 10] are worth mentioning. Most of them have the background of the corresponding fractional integral in the Riemann-Liouville sense. But they are non local and they do not have the *fundamental assets* of the ordinary differentiations.

To overcome this, Khalil et al. [6], Katugampola [5], Sousa and Oliveira [14] and Anastassiou [2] have worked in this direction and gave the following

Received March 19, 2021; Revised May 11, 2021; Accepted May 17, 2021.
2010 *Mathematics Subject Classification.* 26A06, 26A24, 26A33, 26A42, 33E12.
Key words and phrases. Generalized derivatives, mean value theorems, truncated Mittag-Leffler function, conformable fractional derivative, alternative fractional derivative, truncated M-fractional derivative, right local general M-fractional derivative.

fractional derivatives in terms of the conformable sense which encompasses the classical properties of integer order calculus. Khalil et al. [6] defined the right conformable fractional derivative of order α as:

Definition 1. Let $f : (-\infty, b] \to \mathbb{R}$, $b \in \mathbb{R}$. Then the right conformable fractional derivative terminating at b of a function f of order α, $\alpha \in (0,1)$, is given by

$$T_b^{(\alpha)} f(t) = -\lim_{\xi \to 0} \frac{f(t + \xi(b-t)^{1-\alpha}) - f(t)}{\xi}. \tag{1}$$

If $T_b^{(\alpha)} f(t)$ exists on (a,b), $a < b$ and $\lim_{t \to b^-} T_b^{(\alpha)} f(t)$ exists, then $T_b^{(\alpha)} f(b) = \lim_{t \to b^-} T_b^{(\alpha)} f(t)$.

Many of the researchers have studied the conformable fractional derivative with various applications [1,4]. Moreover, in 2014, Katugampola [5] has proposed a new fractional derivative with classical properties similar to the conformable fractional derivative as follows:

Definition 2. Let $f : [0,\infty) \to \mathbb{R}$. Then the alternative fractional derivative of order α is defined as

$$\mathbf{D}^\alpha f(t) = \lim_{\xi \to 0} \frac{f(t\, e^{\xi t^{-\alpha}}) - f(t)}{\xi} \tag{2}$$

for all $t > 0$ and $\alpha \in (0,1)$.

In 2017, Sousa and Oliveira [14] have defined a generalization of the usual definition of a derivative as follows:

Definition 3. Let $f : [0,\infty) \to \mathbb{R}$. Then for all $t > 0$ and $\alpha \in (0,1)$, the local M-derivative of order α of f is defined as

$$\mathbb{D}_M^{\alpha,\beta} f(t) = \lim_{\xi \to 0} \frac{f\left(t\, \mathbb{E}_\beta\left(\xi t^{-\alpha}\right)\right) - f(t)}{\xi}, \tag{3}$$

where $\mathbb{E}_\beta(\cdot)$, $\beta > 0$ is the Mittag-Leffler function with one parameter [3,8].

Sousa and Oliveira [12,13] have defined the truncated M-fractional derivative with the aid of the truncated Mittag-Leffler function of one parameter defined by

$$_i\mathbb{E}_\beta(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(\beta k + 1)}, \tag{4}$$

with $\beta > 0$ and $z \in \mathbb{C}$ as follows:

Definition 4. Let $f : [0,\infty) \to \mathbb{R}$. Then for all $t > 0$ and $\alpha \in (0,1)$, a truncated M-fractional derivative of order α of f is defined as

$$_i\mathcal{D}_M^{\alpha,\beta} f(t) = \lim_{\xi \to 0} \frac{f\left(t\, {}_i\mathbb{E}_\beta\left(\xi t^{-\alpha}\right)\right) - f(t)}{\xi}, \tag{5}$$

where ${}_i\mathbb{E}_\beta(\cdot),\ \beta > 0$ is a truncated Mittag-Leffler function with one parameter.

As a generalization of a truncated M-fractional derivative, Sousa and Oliveira have defined a truncated ν-fractional derivative [13]. In 2019, Anastassiou [2] has defined the right local general M-fractional derivative as follows:

Definition 5. Let $f : (-\infty, b] \to \mathbb{R}$ and $t < b,\ b \in \mathbb{R}$. For $\alpha \in (0, 1]$, the right local general M-fractional derivative of order α of f is defined as

$$\mathfrak{D}^{\alpha,\beta}_{M,b} f(t) = -\lim_{\xi \to 0} \frac{f\left(t\ \mathbb{E}_\beta\left(\xi (b-t)^{-\alpha}\right)\right) - f(t)}{\xi}, \tag{6}$$

where $\mathbb{E}_\beta(\cdot),\ \beta > 0$ is a Mittag-Leffler function with one parameter.

By focusing on all these definitions, we now generalize the right local general M-fractional derivative given in (6) by adding a flavour of truncated Mittag-leffler function (4).

2. Main results

In this section, we first introduce the new structure of the fractional derivative using which the various results having likeness to the results of classical calculus are obtained. We begin with the following definition, which is the generalization of (6).

Definition 6. Let $f : (-\infty, b] \to \mathbb{R}$ and $t < b,\ b \in \mathbb{R}$. For $0 < \alpha \leq 1$, we define the right local general truncated M-fractional derivative of order α of f (α-RLGT M-fractional derivative) as

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(t) := -\lim_{\xi \to 0} \frac{f\left(t\ {}_i\mathbb{E}_\beta(\xi (b-t)^{-\alpha})\right) - f(t)}{\xi}, \tag{7}$$

where ${}_i\mathbb{E}_\beta(\cdot)$ is the truncated Mittag-Leffler function of one parameter as defined in (4).

Taking the limit $i \to \infty$ on both the sides of (7), we get

$${}_\infty\mathfrak{D}^{\alpha,\beta}_{M,b} f(t) = -\lim_{\xi \to 0} \frac{f\left(t\ {}_\infty\mathbb{E}_\beta(\xi (b-t)^{-\alpha})\right) - f(t)}{\xi}.$$

But from (4)

$${}_\infty\mathbb{E}_\beta\left(\xi (b-t)^{-\alpha}\right) = \sum_{k=0}^{\infty} \frac{(\xi (b-t)^{-\alpha})^k}{\Gamma(\beta k + 1)} = \mathbb{E}_\beta(\xi (b-t)^{-\alpha}).$$

Thus, we conclude that

$${}_\infty\mathfrak{D}^{\alpha,\beta}_{M,b} f(t) = -\lim_{\xi \to 0} \frac{f\left(t\ \mathbb{E}_\beta(\xi (b-t)^{-\alpha})\right) - f(t)}{\xi} = \mathfrak{D}^{\alpha,\beta}_{M,b} f(t), \tag{8}$$

which is the right local general M-fractional derivative given in (6).

Now, if α-RLGT M-fractional derivative exists in some open interval (δ, b), $\delta \in \mathbb{R}$, $\delta < b$ and $\lim_{t \to b^-} {}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(t)$ exists, then

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(b) = \lim_{t \to b^-} {}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(t). \tag{9} $$

Next, we try to establish the generalization of the result "Every differentiable function is continuous." in the following theorem in context of α-RLGT M-fractional derivative.

Theorem 2.1. *If a function $f : (\infty, b] \to \mathbb{R}$ is α-RLGT M-fractional differentiable at t_0, $t_0 < b$, then f is continuous at t_0.*

Proof. For $\xi \neq 0$, consider

$$ \begin{aligned} & - \left[f\left(t_0 \; {}_i\mathbb{E}_\beta(\xi(b-t_0)^{-\alpha})\right) - f(t_0) \right] \\ = & - \left(\frac{f\left(t_0 \; {}_i\mathbb{E}_\beta(\xi(b-t_0)^{-\alpha})\right) - f(t_0)}{\xi} \right) \xi. \end{aligned} \tag{10} $$

Now, applying the limit $\xi \to 0$ on both the sides of (10), we have

$$ \begin{aligned} & - \lim_{\xi \to 0} \left(f\left(t_0 \; {}_i\mathbb{E}_\beta(\xi(b-t_0)^{-\alpha})\right) - f(t_0) \right) \\ = & - \lim_{\xi \to 0} \left(\frac{f\left(t_0 \; {}_i\mathbb{E}_\beta(\xi(b-t_0)^{-\alpha})\right) - f(t_0)}{\xi} \right) \times \lim_{\xi \to 0} \xi \\ = & \; {}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(t_0) \lim_{\xi \to 0} \xi \\ = & \; 0. \end{aligned} $$

Hence, f is continuous at t_0. □

Using the truncated Mittag-Leffler function of one parameter, we have the following lemma.

Lemma 2.2. *Let $f : (-\infty, b] \to \mathbb{R}$ be continuous and $t < b, b \in \mathbb{R}$. For $0 < \alpha \leq 1$ and $\beta > 0$,*

$$ \lim_{\xi \to 0} f\left(t \; {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})\right) = f(t). $$

Proof. Using (4), we have

$$ f\left(t \; {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})\right) = f\left(t \sum_{k=0}^{i} \frac{(\xi(b-t)^{-\alpha})^k}{\Gamma(\beta k + 1)}\right). \tag{11} $$

By applying the limit $\xi \to 0$ on both the sides of (11), continuity of f yields

$$ \begin{aligned} & \lim_{\xi \to 0} f\left(t \; {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})\right) \\ = & \lim_{\xi \to 0} f\left(t \sum_{k=0}^{i} \frac{(\xi(b-t)^{-\alpha})^k}{\Gamma(\beta k + 1)}\right) \end{aligned} $$

$$= f\left(t \lim_{\xi\to 0}\sum_{k=0}^{i}\frac{(\xi(b-t)^{-\alpha})^k}{\Gamma(\beta k+1)}\right)$$

$$= f\Bigg(\lim_{\xi\to 0}\Bigg\{t+\frac{t\ \xi(b-t)^{-\alpha}}{\Gamma(\beta+1)}+\frac{t\left(\xi(b-t)^{-\alpha}\right)^2}{\Gamma(2\beta+1)}+\frac{t\left(\xi(b-t)^{-\alpha}\right)^3}{\Gamma(3\beta+1)}$$
$$+\cdots+\frac{t\left(\xi(b-t)^{-\alpha}\right)^i}{\Gamma(i\beta+1)}\Bigg\}\Bigg)$$
$$= f(t).$$
□

In the next theorem, a relation between α-RLGT M-fractional derivative and classical derivative is obtained.

Theorem 2.3. *If $f : (-\infty, b] \to \mathbb{R}$ has the α-RLGT M-fractional derivative at $t,\ t < b$ with $\beta > 0$, then*

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t) = -\frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}f'(t). \tag{12}$$

Proof. For $t < b$, from (4), we have

$$t\ {}_i\mathbb{E}_\beta\left(\xi(b-t)^{-\alpha}\right) = t+\frac{t\ \xi(b-t)^{-\alpha}}{\Gamma(\beta+1)}+\frac{t\left(\xi(b-t)^{-\alpha}\right)^2}{\Gamma(2\beta+1)}+\frac{t\left(\xi(b-t)^{-\alpha}\right)^3}{\Gamma(3\beta+1)}$$
$$+\cdots+\frac{t\left(\xi(b-t)^{-\alpha}\right)^i}{\Gamma(i\beta+1)}$$
$$= t+\frac{t\ \xi(b-t)^{-\alpha}}{\Gamma(\beta+1)}+\mathcal{O}(\xi^2). \tag{13}$$

Let

$$h := t\ \xi(b-t)^{-\alpha}\left(\frac{1}{\Gamma(\beta+1)}+\mathcal{O}(\xi^2)\right). \tag{14}$$

Then

$$\xi = \frac{h}{t(b-t)^{-\alpha}\left(\frac{1}{\Gamma(\beta+1)}+\mathcal{O}(\xi^2)\right)}$$
$$= \frac{h\ (b-t)^{\alpha}\Gamma(\beta+1)}{t\left(1+\Gamma(\beta+1)\mathcal{O}(\xi^2)\right)}.$$

Now, from Definition 6 and (13), we have

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t) = -\lim_{\xi\to 0}\frac{f\left(t+\frac{t\ \xi(b-t)^{-\alpha}}{\Gamma(\beta+1)}+\mathcal{O}(\xi^2)\right)-f(t)}{\xi}.$$

Then from (14), the above expression becomes

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t) = -\lim_{\xi\to 0}\frac{f(t+h)-f(t)}{\xi}$$

$$= -\lim_{\xi\to 0} \frac{(f(t+h)-f(t))\left(t\left(1+\Gamma(\beta+1)\mathcal{O}(\xi^2)\right)\right)}{h(b-t)^{\alpha}\Gamma(\beta+1)}$$
$$= -\frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\left[\lim_{\xi\to 0}\left(\frac{f(t+h)-f(t)}{h}\right)\lim_{\xi\to 0}\left(1+\Gamma(\beta+1)\mathcal{O}(\xi^2)\right)\right]$$
$$= -\frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}f'(t) \quad \text{as if } \xi\to 0, \text{ then } h\to 0. \qquad \square$$

Remark 2.4. From Theorem 2.3, if $f(t) = c$, where c is any constant, then ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t) = 0$ as $f'(t) = 0$.

Remark 2.5. For $\alpha = 1, b = 0$ and $\beta = 0$ or 1, (12) becomes ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t) = -f'(t)$.

Now, we will derive a theorem that encompasses the classical properties of integer order derivatives.

Theorem 2.6. *Let $f_1, f_2 : (-\infty, b] \to \mathbb{R}$ be α-RLGT M-fractional differentiable at t, $t < b$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\beta > 0$. Then*

(1) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(\mu_1 f_1 + \mu_2 f_2)(t) = \mu_1\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(t) + \mu_2\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t)$.
(2) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(f_1\cdot f_2)(t) = f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t) + f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(t)$.
(3) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\left(\frac{f_1}{f_2}\right)(t) = \frac{f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(t) - f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t)}{[f_2(t)]^2}$.
(4) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(k) = 0$, *where k is a constant.*
(5) *If $f_1(t)$ is differentiable at $f_2(t)$, then*
$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(f_1 o f_2)(t) = f_1'(f_2(t))\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t).$$

Proof. (1) From Definition 6, we have

$$\begin{aligned}
&{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(\mu_1 f_1 + \mu_2 f_2)(t)\\
&= -\lim_{\xi\to 0}\left(\frac{(\mu_1 f_1+\mu_2 f_2)(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - (\mu_1 f_1+\mu_2 f_2)(t)}{\xi}\right)\\
&= -\lim_{\xi\to 0}\left(\frac{\mu_1\ f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) + \mu_2\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - \mu_1 f_1(t) - \mu_2 f_2(t)}{\xi}\right)\\
&= -\left(\lim_{\xi\to 0}\frac{\mu_1\ f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - \mu_1 f_1(t)}{\xi}\right)\\
&\quad + \left(-\left(\lim_{\xi\to 0}\frac{\mu_2\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - \mu_2 f_2(t)}{\xi}\right)\right)\\
&= \mu_1\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(t) + \mu_2\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t).
\end{aligned}$$

(2) From Definition 6, we have

$$\begin{aligned}
&{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(f_1\cdot f_2)(t)\\
&= -\lim_{\xi\to 0}\frac{f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))\cdot f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - f_1(t)\cdot f_2(t)}{\xi}
\end{aligned}$$

$$= -\lim_{\xi\to 0}\left\{\begin{array}{c} f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))\cdot f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))-f_1(t)\cdot f_2(t)\\ +f_1(t)f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))-f_1(t)f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))\end{array}\right\}/\xi$$

$$= \left(-\lim_{\xi\to 0}\frac{f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))-f_1(t)}{\xi}\right)\lim_{\xi\to 0} f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))$$
$$+\left(-\lim_{\xi\to 0}\frac{f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))-f_2(t)}{\xi}\right)\lim_{\xi\to 0} f_1(t).$$

Using Lemma 2.2 and from Definition 6, we have

$$\begin{aligned}{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(f_1\cdot f_2)(t) &= {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(t)\ f_2(t) + {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t)\ f_1(t)\\ &= f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t)+f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(t).\end{aligned}$$

(3) Again with the aid of Definition 6, we have

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\left(\frac{f_1}{f_2}\right)(t)$$
$$= -\lim_{\xi\to 0}\frac{\frac{f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))}{f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))}-\frac{f_1(t)}{f_2(t)}}{\xi}$$
$$= -\lim_{\xi\to 0}\frac{f_2(t)\ f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))-f_1(t)\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))}{\xi f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))\ f_2(t)}$$
$$-\lim_{\xi\to 0}\frac{f_1(t)f_2(t)}{\xi f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))\ f_2(t)}+\lim_{\xi\to 0}\frac{f_1(t)f_2(t)}{\xi f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))\ f_2(t)}$$
$$= \frac{-\lim\limits_{\xi\to 0}\frac{f_2(t)\left(f_1(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))-f_1(t)\right)}{\xi}-\lim\limits_{\xi\to 0}\frac{f_1(t)\left(f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))-f_2(t)\right)}{\xi}}{\lim\limits_{\xi\to 0}f_2(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))f_2(t)}$$
$$= \frac{f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(t)-f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(t)}{[f_2(t)]^2}\ \text{as}\ \lim_{\xi\to 0}f_2\left(t\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})\right)=f_2(t).$$

(4) In this case, the proof is directly follows from Remark 2.4.

(5) This result is proved in two cases: (I) f_2 is constant and (II) f_2 is non-constant.

Case-I: Let $f_2(t)=c$, where c is any constant.
Then from Remark 2.4, we have

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\ (f_1 o f_2)(c) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\ f_1(f_2(t)) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\ f_1(c)=0.$$

Case-II: Let f_2 be not a constant in the neighborhood of c.
Since f_2 is continuous at c, for ξ to be small enough, we have

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(f_1 o f_2)(c)$$
$$= -\lim_{\xi\to 0}\frac{f_1(f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})))-f_1(f_2(c))}{\xi}$$

$$= -\lim_{\xi\to 0}\frac{f_1(f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))) - f_1(f_2(c))}{\xi}\ \frac{f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - f_2(c)}{f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - f_2(c)}$$
$$= -\lim_{\xi\to 0}\frac{f_1(f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha}))) - f_1(f_2(c))}{f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - f_2(c)}$$
$$\times \lim_{\xi\to 0}\frac{f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - f_2(c)}{\xi}.$$

Now, let

$$\xi_1 = f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - f_2(c).$$

Then

$$f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) = \xi_1 + f_2(c).$$

Also, it is observed that if $\xi \to 0$, then $\xi_1 \to 0$.
Therefore,

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(f_1 o f_2)(c)$$
$$= \lim_{\xi_1\to 0}\frac{f_1(f_2(c)+\xi_1) - f_1(f_2(c))}{\xi_1} \times -\lim_{\xi\to 0}\frac{f_2(c\ {}_i\mathbb{E}_\beta(\xi(b-t)^{-\alpha})) - f_2(c)}{\xi}$$
$$= f_1^{'}(f_2(c))\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f_2(c).$$

Hence,

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(f_1 o f_2)(t) = f_1^{'}(f_2(t))\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f_2(t). \qquad \square$$

Now, as a consequence of Theorem 2.3, we have the following α-RLGT M-fractional derivatives of various functions.

Theorem 2.7. *Let $\mu \in \mathbb{R}$, $\beta > 0$, $\alpha \in (0,1]$ and $t < b$. Then*

(1) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(1) = 0$;

(2) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(e^{\mu t}) = -\frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\mu\ e^{\mu t}$;

(3) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(\sin \mu t) = -\frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\mu \cos \mu t$;

(4) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(\cos \mu t) = \frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\mu \sin \mu t$;

(5) ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(t^\mu) = -\frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\mu t^{\mu-1} = -\frac{(b-t)^{-\alpha}\mu\ t^\mu}{\Gamma(\beta+1)}$.

Proof. The proof is directly follows from Theorem 2.3. $\square$

2.1. Generalization of fundamental results of calculus through α-RLGT M-fractional derivative

Further, we have observed that the α-RLGT M-fractional derivative also has various important theorems similar to the classical integer order calculus. We have derived the Rolle's theorem, the mean value theorem and its extension using this newly defined fractional derivative in the next three theorems.

Theorem 2.8. *Let $f : [\gamma, \rho] \to \mathbb{R}$, where $\rho < b$. If*

(1) *f is continuous on $[\gamma, \rho]$,*
(2) *f is α-RLGT M-fractional differentiable on (γ, ρ),*
(3) *$f(\gamma) = f(\rho)$,*

then there exists $c \in (\gamma, \rho)$ such that ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(c) = 0,\ \beta > 0$.

Proof. We will prove this theorem in three cases:

Case-I: When $f(x) = k$ on $[\gamma, \rho]$, where k is any constant.
Then from Remark 2.4, ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(x) = 0$ for all $x \in [\gamma, \rho]$. That is, in other words, we can say that there exists $c \in (\gamma, \rho)$ such that

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(c) = 0. $$

Case-II: Let f be non-constant. In this case, suppose that there is some $d \in (\gamma, \rho)$ such that $f(d) > f(\gamma)$.
Since f is continuous on $[\gamma, \rho]$, by the extreme value theorem [11], $f(x)$ has maximum in $[\gamma, \rho]$. Also, as $f(\gamma) = f(\rho)$ and $f(d) > f(\gamma)$, we have the maximum value of f at some c in (γ, ρ). Here, c occurs in the interior of the interval means that $f(x)$ has relative maximum at $x = c$ and by the second hypothesis, ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(x)$ exists. Therefore, ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(c) = 0$.

Case-III: Let f be non-constant, but in this case, suppose that there is some $d \in (\gamma, \rho)$ such that $f(d) < f(\gamma)$.
Now, in the similar manner of Case-II, by extreme value theorem [11], $f(x)$ has minimum in $[\gamma, \rho]$. Also, as $f(\gamma) = f(\rho)$ and $f(d) < f(\gamma)$, we have the minimum value of f at some c in (γ, ρ). Hence, ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(c) = 0$. □

Theorem 2.9. *Let $f : [\gamma, \rho] \to \mathbb{R}$, where $\rho < b$, $0 \notin [\gamma, \rho]$. If*

(1) *f is continuous on $[\gamma, \rho]$,*
(2) *f is α-RLGT M-fractional differentiable on (γ, ρ),*

then there exists $c \in (\gamma, \rho)$ such that

$$ \frac{f(\rho) - f(\gamma)}{\rho - \gamma} = \left(-{}_i\mathfrak{D}^{\alpha,\beta}_{M,b} f(c) \right) \frac{\Gamma(\beta+1)(b-c)^{\alpha}}{c}. \tag{15} $$

Proof. For $x \in [\gamma, \rho]$, consider

$$ g(x) := f(x) - f(\gamma) - \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma} \right)(x - \gamma). \tag{16} $$

Since f is continuous on $[\rho, \gamma]$, g is continuous on $[\rho, \gamma]$ too. Also, it can be easily verified that $g(\gamma) = 0 = g(\rho)$. Therefore, from Theorem 2.6, we can say that g is α-RLGT M-fractional differentiable on (γ, ρ).

Now, from Theorem 2.8, there exists $c \in (\gamma, \rho)$ such that

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} g(c) = 0. \tag{17} $$

Taking ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}$ on both the sides of (16), we get

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}g(x) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(x) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(\gamma) - \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}(x-\gamma). $$

Applying Theorem 2.3 by taking f to be linear function, we obtain

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}g(x) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(x) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(\gamma) + \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{x(b-x)^{-\alpha}}{\Gamma(\beta+1)}. $$

Whence at $x=c$,

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}g(c) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(c) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(\gamma) + \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{c(b-c)^{-\alpha}}{\Gamma(\beta+1)}. $$

Then using (17), we get

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(c) - 0 + \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{c(b-c)^{-\alpha}}{\Gamma(\beta+1)} = 0. $$

Therefore,

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(c) = -\left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{c(b-c)^{-\alpha}}{\Gamma(\beta+1)}. $$

Hence,

$$ \frac{f(\rho)-f(\gamma)}{\rho-\gamma} = \left(-{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(c)\right) \frac{\Gamma(\beta+1)(b-c)^{\alpha}}{c}. $$

□

Theorem 2.10. *Let $\rho < b$, $0 \notin [\gamma,\rho]$ and $f_1, f_2 : [\gamma,\rho] \to \mathbb{R}$. If*

(1) *f_1, f_2 are continuous on $[\gamma,\rho]$ and $f_2(\gamma) \neq f_2(\rho)$,*

(2) *f_1, f_2 is α-RLGT M-fractional differentiable on (γ,ρ),*

then there exists $c \in (\gamma,\rho)$ such that

$$ \frac{{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(c)}{{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(c)} = \frac{f_1(\rho)-f_1(\gamma)}{f_2(\rho)-f_2(\gamma)} \quad \textit{with } \beta > 0. $$

Proof. For $x \in [\gamma,\rho]$, define

$$ G(x) := f_1(x) - f_2(\gamma) - \left(\frac{f_1(\rho)-f_1(\gamma)}{f_2(\rho)-f_2(\gamma)}\right)(f_2(x)-f_2(\gamma)). \tag{18} $$

Since f_1, f_2 are continuous on $[\rho,\gamma]$, G is continuous on $[\rho,\gamma]$ too. Also, it can be easily seen that $G(\gamma) = 0 = G(\rho)$. Therefore, from Theorem 2.6, we can say that f_1, f_2 are α-RLGT M-fractional differentiable functions on (γ,ρ).

Now, from Theorem 2.8, there exists $c \in (\gamma,\rho)$ such that

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}G(c) = 0. \tag{19} $$

Taking ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}$ on both the sides of (18), we get

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}G(x) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(x) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(\gamma) $$

$$-\left(\frac{f_1(\rho)-f_1(\gamma)}{f_2(\rho)-f_2(\gamma)}\right)\,{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\left(f_2(x)-f_2(\gamma)\right).$$

Writing the above expression at $x=c$ and then applying Remark 2.4, we obtain

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}G(c)={}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(c)-0-\left(\frac{f_1(\rho)-f_1(\gamma)}{f_2(\rho)-f_2(\gamma)}\right){}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(c)-0,$$

which implies from (19) as

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(c)-\left(\frac{f_1(\rho)-f_1(\gamma)}{f_2(\rho)-f_2(\gamma)}\right){}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(c)=0.$$

Therefore,

$$\frac{{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(c)}{{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(c)}=\frac{f_1(\rho)-f_1(\gamma)}{f_2(\rho)-f_2(\gamma)}.$$

□

Definition 7. Let $\beta>0$, $\alpha\in(n,n+1]$, where $n\in\mathbb{N}\cup\{0\}$ and f is n-times differentiable for t, $t<b$. Then the general form of α-RLGT M-fractional derivative of order α of function f is defined by

$$(20)\qquad {}_i\mathfrak{D}^{\alpha,\beta;n}_{M,b}f(t):=\lim_{\xi\to 0}\frac{f^{(n)}\left(t\,{}_i\mathbb{E}_\beta(\xi(b-t)^{n-\alpha})\right)-f^{(n)}(t)}{\xi},$$

if the limit exists.

Now, from the above definition, Theorem 2.3 and by the principle of mathematical induction on n, we have for $t<b$

$$(21)\qquad {}_i\mathfrak{D}^{\alpha,\beta;n}_{M,b}f(t)=(-1)^{n+1}\frac{t(b-t)^{n-\alpha}}{\Gamma(\beta+1)}f^{(n+1)}(t),$$

for f to be $(n+1)$-times differentiable.

Further, this α-RLGT M-fractional derivative has a corresponding right M-integral.

Definition 8. Let $t<b$ and f be a function defined in $[t,b)$ and $\alpha\in(0,1]$. Then the right M-integral of order α of f is defined by

$$(22)\qquad \mathfrak{I}^{\alpha,\beta}_{M,b}f(t)=\Gamma(\beta+1)\int_t^b\frac{f(x)}{x(b-x)^{-\alpha}}\,dx,$$

with $\beta>0$.

In context with the above definition, we have generalized the inverse property, fundamental theorem of calculus and the theorem of integration by parts in the upcoming theorems.

Theorem 2.11. *Let $b\in\mathbb{R}$, $\alpha\in(0,1]$ and f be a continuous function such that there exists $\mathfrak{I}^{\alpha,\beta}_{M,b}f$. Then*

$$(23)\qquad {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\,\mathfrak{I}^{\alpha,\beta}_{M,b}f(t)=f(t),$$

with $0\neq t<b$ and $\beta>0$.

Proof. From Theorem 2.3, we have

$$\begin{aligned}
{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}\left(\mathfrak{I}^{\alpha,\beta}_{M,b}f(t)\right) &= -\frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\left(\mathfrak{I}^{\alpha,\beta}_{M,b}f(t)\right)' \\
&= \frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\frac{d}{dt}\left(\Gamma(\beta+1)\int_b^t \frac{f(x)}{x(b-x)^{-\alpha}}\,dx\right) \\
&= \frac{t(b-t)^{-\alpha}}{\Gamma(\beta+1)}\ \Gamma(\beta+1)\ \frac{f(t)}{t(b-t)^{-\alpha}} \\
&= f(t).
\end{aligned}$$

□

Theorem 2.12. *Let $f:(-\infty,b]\to\mathbb{R}$ be a continuously differentiable function such that ${}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f$ exists and $\alpha\in(0,1]$. Then for all $t<b$,*

$$\mathfrak{I}^{\alpha,\beta}_{M,b}\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t) = f(t)-f(b), \tag{24}$$

with $\beta>0$.

Proof. From Definition 8 and then applying Theorem 2.3, we have

$$\begin{aligned}
\mathfrak{I}^{\alpha,\beta}_{M,b}\left({}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t)\right) &= \Gamma(\beta+1)\int_t^b \frac{{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(x)}{x(b-x)^{-\alpha}}\,dx \\
&= \Gamma(\beta+1)\int_t^b \frac{1}{x(b-x)^{-\alpha}}\left(\frac{-x(b-x)^{-\alpha}}{\Gamma(\beta+1)}f'(x)\right)dx \\
&= \int_b^t f'(x)\,dx \\
&= f(t)-f(b),
\end{aligned}$$

by the classical fundamental theorem of calculus. □

It can be easily observed that, if $f(b)=0$, then by (24) for all $t<b$, we have

$$\mathfrak{I}^{\alpha,\beta}_{M,b}\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f(t) = f(t).$$

Now, for the sake of brevity, we denote

$$\mathfrak{I}^{\alpha,\beta}_{M,b}f(t) = -\int_t^b f(x)\,d_{\alpha,\beta}\ x, \quad \text{where}\, d_{\alpha,\beta}\ x = -\frac{\Gamma(\beta+1)}{x(b-x)^{-\alpha}}\,dx.$$

In this notation, we derive the generalization of the integration by parts in the following theorem for the right M-integral.

Theorem 2.13. *Let $f_1,f_2:[c,d]\to\mathbb{R}$ be continuously differentiable and $\alpha\in(0,1]$. Then*

$$\int_c^d f_1(x)\,{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_2(x)\,d_{\alpha,\beta}x = [f_1(x)f_2(x)]_c^d - \int_c^d f_2(x)\,{}_i\mathfrak{D}^{\alpha,\beta}_{M,b}f_1(x)d_{\alpha,\beta}x,\ \beta>0.$$

Proof. In the stated notations,

$$\begin{aligned}
&\int_c^d f_1(x)\ {}_i\mathfrak{D}_{M,b}^{\alpha,\beta} f_2(x)\, d_{\alpha,\beta}\ x \\
&= \int_c^d f_1(x)\ {}_i\mathfrak{D}_{M,b}^{\alpha,\beta} f_2(x) \left(-\frac{\Gamma(\beta+1)}{x(b-x)^{-\alpha}}\right) dx \\
&= \int_c^d f_1(x) \left(-\frac{x(b-x)^{-\alpha}}{\Gamma(\beta+1)}\right) f_2^{'}(x) \left(-\frac{\Gamma(\beta+1)}{x(b-x)^{-\alpha}}\right) dx,
\end{aligned}$$

by Theorem 2.3.

Now, applying the classical integration by parts, we get

$$\begin{aligned}
&\int_c^d f_1(x)\ {}_i\mathfrak{D}_{M,b}^{\alpha,\beta}\ f_2(x)\, d_{\alpha,\beta} x \\
&= \int_c^d f_1(x)\ f_2^{'}(x) dx \\
&= [f_1(x) f_2(x)]_c^d - \int_c^d f_1^{'}(x)\ f_2(x)\, dx \\
&= [f_1(x) f_2(x)]_c^d - \int_c^d f_2(x) \left(-\frac{x(b-x)^{-\alpha}}{\Gamma(\beta+1)}\right) f_1^{'}(x) \left(-\frac{\Gamma(\beta+1)}{x(b-x)^{-\alpha}}\right) dx \\
&= [f_1(x) f_2(x)]_c^d - \int_c^d f_2(x)\ {}_i\mathfrak{D}_{M,b}^{\alpha,\beta}\ f_1(x)\, d_{\alpha,\beta} x,
\end{aligned}$$

by again using Theorem 2.3. □

The general form of the right M-integral is as follows:

Definition 9. Let $t < b$ and f be a function defined in $[t, b)$ and $\alpha \in (n, n+1]$, $n \in \mathbb{N} \cup \{0\}$. Then the general right M-integral of order α of f is defined as

$$\mathfrak{I}_{M,b}^{\alpha,\beta;n} f(t) = \frac{\Gamma(\beta+1)}{n!} \int_t^b \frac{(x-t)^n}{x(b-x)^{n-\alpha}} f(x)\, dx. \tag{25}$$

Clearly, for $n = 0$, $\mathfrak{I}_{M,b}^{\alpha,\beta;0} f(t) = \mathfrak{I}_{M,b}^{\alpha,\beta} f(t)$.

Next, we derive a right fractional Taylor's theorem with integral remainder associated to the above definition.

Theorem 2.14. *Let $f : (-\infty, b] \to \mathbb{R}$ be $(n+1)$ times continuously differentiable for $t < b$ with $\beta > 0$ and $\alpha \in (n, n+1]$, $n \in \mathbb{N}$. Then for all $t < b$,*

$$\mathfrak{I}_{M,b}^{\alpha,\beta;n} \left({}_i\mathfrak{D}_{M,b}^{\alpha,\beta;n} f(t)\right) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(b)(t-b)^k}{k!}. \tag{26}$$

Proof. From Definition 9, we have

$$
\begin{aligned}
&\mathfrak{I}_{M,b}^{\alpha,\beta;n}\left({}_i\mathfrak{D}_{M,b}^{\alpha,\beta;n} f(t)\right)\\
&= \frac{\Gamma(\beta+1)}{n!}\int_t^b \frac{(x-t)^n}{x(b-x)^{n-\alpha}}\ {}_i\mathfrak{D}_{M,b}^{\alpha,\beta;n} f(x)\,dx\\
&= \frac{\Gamma(\beta+1)}{n!}\int_t^b \frac{(x-t)^n}{x(b-x)^{n-\alpha}}\ \left((-1)^{n+1}\ \frac{x(b-x)^{n-\alpha}}{\Gamma(\beta+1)} f^{(n+1)}(x)\right)\,dx\\
&= \frac{(-1)^{n+1}}{n!}\int_t^b (x-t)^n\ f^{(n+1)}(x)\,dx\\
&= -\ \frac{(-1)^{n+1}}{n!}\ (-1)^n\int_b^t (t-x)^n\ f^{(n+1)}(x)\,dx\\
&= \frac{1}{n!}\int_b^t (t-x)^n\ f^{(n+1)}(x)\,dx.
\end{aligned}
$$

Now, taking one by one integer order integration, we get

$$\mathfrak{I}_{M,b}^{\alpha,\beta;n}\left({}_i\mathfrak{D}_{M,b}^{\alpha,\beta;n} f(t)\right) = f(t) - \sum_{k=0}^{n}\frac{f^{(k)}(b)(t-b)^k}{k!}.$$

□

3. Application

In this section, we have generalized the Kirchoff's voltage law in terms of α-RLGT M-fractional derivative which is represented by

$$ {}_i\mathfrak{D}_{M,0}^{\alpha,\beta}\ I + \frac{R}{L}\ I = \frac{E}{L}, \tag{27}$$

where I is the current with $I(0) = I_0$, R is the resistance, L is the inductance and E is the emf of the circuit.

Now, with the use of Theorem 2.3 in (27), we have

$$-\frac{dI}{dt}\frac{t(-t)^{-\alpha}}{\Gamma(\beta+1)} + \frac{R}{L}\ I = \frac{E}{L}.$$

Then

$$\frac{dI}{dt} + \frac{\Gamma(\beta+1)}{(-t)^{1-\alpha}}\frac{R}{L}\ I = \frac{\Gamma(\beta+1)}{(-t)^{1-\alpha}}\ \frac{E}{L}. \tag{28}$$

Now, we take emf $E = 0$ and replace t by $-t$ in (28), we get

$$\frac{dI}{dt} + \frac{\Gamma(\beta+1)}{t^{1-\alpha}}\frac{R}{L}\ I = 0,$$

which is a linear differential equation in I whose integrating factor is given by

$$
\begin{aligned}
I.F. &= e^{\int \frac{\Gamma(\beta+1)}{t^{1-\alpha}}\frac{R}{L}\,dt}\\
&= e^{\Gamma(\beta+1)\frac{R}{L}\int t^{\alpha-1}\,dt}\\
&= e^{\frac{R}{L\alpha}\Gamma(\beta+1)t^{\alpha}}.
\end{aligned}
$$

Therefore the solution is given by

$$I\left(e^{\frac{R}{L\alpha}\Gamma(\beta+1)t^{\alpha}}\right)=c,$$
$$I=c\ e^{-\frac{R}{L\alpha}\Gamma(\beta+1)t^{\alpha}},$$

where c is an arbitrary constant.

Now, for $I(0)=I_0$, we have

$$\begin{aligned} I(t) &= I_0\ e^{-\frac{R}{L\alpha}\Gamma(\beta+1)t^{\alpha}} \\ &= I_0 \sum_{k=0}^{\infty} \frac{\left(-\frac{R}{L\alpha}\Gamma(\beta+1)t^{\alpha}\right)^k}{k!} \\ &= I_0\ \mathbb{E}_1\left(-\frac{R}{L\alpha}\Gamma(\beta+1)t^{\alpha}\right). \end{aligned}$$

By restricting the parameters $\alpha=1, a=0$ and $\beta=1$ of the α-RLGT M-fractional derivative and then applying Theorem 2.3, for $E=0$, (27) reduces to the classical Kirchoff's voltage law

$$\frac{dI}{dt}+\frac{R}{L}I=0,\ I(0)=I_0, \tag{29}$$

whose solution is given by $I(t)=I_0e^{\frac{R}{L}t}$.

The comparison of the α-RLGT M-fractional derivative with the classical integer order derivative has been carried out in the following graphs in which the solid line represents the classical solution whereas the other lines show the solution corresponds to the α-RLGT M-fractional derivative with different values of α as shown in the Figures 1, 2 and 3.

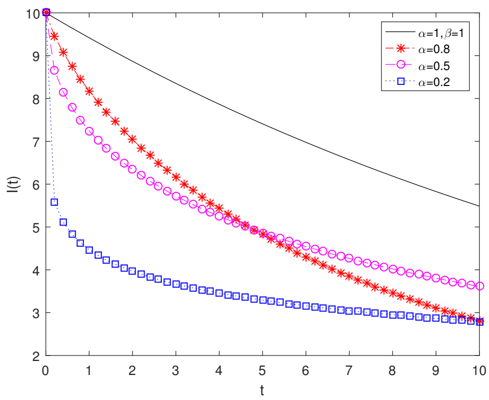

FIGURE 1. Solutions of (27) for $E=0$, $\beta=2.3$, $R=4\ \Omega$, $L=60\ mH$ and $I_0=10$

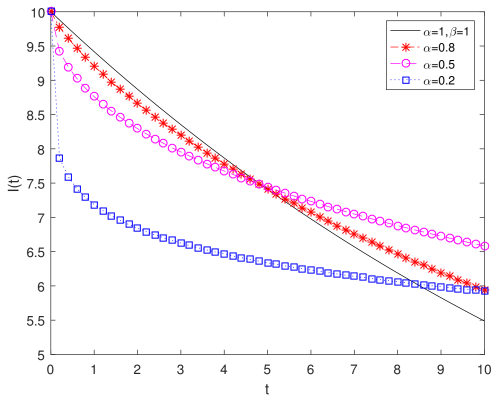

FIGURE 2. Solutions of (27) for $E = 0$, $\beta = 1.2$, $R = 4\ \Omega$, $L = 60\ mH$ and $I_0 = 10$

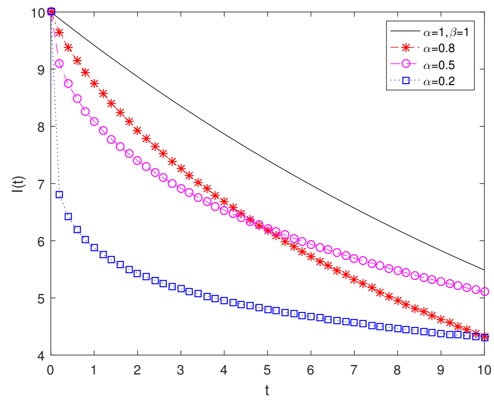

FIGURE 3. Solutions of (27) for $E = 0$, $\beta = -0.5$, $R = 4\ \Omega$, $L = 60\ mH$ and $I_0 = 10$

4. Conclusion

We have established a new fractional order derivative and its analogue as the right M-integral. In corresponds with the fractional derivatives, we have shown that the α-RLGT M-fractional derivatives responds well with respect to the classical results of the integer order calculus. Additionally, we could find the associations between the α-RLGT M-fractional derivative and right M-integral. The well known results of the calculus like the Rolle's theorem, the mean value theorem, the fundamental theorem of calculus and the theorem containing integration by parts are also generalized for our newly defined fractional derivative.

We have obtained a relation between our newly defined fractional derivative and the available fractional derivative in the literature hitherto. Also, using the proved result in the previous sections, we have obtained the generalized version of the well known physical problem, Kirchoff's voltage law, by our newly defined α-RLGT M-fractional derivative and with the use of MATLAB software, we have compared its solution with the ordinary version of the same. From the Figures 1, 2 and 3 it can be conclude that if any fractional ordered ($\alpha \in (0, 1)$) physical problem is describe by the α-RLGT M-fractional derivative, then by assigning appropriate parametric value of the parameter β from the truncated Mittag-Leffler function, one can easily approach to the existing ordinary solution.

Acknowledgement. Authors sincerely thank the referees for going through the manuscript critically and giving the valuable comments for the improvement of the manuscript.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016

[2] G. A. Anastassiou, *Intelligent Analysis*: *Fractional Inequalities and Approximations Expanded*, Springer, Cham, 2020.

[3] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer Monographs in Mathematics, Springer, Heidelberg, 2014. https://doi.org/10.1007/978-3-662-43930-2

[4] O. S. Iyiola and E. R. Nwaeze, *Some new results on the new conformable fractional calculus with application using D'Alambert approch*, Progress in Fractional Differentiation and Applications **2** (2016), no. 2, 115–122.

[5] U. N. Katugampola, *A new fractional derivative with classical properties*, arXiv:1410.6535v2, 2014.

[6] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002

[7] G. W. Leibniz, *Letter from Hanover*; *Germany*; *to G.F.A L'Hospital, September 30*; *1695*; *in Mathematische Schriften, 1849*; *reprinted 1962*, **2** (1965), 301–302.

[8] G. Mittag-Leffler, *Sur la nouvelle fonction* $\mathbb{E}_\alpha(x)$, Comptes Rendus de l'Academie des Sciences Paris **137** (1903), 554–558.

[9] E. C. de Oliveira and J. A. Tenreiro Machado, *A review of definitions for fractional derivatives and integral*, Math. Probl. Eng. **2014** (2014), Art. ID 238459, 6 pp. https://doi.org/10.1155/2014/238459

[10] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, 198, Academic Press, Inc., San Diego, CA, 1999.

[11] W. Rudin, *Principles of Mathematical Analysis*, third edition, McGraw-Hill Book Co., New York, 1976.

[12] J. Vanterler da C. Sousa and E. Capelas de Oliveira, *A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties*, arXiv:1704.08187v4, 2017.

[13] J. Vanterler da C. Sousa and E. Capelas de Oliveira, *Mittag-Leffler functions and the truncated $\mathcal{V}$-fractional derivative*, Mediterr. J. Math. **14** (2017), no. 6, Paper No. 244, 26 pp. `https://doi.org/10.1007/s00009-017-1046-z`

[14] J. Vanterler da C. Sousa and E. Capelas de Oliveira, *On the local M-derivative*, Progress in Fractional Differentiation and Applications **4** (2018), no. 4, 479–492.

Rajendrakumar B. Chauhan
Department of Mathematical Sciences
P. D. Patel Institute of Applied Sciences
Charotar University of Science and Technology
Changa, Anand-388421, Guajarat, India
Email address: `rajendrachauhan.research@gmail.com`

Meera H. Chudasama
Department of Mathematical Sciences
P. D. Patel Institute of Applied Sciences
Charotar University of Science and Technology
Changa, Anand-388421, Guajarat, India
Email address: `meera.chudasama@yahoo.co.in`